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Vague Ideals and Normal Vague Ideals in Γ-Semirings

Y. Bhargavi
Research Scholar, Department of Mathematics, K. L. University,
Vaddeswaram, Guntur District, Andhra Pradesh, India
T. Eswarlal
Associate Professor, Department of Mathematics, K. L. University,
Vaddeswaram, Guntur District, Andhra Pradesh, India

Abstract:
The notions of left(right) vague ideal and normal left(right) vague ideals of Γ-semirings with membership and non-membership functions taking values in unit interval of real numbers are introduced which generalize the existing notions left(right) fuzzy ideal and normal left(right) fuzzy ideal of Γ-semiring and studied various properties. Further we proved, a non-constant maximal element in set of all normal left (right) vague idea of a Γ-smearing takes only two values [0,0] and [1,1].
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1. Introduction

Gau.W. L and Buehrer. D. J [6] proposed the theory of vague sets as an improvement of theory of fuzzy sets in approximating the real life situations. According to them a vague set A in the universe of discourse U is a pair (t_A, f_A), where t_A and f_A are fuzzy subsets of U satisfying the condition $t_A(u) + f_A(u) \leq 1, \forall\ u \in U$. Even though the vague sets of Gau. W.L and Buehrer. D.J[6] and Attanasov. K.T.[1]'s intuitionist fuzzy sets are mathematically equivalent objects, there is a controversy about the name intuitionistic fuzzy sets. Without entering into that controversy we prefer the terminology of vague sets. Ranjit Biswas [15] initiated the study of vague algebra by introducing the concepts of vague groups, vague cuts, vague normal groups etc. Further, Ramakrishna. N[14] and Eswarlal.T[5] continued the study of vague algebra by studying the characterization of cyclic groups in terms of vague groups, vague normal groups, vague normalizer, vague centralizer, vague ideals, normal vague ideals, vague fields, vague vector spaces etc.
Dutta. T. K and Biswas. B. K[4] studied fuzzy ideals, fuzzy prime ideals of semirings, to continue this Jun.Y.B, Neggers.J and Kim.H.S[8] extended the concept of an L-fuzzy (characteristic) left(right) ideal of a ring to a semiring. Further they introduced and studied normal fuzzy ideals in semirings and established various properties related to them. In 1995, the concept of Γ-semiring was introduced by M. K. Rao[12] as a generalization of Γ-ring by Nobusawa.N[13] as well as semiring. The concepts of Γ-semirings and its sub Γ-semirings with a left(right) unity was studied by Luh.J[11] and M.K.Rao[12], further the ideals, prime ideals, semiprime ideals, k-ideals and h-ideals of a Γ-semiring, regular Γ-semiring were extensively studied by Kyuno.S[10] and M.K.Rao[12]. The properties of an ideal in semirings and Γ-semirngs were somewhat different from the properties of the usual ring ideals. Moreover the notion of Γ-semiring not only generalizes the notions of semiring and Γ-ring but also the notion of ternary semiring. In this paper, we introduced and studied the concept of left(right) vague ideal of a Γ-semiring and established a one-one correspondence between left(right) vague ideal A and its vague cut, $A_{(\alpha,\beta)}$, where $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$ and also we studied homomorphic property of left(right) vague ideals of a Γ-semiring. Further we introduced and studied the concept of normal left(right) vague ideal of a Γ-semiring and proved, a non-constant maximal element in set of all normal left(right) vague ideal of a Γ-semiring takes only two values [0, 0] and [1, 1].
Throughout this paper, R stands for Γ-semiring. That is let R and Γ be two additive commutative semigroups. Then R is called Γ-semiring if there exists a mapping
$R \times \Gamma \times R \to R$ image to be denoted by aαb for a, b ∈ R and α ∈ Γ satisfying the following conditions.

1. $a\alpha(b + c) = a\alpha b + a\alpha c$
2. $(a + b)\alpha c = a\alpha c + b\alpha c$
3. $a(\alpha + \beta)c = a\alpha c + a\beta c$
4. $a\alpha(b\beta c) = (a\alpha b)\beta c,\ \forall\ a, b, c \in R;\ \alpha, \beta \in \Gamma$.

