INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH & DEVELOPMENT

ISSN 2278 – 0211 (Online)

A Comparison of Methods of Estimating the Parameters of the Three-Parameter Weibull Distribution with Application to Reliability Analysis

Ogbonna Uchenna Austine
Student, Department of Statistics,
Nnamdi Azikiwe University Awka, Anambra State, Nigeria
Obiora-Ilouno Happiness
Associate Professor, Department of Statistics,
Nnamdi Azikiwe University Awka, Anambra State, Nigeria

Abstract:
In this work, eight methods of estimating the parameters of a three-parameter Weibull distribution were discussed and compared. The methods are; method of moments (MOM), maximum likelihood method (MLE), percentile method (PM), method of L-moments (LM), Teimouri and Gupta Beta (MGB), maximum product of spacing method (MPS), modified method of moments (MMM) and the Goda's polynomial method (GPM). A simulation study was carried out were samples of different sizes with different shape parameter (β) values were generated and the five methods were applied to the samples. The root means square error (RMSE) was the basis for comparison of the methods based on their ability to estimate each parameter accurately. The Euclidean norm was also used to compare performance of methods based on their ability to accurately estimate the three parameters. The results show that the Mahdi and Gupta method is the best method for estimating the parameters of a three-parameter Weibull distribution in almost all the simulation conditions. The maximum product of spacing performed second best. It was also discovered that sample size does not really affect the choice of method but the accuracy of all the methods increases with sample. An application of these methods to real life data was also demonstrated here.

Keywords: *Weibull distribution, estimation, simulation, comparison, L-moments*

1. Introduction

Weibull distribution has proven to be a successful model for many product failure mechanisms because it is a flexible distribution given that it can for example take the form of either the exponential distribution or the appropriate normal distribution and can be skewed either positively or negatively. It is a distribution with a wide variety of possible failure rate curves. However, Lloyd (1967), ku et al (1972), Mc cool (1998) and Hammit (2004) as well as so many others have expanded the scope and usefulness of the Weibull distribution to other branches of statistics such as quality control.
The Weibull distribution has the following pdf;

$$f(x) = \frac{\beta}{\theta}\left(\frac{x-\gamma}{\theta}\right)^{\beta-1} e^{\left(\frac{x-\gamma}{\theta}\right)^{\beta}} \quad x > \gamma, \beta > 0, \theta > 0 \qquad (1.1)$$

The Weibull distribution has three parameters; the shape parameter (β), the scale parameter (θ) and the location parameter (γ).

The Maximum Likelihood Estimator is not available in closed form for two of the parameters of a three parameter Weibull distribution, therefore raising the need to review and compare alternative methods. In this paper, we review and compare eight methods of estimating the parameters of the three parameter-Weibull distribution.
The aim of this paper is to find the best method for estimating the parameters of the three-parameter Weibull distribution under different conditions.

2. Review of Methods

2.1. Maximum Likelihood Method

The maximum likelihood estimates are gotten by maximizing the log likelihood function of the distribution. The log-likelihood function of the three-parameter Weibull distribution which can be gotten from the pdf (equation 1.1) as;

$$\boldsymbol{L}(x_1,x_2,\dots,x_n,\beta,\theta,\gamma) = n(\log\beta - \beta\log\theta) + (\beta-1)\sum_{i=1}^{n}\log(x_i-\gamma) - \frac{1}{\theta^\beta}\sum_{i=1}^{n}\log(x_i-\gamma)^\beta \quad (2.1)$$

The estimates of $\beta, \theta and \gamma$ are gotten by maximizing the equation above. This is done by differentiating the equations with respect to $\beta, \theta and \gamma$ respectively, and equating the resulting expressions to zero.
The resulting equations are;
Differentiating equation 3.0 with respect to $\beta, \theta and \gamma$respectively gives;

$$\frac{n}{\beta} + \sum_{i=1}^{n}\log\left(\frac{x_i-\gamma}{\theta}\right) - \sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\theta}\right)^\beta \log\left(\frac{x_i-\gamma}{\theta}\right) = 0 \quad (2.2)$$

$$-\frac{n\beta}{\theta} + \frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\theta}\right)^\beta = 0 \quad (2.3)$$

$$-(\beta-1)\sum_{i=1}^{n}\frac{1}{x_i-\gamma} + \frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\theta}\right)^{\beta-1} = 0 \quad (2.4)$$

The shape parameter cannot be isolated from any of these equations, which means the maximum likelihood estimate for the shape parameter cannot be expressed in closed form. The estimate for the shape parameter is gotten by solving these equations iteratively using Newton Raphson method.

