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U-Ternary Semigroups And V-Ternary Semigroup

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Abstract:

In this paper the term U-ternary semigroup is introduced. It is proved that a ternary semigroup T is a U-ternary semigroup if either T has a left (lateral, right) identity or T is generated by an idempotent. It is proved that a ternary semigroup is U-ternary semigroup if and only if (1) every proper ideal of T is contained in a proper prime ideal of T, (2) every ideal A of T is semiprime ideal of T, (3) every ideal A of T is the intersection of all prime ideal of T contains A, (4) T\A is an n-system of T or empty where A is an ideal of T, (5) T\A is an m-system of T where A is an ideal of T. Further it is proved that if T be a U-ternary semigroup. Then $T = T^3$ *and hence every maximal ideal is prime. Conversely if* $\{P_\alpha\}$ *is a collection of all prime ideals in T and if P is a maximal element in this collection, then P is a maximal ideal of T. The term dimension n is introduced and it is proved that if A is a proper ideal of the finite dimensional U-ternary semigroup T. Then A is contained in maximal ideal. The term V-ternary semigroup is introduced and proved that a ternary semigroup T is a V-ternary semigroup if and only if T has at least one proper prime ideal and if* $\{P_\alpha\}$ *is the family of all proper prime ideals, then* $< x > = T$ *for* $x \in T \setminus \cup P_\alpha$ *or T is a simple ternary semigroup.*

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Key words : *U-ternary semigroup, ideal, prime ideal, semiprime ideal, n-system, m-system, V-ternary semigroup.*

1.Introduction

Anjaneyulu (1980) made a study on primary ideals in semigroups. Later Anjanetulu (1981) made a study on semigroups in which prime ideals are maximal. The study of ternary algebraic systems had been made by Lehmer (1932), but earlier ternary structure was studied by Kerner (2000) who give the idea of n-ary algebras. Aiyared Iampan (2007) characteriae the relationship between the 0-minimal and maximal lateral ideals and the lateral 0-simple ternary semigroups. Los (1955) studied some properties of ternary semigroup and proved that every ternary semigroup can be embedded in a semigroup. Sioson (1965) introduced the ideal theory in ternary semigroups. Shabir and Bashir (2009) launched prime ideals in ternary semigroups. Sarala. Y, Anjaneyulu. A and Madhusudhana Rao. D (2013) studied about globally idempotent ternary semigroups and proved that every maximal ideal of a globally idempotent ternary semigroup T is a prime ideal of T and introduced d-system, n-system and m-system in ternary semigroup. In this paper we study the notion of U-ternary semigroup and characterize U-ternary semigroups.

2.Preliminaries

- DEFINITION 2.1 : [7] . Let T be a non-empty set. Then T is said to be a ternary semigroup if there exist a mapping from T×T×T to T which maps $(x_{1,}x_{2,}x_3) \rightarrow [x_1x_2x_3]$ satisfying the condition : $\left[\left(x_1x_2x_3\right)x_4x_5\right]=\left[x_1\left(x_2x_3x_4\right)x_5\right]=\left[x_1x_2\left(x_3x_4x_5\right)\right]\ \forall\ x_i \in$ T, $1\leq i\leq 5$.
- DEFINITION 2.2 : [7] . An element a of a ternary semigroup T is said to be an identity provided aat = taa = ata = t $\forall$ t$\in$T.
- DEFINITION 2.3 : [7] . An element a of a ternary semigroup T is said to be an idempotent element provided $a^3 = a$.
- DEFINITION 2.4 : [7] . A ternary semigroup T is said to be globally idempotent if T^3 = T.
- DEFINITION 2.5 : [7] . A nonempty subset A of a ternary semigroup T is a ternary ideal or simply ideal of T if b, c $\in$ T , a $\in$ A implies bca $\in$ A, bac $\in$ A, abc $\in$ A.
- DEFINITION 2.6 : [7] . A ternary semigroup T is said to be simple ternary semigroup if T is its only ideal of T.