2. Preliminaries

- Definition 2.1: A nonempty subset S of a Γ-semiring R is said to be a left(right) ideal of R if (S, +) is a sub semigroup of (R, +) and $x\alpha a \in S$ ($a\alpha x \in S$), $\forall a \in S; x \in R; \alpha \in \Gamma$.
- Definition 2.2: Let X be any non-empty set. A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy subset of R.
- Definition 2.3: A vague set A in the universe of discourse U is a pair (t_A, f_A), where $t_A : U \rightarrow [0, 1]$, $f_A : U \rightarrow [0, 1]$ are mappings such that $t_A(u) + f_A(u) \leq 1$, $\forall u \in U$. The functions t_A and f_A are called true membership function and false membership function respectively.
- Definition 2.4: The interval $[t_A(u), 1 - f_A(u)]$ is called the vague value of u in A and it is denoted by $V_A(u)$ i.e., $V_A(u) = [t_A(u), 1 - f_A(u)]$.
- Definition 2.5: A vague set A is contained in the other vague set B, $A \subseteq B$ if and only if $V_A(u) \leq V_B(u)$ i.e., $t_A(u) \leq t_B(u)$ and $1 - f_A(u) \leq 1 - f_B(u)$, $\forall u \in U$.
- Definition 2.6: Two vague sets A and B are equal written as A = B, if and only if $A \subseteq B$ and $B \subseteq A$ i.e., $V_A(u) \leq V_B(u)$ and $V_B(u) \leq V_A(u)$, $\forall u \in U$.
- Definition 2.7: The union of two vague sets A and B with respective truth membership and membership functions t_A, f_A : t_B, f_B is a vague set C, written as $C = A \cup B$, whose truth membership and false membership functions are related to those of A and B by $t_C = \max\{t_A, t_B\}$ and $1 - f_C = \max\{1 - f_A, 1 - f_B\} = 1 - \min\{f_A, f_B\}$.
- Definition 2.8: The intersection of two vague sets A and B with respective truth membership and membership functions t_A, f_A : t_B, f_B is a vague set C, written as $C = A \cap B$, whose truth membership and false membership functions are related to those of A and B by $t_C = \min\{t_A, t_B\}$ and $1 - f_C = \min\{1 - f_A, 1 - f_B\} = 1 - \max\{f_A, f_B\}$.
- Definition 2.9: A vague set A of a set U with $t_A(u) = 0$ and $f_A(u) = 1$, $\forall u \in U$ is called zero vague set of U.
- Definition 2.10: A vague set A of a set U with $t_A(u) = 1$ and $f_A(u) = 0$, $\forall u \in U$ is called unit vague set of U.
- Definition 2.11: Let A be a vague set of a universe U with true membership function t_A and false membership function f_A. For $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, the (α, β)- cut or vague cut of a vague set A is the crisp subset of U is given by $A_{(\alpha,\beta)} = \{x \in U / V_A(x) \geq [\alpha, \beta]\}$ i.e., $A_{(\alpha,\beta)} = \{x \in U / t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta\}$.
- Definition 2.12: The α-cut, A_α of the vague set A is the (α, α)-cut of A and hence given by $A_\alpha = \{x \in U / t_A(x) \geq \alpha\}$.
- Definition 2.13: Let f be a mapping from a set X into a set Y. Let B be a vague set in Y. Then the inverse image of B, f^{-1} is the vague set in X by $V_{f^{-1}(B)}(x) = V_B(f(x))$, $\forall x \in X$.
- Definition 2.14: Let f be a mapping from a set X into a set Y . Let A be a vague set in X with vague value V_A. Then the image of A, f(A) is the vague set in Y defined by

$$V_{f(A)}(y) = \begin{cases} \sup V_A(z), z \in f^{-1}(y) if \quad f^{-1}(y) \neq \phi \\ 0 \quad otherwise \end{cases}$$, $\forall y \in Y$, where $f^{-1}(y) = \{x / f(x) = y\}$.

- Definition 2.15: Let R_1 be a Γ_1-semiring and R_2 be a Γ_2-semiring. Then $(f, g) : (R_1, \Gamma_1) \rightarrow (R_2, \Gamma_2)$ is called a homomorphism if $f : R_1 \rightarrow R_2$ and $g : \Gamma_1 \rightarrow \Gamma_2$ are homomorphisms of semigroups such that $f(x\gamma y) = f(x)g(\gamma)f(y)$, $\forall x, y \in R_1; \gamma \in \Gamma_1$.
- Notation: Let I[0, 1] denote the family of all closed sub intervals of [0, 1]. I1 = $[a_1, b_1]$ and $[a_2, b_2]$ are two elements of I[0, 1]. We call $I_1 \geq I_2$, if $a_1 \geq a_2$ and $b_1 \geq b_2$, with the order in I[0, 1] is a lattice with the operations min. or inf and max. or sup given by $\text{imin}\{I_1, I_2\} = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$ and $\text{imax}\{I_1, I_2\} = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$. Also we denote $I_1 + I_2 = [a_1 + a_2, b_1 + b_2]$ and $(I_1 + I_2)/2 = [(a_1 + a_2)/2, (b_1 + b_2)/2]$