From equation 3.7, the scale parameter is estimated using the following expression;

$$\hat{\theta} = \left(\frac{\sum(x_i-\hat{\gamma})^{\hat{\beta}}}{n}\right)^{1/\hat{\beta}} \quad (2.5)$$

The MLE for the location is given by;
$\hat{\gamma} = x_{(1)} = min\{x_1, x_2, \dots, x_n\}$

2.2. Method of Moments
The first three central moment are gotten from equations are given by;

$$\mu_1 = \gamma + \theta\Gamma\left(1+\frac{1}{\beta}\right) \quad (2.6)$$

$$\mu_2 = \theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right] \quad (2.7)$$

$$\mu_3 = 3\,\Gamma\left(1+\frac{3}{\beta}\right) - 3\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{2}{\beta}\right) + 2\Gamma^3\left(1+\frac{1}{\beta}\right) \quad (2.8)$$

The coefficient of skewness is used to find an estimator for β. The coefficient of skewness is given by the expression;

$$S_k = \sqrt{\frac{\mu_3^2}{\mu_2^3}} = \sqrt{\frac{\left[\Gamma\left(1+\frac{3}{\beta}\right) - 3\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{2}{\beta}\right) + 2\Gamma^3\left(1+\frac{1}{\beta}\right)\right]^2}{\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right]^3}} \quad (2.9)$$

From equation 2.6 and 2.7 the estimators for $\theta and \gamma$ are given by;

$$\hat{\theta} = \sqrt{\frac{\mu_2}{\Gamma\left(1+\frac{2}{\hat{\beta}}\right) - \Gamma^2\left(1+\frac{1}{\hat{\beta}}\right)}} \quad and \quad \hat{\gamma} = \mu_1 - \hat{\theta}\Gamma\left(1+\frac{1}{\beta}\right) \quad (2.10)$$

2.3. Percentile Method
As the name implies, this method involves the use of quantiles. This method was derived by Dubey (1967). He proposed an estimator based on the 17th and 97th percentiles for the shape parameter and 45th and 82nd percentiles for the scale parameter.
Let $P_1 = 0.167$ and $P_2 = 0.9737$
And define $K_1 = log\left(-log(1-P_1)\right) - log(-log(1-P_2))$
Let Y_1 *and* Y_2 represent the $100(P_i$th) percentile from a sample given sample, then,
$\hat{\beta} = \frac{-K_1}{\log(Y_1) - \log(Y_2)}$(2.11)
To estimate the scale parameter,

Let $P_3 = 0.3978$ and $P_4 = 0.8211$
Define $K_2 = log(-log(1-P_1)) - log(-log(1-P_2))$ and $K_3 = -\log(1-P_3)$
Also define $w = 1 - \frac{log(K_3)}{K_2}$
Then, $\hat{\theta} = e^{(wlog(Y_3)+(1-w)\log(Y_4))}$ (2.12)
The location parameter is estimated using;
$\hat{\gamma} = 100(0.1)th$ percentile

2.4. Teimouriand Gupta Beta

Teimouri and Gupta (2013) gave a useful theorem for constructing a simple, consistent and closed form estimator for β. And interestingly this estimator is independent of θ. The theorem states; suppose $x_1, x_2, \dots, x_n$ is a random sample from a Weibull distribution. Let ρ denote the sample correlation coefficient between x_i and their ranks. Let C and S denote the sample coefficient of variation and the sample standard deviation respectively. Then,
$\rho = \left(\frac{\mu_x-\gamma}{\sigma_x}\right)\left(\frac{1}{2} - \frac{1}{2^{1+\frac{1}{\beta}}}\right)\sqrt{\frac{12(n-1)}{n+1}}$ (3.4)$\rho = \left(\frac{\mu_x-\gamma}{\sigma_x}\right)\left(\frac{1}{2} - \frac{1}{2^{1+\frac{1}{\beta}}}\right)\sqrt{\frac{12(n-1)}{n+1}}$ … (2.13) Where $\mu_x = E(x)$ and $\sigma_x = standard\ deviation\ (x)$
Based on a corollary of this theorem they stated $\hat{\beta} = \frac{-\ln 2}{\ln\left[1-\frac{\rho}{\sqrt{3}}\left(\frac{1}{C}-\frac{\gamma}{S}\right)^{-1}\sqrt{\frac{n+1}{n-1}}\right]}$ (2.14), where γ is the location parameter.