- DEFINITION 2.7 : [7] .An ideal A of a ternary semigroup T is said to be a maximal ideal provided A is a proper ideal of T and is not properly contained in any proper ideal of T.
- DEFINITION 2.8 : [7] . An ideal A of a ternary semigroup T is said to be a completely prime ideal of T provided x, y, z $\in$ T and xyz $\in$ A implies either x $\in$ A or y $\in$ A or z $\in$ A.
- DEFINITION 2.9 : [7] . An ideal A of a ternary semigroup T is said to be a prime ideal of T provided X,Y,Z are ideals of T and XYZ $\subseteq$ A $\Rightarrow$ X $\subseteq$ A or Y $\subseteq$ A or Z $\subseteq$ A.
- DEFINITION 2.10 : [7] . A nonempty subset A of a ternary semigroup T is said to be an m-system provided for any a, b, c $\in$ A implies that $T^1T^1aT^1T^1bT^1T^1c\,T^1T^1 \cap A \neq \emptyset$.
- THEOREM 2.11 : [7] . An ideal A of a ternary semigroup T is a prime ideal of T if and only if T\A is an m-system of T or empty.
- DEFINITION 2.12 : [7] . An ideal A of a ternary semigroup T is said to be a completely semiprime ideal provided x $\in$ T, $x^n \in$ A for some odd natural number n >1 implies x $\in$ A.
- DEFINITION 2.13 : [7] . An ideal A of a ternary semigroup T is said to be semiprime ideal provided X is an ideal of T and $X^n \subseteq$ A for some odd natural number n implies X $\subseteq$ A.
- DEFINITION 2.14 : [7] . A non-empty subset A of a ternary semigroup T is said to be an n-system provided for any a $\in$ A implies that $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \cap A \neq \emptyset$.
- THEOREM 2.15 : [7] . Every m-system in a ternary semigroup T is an n-system.
- THEOREM 2.16 : [7] . An ideal Q of a ternary semigroup T is a semiprime ideal if and only if T\Q is an n-system of T (or) empty.
- DEFINITION 2.17 : [7] . If A is an ideal of a ternary semigroup T , then the intersection of all prime ideals of T containing A is called prime radical or simply radical of A and it is denoted by $\sqrt{A}$ or rad A.
- THEOREM 2.18 : [7] . An ideal Q of ternary semigroup T is a semiprime ideal of T if and only if $\sqrt{Q}$ = Q.

- COROLLARY 2.19 : [7] . An ideal Q of a ternary semigroup T is a semiprime ideal if and only if Q is the intersection of all prime ideal of T contains Q.

3.U-Ternary Semigroup

- DEFINITION 3.1 : A ternary semigroup T is said to be U-ternary semigroup provided for any ideal A in T, $\sqrt{A} = T$ implies $A = T$.
- EXAMPLE 3.2 : Let T be the ternary semigroup under the multiplication given in the following table.

.	a	b	c	d
a	a	a	a	a
b	a	a	a	b
c	a	a	a	a
d	a	a	c	d

Table 1

It can be easily verified that T is U-ternary semigroup.

- THEOREM 3. 3 : A ternary semigroup T is a U-ternary semigroup if either T has a left identity or T is generated by an idempotent.
- Proof : Suppose T has a left identity e. Let A be any proper ideal such that $\sqrt{A} = T$. Since $\sqrt{A} = \{ x \in T : x^n \in A$ for some odd natural number n $\} = T$. So there is an odd natural number n such that $e^n \in A$ and hence $e \in A$. Thus $T = eeT \subseteq A$. This is a contradiction. Therefore T is a U-ternary semigroup.
- THEOREM 3. 4 : A ternary semigroup T is a U-ternary semigroup if either T has a lateral identity or T is generated by an idempotent.
- Proof : Similar to theorem 3.3.
- THEOREM 3. 5 : A ternary semigroup T is a U-ternary semigroup if either T has a lateral identity or T is generated by an idempotent.
- Proof : Similar to theorem 3.3.
- NOTE 3.6 : One can note that there are U-ternary semigroups neither containing left (lateral, right) identity nor generated by an idempotent.
- EXAMPLE 3.7 : In example 3.2, we remark that the U-ternary semigroup T neither containing left (lateral, right) identity nor generated by an idempotent.

- THEOREM 3.8 : A ternary semigroup T is U-ternary semigroup if and only if every proper ideal of T is contained in a proper prime ideal of T.
- Proof : Suppose T is a U-Ternary semigroup. Let A be any proper ideal of T. If A is not contained in any proper prime ideal of T, then $\sqrt{A} = T$. Since T is a U-ternary semigroup, we have A = T, this is a contradiction. So every proper ideal is contained in a proper prime ideal of T.
  - Conversely suppose that every proper ideal is contained in a proper prime ideal of T, then clearly T is a U-ternary semigroup.
- THEOREM 3.9 : A ternary semigroup T is U-ternary semigroup if and only if every ideal A of T is semiprime ideal of T.
- Proof : Suppose that T is U-ternary semigroup. Let A is a ideal of T and $\sqrt{A} = T$. $\sqrt{A} = T$ and T is U-ternary semigroup implies that A = T. Therefore $\sqrt{A} = A$. By theorem 2.18, A is semiprime ideal of T.
  - Conversely suppose that A is semiprime ideal of ternary semigroup T and $\sqrt{A} = T$. By theorem 2.18, $\sqrt{A} = A$ and hence $\sqrt{A} = T$ implies that A = T. Therefore T is U-ternary semigroup.
- COROLLARY 3.10 : A ternary semigroup T is U-ternary semigroup if and only if every ideal A of T is the intersection of all prime ideal of T contains A.
- Proof : Suppose that T is U-ternary semigroup. Let A is a ideal of T. By theorem 3.9, A is semiprime ideal of T. By theorem 2.19, A is the intersection of all prime ideals of T contains A.
- Conversely suppose that every ideal A of ternary semigroup T is the intersection of all prime ideal of T contains A. By theorem 2.19, A is semiprime ideal of T. Therefore by theorem 3.9, T is U-ternary semigroup.
- THEOREM 3.11 : If T is a ternary semigroup and A is an ideal of T, then T is U-ternary semigroup if and only if T\A is an n-system of T or empty.
- Proof : Suppose that T is U-ternary semigroup. By theorem 3.9, an ideal A of ternary semigroup T is semiprime ideal of T. By theorem 2.16, T\A is an n-system of T or empty.
- Conversely suppose that T\A is an n-system of T or empty. By theorem 2.16, A is semiprime ideal of T. By theorem 3.9, T is U-ternary semigroup.
- COROLLARY 3.12 : If T is U-ternary semigroup and A is an ideal of T, then T\A is an m-system of T .