3. Vague Ideals of Γ–Semirings

In this section we introduce the notion of left(right) vague ideal of R. We established a one-to-one correspondence between left(right) vague ideals of R and crisp left(right) ideals of R.

- Definition 3.1: A vague set $A = (t_A, f_A)$ on R is said to be vague left (right) ideal of R if the following conditions are true:

For all $x, y \in R; \gamma \in \Gamma$,
$V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$ and
$V_A(x\gamma y) \geq V_A(y)(V_A(x\gamma y) \geq V_A(x))$
i.e., (i). $t_A(x + y) \geq \min\{t_A(x), t_A(y)\}$,
$1 - f_A(x + y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$ and
(ii). $t_A(x\gamma y) \geq t_A(y)(t_A(x\gamma y) \geq t_A(x))$,
$1 - f_A(x\gamma y) \geq 1 - f_A(y)$ $(\geq 1 - f_A(x))$.

- Example 3.2: Let R be the set of 2 x 2 matrices over the set of all non-negative integers and be the set of 2 x 2 matrices over the set of all non-negative integers. Then R, Γ are additive commutative semigroups.

Define the mapping $R \times \Gamma \times R \rightarrow R$ by XPY is the usual matrix multiplication of X, P, Y, $\forall X, Y \in R; P \in \Gamma$.
Then R is a -semiring.
Let $A = (t_A, f_A)$, where t_A: $R \rightarrow [0, 1]$ and f_A: $R \rightarrow [0, 1$ defined by

$$t_{(A)}(X) = \begin{cases} 0.8, & if \quad X \quad is \quad of \quad the \quad form \quad \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \\ 0.6 & otherwise \end{cases}$$ and

$$f_{(A)}(X) = \begin{cases} 0.1, & if \quad X \quad is \quad of \quad the \quad form \quad \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \\ 0.4 & otherwise \end{cases}$$

Then A is left vague ideal but not right vague ideal of R.

- Example 3.3: Let R be the set of 2 x 2 matrices over the set of all non-negative integers and be the set of 2 x 2 matrices over the set of all non-negative integers. Then R, Γ are additive commutative semigroups.

Define the mapping R × Γ × R → R by XP Y matrix multiplication of X, P, Y, ∀ X, Y ∈ R; P∈ Γ.
Then R is a -semiring.
Let A = (t_A, f_A), where t_A: R→ [0, 1] and f_A: R→ [0, 1 defined by

$$t_{(A)}(X) = \begin{cases} 0.8, & if \quad X \quad is \quad of \quad the \quad form \quad \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \\ 0.6 & otherwise \end{cases}$$ and

$$f_{(A)}(X) = \begin{cases} 0.1, & if \quad X \quad is \quad of \quad the \quad form \quad \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \\ 0.4 & otherwise \end{cases}$$

Then A is right vague ideal but not left vague ideal of R.

- Example 3.4: Let R be the set of natural numbers with zero and let be the set of positive even integers. Then R, Γ are additive commutative semigroups.

Define the mapping R × Γ × R → R by aαb usual product of a, α, b, ∀ a, b ∈ R; α ∈ Γ.
Then R is a -semiring.
Let A = (t_A, f_A), where t_A: R→ [0, 1] and f_A: R→ [0, 1 defined by

$$t_{(A)}(x) = \begin{cases} 0.5, & if \;\; x \; is \; even \; or \;\; 0 \\ 0.3 & otherwise \end{cases} \quad \text{and} \quad f_{(A)}(x) = \begin{cases} 0.5, & if \;\; x \; is \; even \; or \;\; 0 \\ 0.6 & otherwise \end{cases}$$

Then A is left vague ideal and right vague ideal of R.

- Example 3.5: Let R be the set of negative integers and be the set of negative even integers. Then R, Γ are additive commutative semigroups.