- Proof : Suppose that T is U-ternary semigroup. By theorem 3.11, T\A is an n-system of T. Therefore by theorem 2.15, T\A is an m-system of T.
- THEOREM 3.13 : Let T be a U-ternary semigroup. Then $T = T^3$ and hence every maximal ideal is prime. Conversely if $\{P_\alpha\}$ is a collection of all prime ideals in T and if P is a maximal element in this collection, then P is a maximal ideal of T.
- Proof : Clearly $\sqrt{T^3} = T$. Since S is a U-ternary semigroup, we have $T^3 = T$ and hence every maximal ideal is prime. If P is not maximal ideal of T, then there exists a proper ideal A of T containing P properly. Since P is a maximal element in the collection of all proper prime ideals in T, we have A is not contained in any proper prime ideal. So $\sqrt{A} = T$. Since T is a U-ternary semigroup, A = T, this is a contradiction. Therefore P is a maximal ideal of T.
- DEFINITION 3. 14 : A ternary semigroup T is said to have dimension n or n-dimensional if there exist a strictly ascending chain $P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_n$ of prime (proper) ideals in T, but no such a chain of n + 2 proper prime ideals exists in S.
- THEOREM 3. 15 : If A is a proper ideal of the finite dimensional U-ternary semigroup T. Then A is contained in maximal ideal.
- Proof : By theorem 3.8, A is contained in a proper prime ideal P_0. If P_0 is not a maximal ideal of T, then by theorem 3.9, there exists a proper prime ideal P_1 such that $P_0 \subset P_1$. If P_1 is maximal we are through. Otherwise P_1 is properly contained in proper prime ideal P_2 of T. The process of choosing P_i's must cease in a finite number of steps because of the finite dimensionality of T. Hence A is contained in a maximal ideal.
- NOTE 3.16 : In a ternary semigroup, every finite dimensional U-ternary semigroup is not a union of maximal ideals.
- EXAMPLE 3.17 : As the ternary semigroup T in example 3.2, is a finite dimensional

  U-ternary semigroup with the unique maximal ideal {a, b, c}.

4.V-Ternary Semigroup

- DEFINITION 4.1 : A Ternary semigroup T is said to be V-Ternary semigroup provided for any element $a \in T$, $\sqrt{< a >} = T$.

- NOTE 4.2 : Every U-ternary semigroup is a V-ternary semigroup. But V-ternary semigroup is not necessarily a U-ternary semigroup.
- EXAMPLE 4.3 : Let T be a ternary semigroup of all natural numbers greater than 1, under usual multiplication. The ideal A = {3, 4, …} is not contained in any proper prime ideal and hence by theorem 3.8, T is not a U-ternary semigroup. Clearly every principal ideal is contained in a proper prime ideal. So T is V-ternary semigroup.
- THEOREM 4.4 : A ternary semigroup T is a V-ternary semigroup if and only if T has at least one proper prime ideal and if $\{P_\alpha\}$ is the family of all proper prime ideals, then $< x > = T$ for $x \in T \backslash \cup P_\alpha$ or T is a simple ternary semigroup.
- Proof : Let T be V-ternary semigroup which is not a simple ternary semigroup. If T has no proper prime ideals, then $\sqrt{< a >} = T$ for every $a \in T$. Thus implies $< a > = T$ and hence T is a simple ternary semigroup. So assume T has no proper prime ideals. Then for any $a \in T \backslash \cup P_\alpha$, $\sqrt{< a >} = T$, since a does not belong to any proper prime ideal. Thus $< a > = T$. Conversely let a is any element of T such that $< a > \neq T$. If $a \in T \backslash \cup P_\alpha$, then $< a > = T$. So $a \in \cup P_\alpha$ and hence $\sqrt{< a >} \neq T$. Therefore T is a V-ternary semigroup.

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