Define the mapping R × Γ × R → R by aαb usual product of a, α, b, ∀ a, b ∈ R; α ∈ Γ. Then R is a -semiring.
Let A = (t_A, f_A), where t_A: R→ [0, 1] and f_A: R→ [0, 1 defined by

$$t_{(A)}(x) = \begin{cases} 0.5, & if \quad x \;\; = -1 \\ 0.6, & if \quad x \;\; = -2 \\ 0.8 & if \quad x < -2 \end{cases} \quad \text{and} \quad t_{(A)}(x) = \begin{cases} 0.6, & if \quad x \;\; = -1 \\ 0.4, & if \quad x \;\; = -2 \\ 0.1 & if \quad x < -2 \end{cases}$$

Then A is left vague ideal and right vague ideal of R.

- Lemma 3.6: Let A = (t_A, f_A) be a left (right) vague ideal of R. For α, β ∈ [0, 1] if α ≥ β, then

1. $A_\alpha \subseteq A_\beta$
2. $A_{(\alpha,\alpha)} = A_\alpha$.

Proof:

1. Let x ∈ A_α.
   i.e., $t_A(x) \geq \alpha$.
   Since α ≥ β, we have $t_A(x) \geq \beta$.
   Hence x ∈ A_β.
   Thus $A_\alpha \subseteq A_\beta$.
2. Let x ∈ $A_{(\alpha,\alpha)}$.
   That implies $V_A(x) \geq [\alpha, \alpha]$.
   i.e., $t_A(x) \geq \alpha$.
   Then x ∈ A_α.
   Let x ∈ A_α.

i.e., $t_A(x) \geq \alpha$.
Then $1 - f_A(x) \geq \alpha$.
So, $V_A(x) \geq [\alpha, \alpha]$.
We have $x \in A_{(\alpha,\alpha)}$.
Hence $A_{(\alpha,\alpha)} = A_\alpha$.

- Theorem 3.7: A necessary and sufficient condition for a vague set $A = (t_A, f_A)$ of R to be a left(right) vague ideal of R is that t_A and $1 - f_A$ are left(right) fuzzy ideals of R.

Proof: Suppose $A = (t_A, f_A)$ is a left(right) vague ideal of R.
Let $x, y \in R$; $\gamma \in \Gamma$.
We have $V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$ and
$V_A(x\gamma y) \geq V_A(y)$ ($\geq V_A(x)$)
i.e., 1. $t_A(x + y) \geq \min\{t_A(x), t_A(y)\}$,
$1 - f_A(x + y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$ and
2. $t_A(x\gamma y) \geq t_A(y)$ ($\geq t_A(x)$),
$1 - f_A(x\gamma y) \geq 1 - f_A(y)$ ($\geq 1 - f_A(x)$).
Hence t_A and $1 - f_A$ are fuzzy Γ-semirings of R.
The converse part is obvious from the definition.

- Theorem 3.8: A vague set A of R is a left (right) vague ideal of R if and only if for all $\alpha, \beta \in [0, 1]$, the (α, β) - cut, $A_{(\alpha,\beta)}$ is a left (right) ideal of R.

Proof: Suppose A is a left(right) vague ideal of Γ-semiring R.
Let $x, y \in A_{(\alpha,\beta)}$; $a \in R$; $\gamma \in \Gamma$.
$\Rightarrow V_A(x) \geq [\alpha, \beta]$ and $V_A(y) \geq [\alpha, \beta]$
We have 1.$V_A(x + y) \geq \min\{V_A(x), V_A(y)\} \geq [\alpha, \beta]$
$\Rightarrow x + y \in A_{(\alpha,\beta)}$.
2. $V_A(a\gamma x) \geq V_A(x)$ ($V_A(x\gamma a) \geq V_A(x)$) $\geq [\alpha, \beta]$.
$\Rightarrow a\gamma x$ ($x\gamma a$) $\in A_{(\alpha,\beta)}$.
Hence $A_{(\alpha,\beta)}$ is a left(right) vague ideal of R.
Conversely suppose that $A_{(\alpha,\beta)}$ is a left(right) vague ideal of R.
Let $x, y \in R$ and $\gamma \in \Gamma$.
Let $V_A(x) = [\alpha_1, \beta_1]$ and $V_A(y) = [\alpha_2, \beta_2]$ with $[\alpha_2, \beta_2] \leq [\alpha_1, \beta_1]$.
put $[\alpha, \beta] = \min\{[\alpha_1, \beta_1], [\alpha_2, \beta_2]\}$.
Then $x, y \in A_{(\alpha,\beta)}$.
$\Rightarrow x + y \in A_{(\alpha,\beta)}$ and $x\gamma y, y\gamma x \in A_{(\alpha,\beta)}$.
$\Rightarrow V_A(x + y) \geq [\alpha, \beta] = \min\{V_A(x), V_A(y)\}$ and
$V_A(x\gamma y) \geq [\alpha, \beta] = V_A(y)$ ($V_A(y\gamma x) \geq [\alpha, \beta] = V_A(y)$).
Hence A is a left (right) vague ideal of R.
Notation**:** We denote the vague values $\mathbf{1} = [1, 1]$ and $\mathbf{0} = [0, 0]$ throughout this paper.

- Definition 3.9: Let $\delta = (t_\delta, f_\delta)$ be a vague set of R. For any subset S of R , the characteristic function of S taking values in [0, 1] of a vague set $\delta_S = (t_{\delta_S}, f_{\delta_S})$ by

$$V_{\delta_S}(x) = \begin{cases} [1,1] & if \ x \in S \\ [0,0] & if \ x \notin S \end{cases}$$

i.e., $t_{\delta_S}(x) = \begin{cases} 1 & if \ x \in S \\ 0 & if \ x \notin S \end{cases}$ and $f_{\delta_S}(x) = \begin{cases} 0 & if \ x \in S \\ 1 & if \ x \notin S \end{cases}$

Then δ_S is called the vague characteristic set of S in [0, 1].

- Theorem 3.10: Let S be a non-empty subset of R. Then δ_S is a left(right) vague ideal of R if and only if S is a left(right) ideal of R.

Proof: Suppose δ_S is a left(right) vague ideal of R.
Let $x, y \in S$ and $\gamma \in \Gamma$.
We have 1. $V_{\delta_S}(x + y) \geq \min\{V_{\delta_S}(x), V_{\delta_S}(y)\} = \mathbf{1}$ and
2. $V_{\delta_S}(x\gamma y) \geq V_{\delta_S}(y)$ ($\geq V_{\delta_S}(x)$) $= \mathbf{1}$.
which implies that $x + y \in S$ and $x\gamma y \in S$.
Hence S is a left(right) ideal of R.
Conversely assume that S is a left (right) ideal of R.

Let x, y ∈ R and γ ∈ Γ.
If x, y ∈ S, then x + y ∈ S and xγy ∈ S.
So, 1. V_{δ_S} (x + y) = **1** = min{V_{δ_S} (x), V_{δ_S} (y)} and

2. V_{δ_S} (xγy) = **1** = V_{δ_S} (y) (=V_{δ_S} (x)).

If x, y ∉ S, then x + y ∉ S and xγy ∉ S.
So, 1. V_{δ_S} (x + y) = **0** = min{V_{δ_S} (x), V_{δ_S} (y)} and

2. V_{δ_S} (xγy) = **0** = V_{δ_S} (y) (= V_{δ_S} (x)).

If x ∉ S and y ∈ S, then x + y ∉ S and xγy ∉ S.
So, 1. V_{δ_S} (x + y) = **0** = min{V_{δ_S} (x), V_{δ_S} (y)} and

2. V_{δ_S} (xγy) = **0** = V_{δ_S} (y)(= V_{δ_S} (x)).

A Similar argument for x ∈ S and y ∉ S.
Hence δ_S is a left(right) vague ideal of R.

- Theorem 3.11: Let A be a left (right) vague ideal of R. Then

1.$V_A(0) \geq V_A(x)$, ∀ x ∈ R.
2.$R_A = \{x \in R / V_A(x) = V_A(0)\}$ is a left(right) ideal of R.
Proof: Let x, y ∈ R; γ ∈ Γ.
1.$V_A(0) = V_A(0.\gamma.x) \geq V_A(x)$
that implies $V_A(0) \geq V_A(x)$.
2. Let x, y ∈ R_A.
that implies $V_A(x) = V_A(0)$ and $V_A(y) = V_A(0)$.
Now, $V_A(x + y) \geq \min\{V_A(x), V_A(y)\} = V_A(0)$ and
$V_A(x\gamma y) \geq V_A(y) = V_A(0)$ ($\geq V_A(x) = V_A(0)$).
Then x + y, xγy ∈ R_A.
Hence R_A is a left(right) ideal of R.

- Theorem 3.12: Let R_1 be a Γ_1-semiring and R_2 be a Γ_2-semiring and let f be a homomorphism of R_1 to R_2. If B is a left(right) vague ideal of R_2, then the inverse image of B, $f^{-1}(B)$ is a left(right) vague ideal of R_1.

Proof: For all x, y ∈ R_1; γ ∈ Γ_1,

1. $V_{f^{-1}(B)}(x + y) = V_B(f(x + y))$
$= V_B(f(x) + f(y))$
$\geq \min\{V_B(f(x)), V_B(f(y))\}$
$= \min\{V_{f^{-1}(B)}(x), V_{f^{-1}(B)}(y)\}$

2. $V_{f^{-1}(B)}(x\gamma y) = V_B(f(x\gamma y))$
$= V_B(f(x)g(\gamma)f(y))$
$\geq V_B(f(y))$ ($\geq V_B(f(x))$)

Hence $f^{-1}(B)$ is a left(right) vague ideal on R_1.

- Definition 3.13: A vague set A of R is said to have the Sup. property if for any subset

S of R, there exists x_0 ∈ S such that $V_A(x_0) = \sup_{x\in S} V_A(x)$.

- Theorem 3.14: Let R_1 be a Γ_1-semiring and R_2 be a Γ_2-semiring and let f be a homomorphism of R_1 onto R_2. If A is a left(right) vague ideal of R_1 with Sup. property, then the homomorphic image of A, f(A) is a left(right) vague ideal of R_2.

Proof: Let x, y ∈ R_2; $\gamma_2 \in \Gamma_2$.
If either $f^{-1}(x)$ or $f^{-1}(y)$ is empty then the result is trivially satisfied.
Suppose neither $f^{-1}(x)$ nor $f^{-1}(y)$ is non-empty.
Let $x_0 \in f^{-1}(x)$ and $y_0 \in f^{-1}(y)$ be such that $V_A(x_0) = \sup V_A(a)$ where a ∈ f−1(x) and $V_A(y_0) = \sup V_A(b)$ where b ∈ $f^{-1}(y)$

then 1.$V_{f(A)}(x+y) = \sup_{z\in f^{-1}(x+y)} V_A(z)$
$\geq V_A(z)$, z ∈ $f^{-1}(x+ y)$
$= V_A(x_0 + y_0)$
$\geq \min\{V_A(x_0), V_A(y_0)\}$
$= \min\{V_{f(A)}(x), V_{f(A)}(y)\}$.

2. $V_{f(A)}(x\gamma_2 y) = \sup_{z\in f^{-1}(x\gamma_2 y)} V_A(z)$
$\geq V_A(z),\ z \in f^{-1}(x\gamma_2 y)$
$= V_A(x_0\gamma_1 y_0),\ \gamma_1 \in \Gamma_1$
$\geq V_A(y_0)$
$= V_{f(A)}(y)$.

Hence f(A) is a left(right) vague ideal of R_2.

4. Normal Vague ideals of Γ–semirings

- Definition 4.1: A vague set $A = (t_A, f_A)$ of R is said to be normal, if $V_A(0) = 1$.

i.e., $t_A(x) = 1$ and $f_A(x) = 0$.

- Lemma 4.2: Let $A = (t_A, f_A)$ be a vague set of R such that $t_A(x) + f_A(x) \leq t_A(0) + f_A(0)$, $\forall\ x \in R$. Define $A^+ = (t_{A^+}, f_{A^+})$, where $t_{A^+}(x) = t_A(x) + 1 - t_A(0)$ and $f_{A^+}(x) = f_A(x) - f_A(0)$, $\forall\ x \in R$. Then A^+ is a normal vague set.

Proof: First we show that A^+ is a vague set.
Let $x \in R$.
Now, $t_{A^+}(x) + f_{A^+}(x) = t_A(x) + 1 - t_A(0) + f_A(x) - f_A(0) \leq 1$.
Thus A^+ is a vague set.
Also $t_{A^+}(0) = 1$ and $f_{A^+}(0) = 0$.
Hence A^+ is a normal vague set.

- Theorem 4.3: Let $A = (t_A, f_A)$ be a left(right) vague ideal of R. Then the vague set A^+ is a normal left(right) vague ideal of R containing A.

Proof: Let $x, y \in R$; $\gamma \in \Gamma$.
Now, $V_{A^+}(x+y) = V_A(x + y) + \mathbf{1} - V_A(0)$
$\geq \min\{V_A(x), V_A(y)\} + \mathbf{1} - V_A(0)$
$= \min\{V_A(x) + \mathbf{1} - V_A(0), V_A(y) + \mathbf{1} - V_A(0)\}$
$= \min\{V_{A^+}(x), V_{A^+}(y)\}$ and

$V_{A^+}(x\gamma y) = V_A(x\gamma y) + \mathbf{1} - V_A(0)$
$\geq V_A(y) + \mathbf{1} - V_A(0)$
$= V_A(y) + \mathbf{1} - V_A(0)\ (\geq V_A(x) + \mathbf{1} - V_A(0))$
$= V_{A^+}(y)\ (= V_{A^+}(x))$

Also $V_{A^+}(0) = V_A(0) + \mathbf{1} - V_A(0) = \mathbf{1}$.
Thus A^+ is a normal left(right) vague ideal of a Γ-semiring R.
Clearly $A \subset A+$.

- Corollary 4.4: If A is a left(right) vague ideal of R satisfying $V_{A^+}(0) = 0$, for some $x \in R$. Then $V_A(x) = 0$.
- Theorem 4.5: A left(right) vague ideal A of a Γ-semiring R is normal if and only if

$A^+ = A$.
Proof: Suppose that A is normal left(right) vague ideal of R.
Let $x \in R$.
Then $t_{A^+}(x) = t_A(x) + 1 - t_A(0) = t_A(x) + 1 - 1 = t_A(x)$.
$f_{A^+}(x) = f_A(x) - f_A(0) = f_A(x) - 0 = f_A(x)$.
Thus $A^+ = A$.
The converse is obvious.

- Theorem 4.6: Let A, B be two left(right) vague ideals of R. Then 1.$(A^+)^+ = A$

2.$(A \cap B)^+ = A^+ \cap B^+$.
3.$(A \cup B)+ = A^+ \cup B^+$.
4.$A \subseteq B \Rightarrow A^+ \subseteq B^+$.
Proof: Let $x \in R$.
1. $t_{(A^+)^+}(x) = t_{A^+}(x) + 1 - t_{A^+}(0)$
$= t_{A^+}(x)$ (since A is left (right) vague ideal $\Rightarrow A^+$ is normal).

$f_{(A^+)^+}(x) = f_{A^+}(x) - f_{A^+}(0)$

$= f_{A^+}(x)$ (since A is left(right) vague ideal ⇒ A^+ is normal).

Thus $(A^+)^+ = A^+ = A$ (from theorem-4.5).

2. $t_{(A\cap B)^+}(x) = t_{A\cap B}(x) + 1 - t_{A\cap B}(0)$

$= \min\{t_A(x), t_B(x)\} + 1 - \min\{t_A(0), t_B(0)\}$

$= \min\{t_A(x) + 1 - t_A(0), t_B(x) + 1 - t_B(0)\}$

$= \min\{t_{A^+}(x), t_{B^+}(x)\}$

$= t_{A^+\cap B^+}(x)$

Similarly, we can prove that $1 - f_{(A\cap B)^+} = 1 - f_{A^+\cap B^+}$.

Hence $(A \cap B)^+ = A^+ \cap B^+$.

3. $t_{(A\cup B)^+}(x) = t_{A\cup B}(x) + 1 - t_{A\cup B}(0)$

$= \max\{t_A(x), t_B(x)\} + 1 - \max\{t_A(0), t_B(0)\}$

$= \max\{t_A(x) + 1 - t_A(0), t_B(x) + 1 - t_B(0)\}$

$= \max\{t_{A^+}(x), t_{B^+}(x)\}$

$= t_{A^+\cup B^+}(x)$

Similarly, we can prove that $1 - f_{(A\cup B)^+} = 1 - f_{A^+\cup B^+}(x)$.

Hence $(A \cup B)^+ = A^+ \cup B+$.

4. $t_{A^+}(x) = t_A(x) + 1 - t_A(0)$

$\leq t_B(x) + 1 - t_B(0)$

$= t_{B^+}(x)$

$f_{A^+}(x) = f_A(x) - fA(0)$

$\leq f_B(x) - f_B(0)$

$= f_{B^+}(x)$

Hence $A^+ \subseteq B^+$.

- Theorem 4.7: Let A be a left (right) vague ideal of R. If there exists a left(right) vague ideal B of R satisfying $B^+ \subseteq A$, then A is normal.

Proof: Assume that there exists a left (right) vague ideal B of R satisfying $B^+ \subseteq A$.

So, $1 = V_{B^+}(0) \leq V_A(0)$.

We get $V_A(0) = 1$.

Thus A is normal.

- Corollary 4.8: Let A be a left (right) vague ideal of R. If there exists a left(right) vague ideal B of R satisfying $B^+ \subseteq A$, then $A^+ = A$.

Let N(R) denotes the set of all normal left (right) vague ideals of R. Then (N(R), ⊆) is a poset.

- Theorem 4.9: Let A ∈ N(R) be a non-constant maximal element of (N(R), ⊆). Then A takes only two values **0** and **1**.

Proof**:** Let A be a normal left (right) vague ideal of R.

Then $V_A(0) = \mathbf{1}$.

Let x ∈ R.

Suppose that $V_A(x) \neq \mathbf{1}$.

We have to show that $V_A(x) = \mathbf{0}$.

Assume that there exists $x_0 \in R$ such that $\mathbf{0} < V_A(x_0) < \mathbf{1}$.

Define a vague set $B = (t_B, f_B)$ on R by $V_B(x) = V_A(x) + V_A(x_0)/ 2$,∀ x ∈ R.

i.e., $t_B(x) = t_A(x) + t_A(x_0)/ 2$ and $f_B(x) = f_A(x) + f_A(x_0)/ 2$,∀ x ∈ R.

Clearly B is well-defined.

Let x, y ∈ R; γ ∈Γ.

then 1.$V_B(x+y) = V_A(x+y) + V_A(x_0)/ 2$

$\geq \min\{V_A(x), V_A(y)\} + V_A(x_0)/ 2$

$= \min\{V_A(x) + V_A(x_0)/ 2 , V_A(y) + V_A(x_0)/ 2 \}$

$= \min\{V_B(x), V_B(y)\}$

2.$V_B(x\gamma y) = V_A(x\gamma y) + V_A(x_0)/ 2$
$\geq V_A(y) + V_A(x_0)/ 2$ ($\geq V_A(x) + V_A(x_0)/ 2$)
$= V_B(y)$ ($= V_B(x)$).

Thus B is a left(right) vague ideal of R.

Now, $V_{B^+}(x) = V_B(x) + \mathbf{1} - V_B(0)$
$= V_A(x) + V_A(x_0)/ 2 + \mathbf{1} - V_A(0) + V_A(x_0)/ 2$
$= V_A(x)+\mathbf{1}/ 2$.

That implies $V_{B^+}(0) = V_A(0) + \mathbf{1}/ 2 = \mathbf{1}$.

Thus B+ is a normal left(right) vague ideal of R.

Now, $V_{B^+}(0) = \mathbf{1} > V_A(x_0)$.

So, B^+ is a non-constant normal left(right) vague ideal of R and $V_{B^+}(x_0) > V_A(x_0)$.

It follows that A is not maximal.
Hence $V_A(x) = \mathbf{0}$.
Thus A takes only two values **0** and **1**.

- Definition 4.10: A normal left(right) vague ideal A of R is said to be complete normal if there exists $x \in R$ such that $V_A(x) = 0$.

Let C(R) denotes the set of all complete normal left (right) vague ideals of R. Clearly $C(R) \subseteq N(R)$ and that $(C(R), \subseteq)$ is a poset.

- Theorem 4.11: Any non-constant maximal element of $(N(R), \subseteq)$ is also a maximal element of $(C(R), \subseteq)$.

Proof: let A be a non-constant maximal element of $(N(R), \subseteq)$.
Then A takes only two values **0** and **1**.
i.e., $V_A(0) = \mathbf{1}$ and $V_A(x) = \mathbf{0}$, for some $x \in R$.
That implies $A \in C(R)$.
Suppose $B \in C(R)$ such that $A \subseteq B$.
So, $B \in N(R)$.
Since A is maximal in N(R) and $B \in N(R)$ with $A \subseteq B$, that gives A = B.
Hence A is maximal element in C(R).

5. Conclusion

In this paper, the notions of left (right) vague ideals and normal left(right) vague ideals of Γ-semirings with membership and non-membership functions taking values in unit interval of real numbers are introduced. In future it is expected that these structures are useful in developing vague homomorphism, vague prime ideals, vague maximal ideals and vague semiprime ideals of a Γ-semiring.

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7. References

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