An estimator for γ is $x_{(1)} - \frac{1}{n}$(Sirvanci and yang 1954).

$$\therefore \hat{\beta} = \frac{-\ln 2}{\ln\left[1-\frac{\rho}{\sqrt{3}}\left(\frac{1}{C}-\frac{x_{(1)}-\frac{1}{n}}{S}\right)^{-1}\sqrt{\frac{n+1}{n-1}}\right]} \quad (2.15)$$

This estimator is now independent of $\gamma\ and\ \theta$.For this method, the maximum likelihood estimator for the scale parameter will be used;
$\hat{\theta} = \left(\frac{\sum(x_i-\gamma)^\beta}{n}\right)^{1/\beta}$ (3.16) Where $\beta = \hat{\beta}$ and $\gamma = \hat{\gamma}$

2.5. Method of L-Moments

This method is based on quantiles and order statistics.
If X is a random variable having a distribution with distribution function F(x) and quantile function x(F) and let $x_1,\ x_2, \dots,\ x_n$ be a random sample of size n from this distribution. Then $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ are the order statistic of the random sample which comes from the distribution of the random variable X. Let X be a real valued random variable with cumulative distribution $F(x)$ and quantile function $x(F)$.For the Weibull distribution $x(F) = \theta[-\ln(1-F(x))]^{1/\beta} + \gamma$ (which is gotten by making x the subject of formula in the distribution function of the Weibull distribution). The expectation of an order statistic is given by;
The first three L-moments can be expressed as;

$$\alpha_1 = \gamma + \theta\Gamma\left(\frac{1}{\beta}+1\right) = E(x) \quad (2.16)$$

$$\alpha_2 = \theta\Gamma\left(\frac{1}{\beta}+1\right)\left[1-\frac{1}{2^{\frac{1}{\beta}}}\right] \quad (2.17)$$

$$\alpha_3 = \theta\Gamma\left(\frac{1}{\beta}+1\right)\left[1-\frac{3}{2^{\frac{1}{\beta}}} + \frac{2}{3^{\frac{1}{\beta}}}\right] \quad (2.18)$$

Equating these to the sample L-moments say $\pi_i$and solving equations 2.16, 2.17 and 2.18 does not yield explicit results for the estimators of parameters. However, Goda et al (2010) provided a solution which is to use the L-skewness to find the estimator of β. The coefficient of L-skewness is given by $\tau_3 = \frac{\widehat{\alpha_3}}{\widehat{\alpha_2}} = \frac{\pi_3}{\pi_2}$

$$\tau_3 = \frac{\theta\Gamma\left(\frac{1}{\beta}+1\right)\left(1 - \frac{3}{2^{1/\beta}} + \frac{2}{3^{1/\beta}}\right)}{\theta\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\frac{1}{2^{1/\beta}}\right)} = \frac{\left(\mathbf{1}-\frac{\mathbf{3}}{\mathbf{2}^{\mathbf{1}/\boldsymbol{\beta}}} + \frac{\mathbf{2}}{\mathbf{3}^{\mathbf{1}/\boldsymbol{\beta}}}\right)}{\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}^{\mathbf{1}/\boldsymbol{\beta}}}\right)} \quad (2.19)$$

This equation is used to estimate β. The estimators for the other two parameters are gotten from equation 3.17 and 3.18 as;$\hat{\theta} = \frac{\pi_2}{\Gamma\left(\frac{1}{\beta}+1\right)\left(1-\frac{1}{2^{1/\beta}}\right)}$ *and* $\hat{\gamma} = \pi_1 - \hat{\theta}\Gamma\left(\frac{1}{\beta}+1\right)$

2.6. Goda's Polynomial

Goda et al (2010) fitted a polynomial function to equation 2.19 and got;

$\hat{\beta} = 285.3\pi_3{}^6 - 658.6\pi_3{}^5 + 622.8\pi_3{}^4 - 317.2\pi_3{}^3 + 98.52\pi_3{}^2 - 21.256\pi_3 + 3.516$ (2.20)

This polynomial was fit with an error of 0.3 for 0.6 $<\beta < 6$.

2.7. Maximum Product of spacing

The MPS method was introduced independently by Cheng and Amin (1983) and Ranneby (1984). The MPS criterion amounts to turning the parameters so that the product of probabilities of a new observation falling between each two neighboring sampling points in the set of order statistics is maximized. The sum of this probability is one.

Given an iid random sample $(x_1, x_2, \ldots, x_n)$ of size n from a univariate distribution $F(x,\ \theta_0)$ where $\theta_0$is an unknown parameter to be estimated.

Let $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ be corresponding ordered sample that is the result of sorting of all observations from the smallest to the largest. Also, for convenience, let $x_{(0)} = -\infty$ *and* $x_{(n+1)} = +\infty$

The spacing D_i are defined as the gaps between the values of the distribution function at adjacent ordered points Say, $D_i = F\left(x_{(i)},\ \theta_0\right) - F\left(x_{(i-1)},\ \theta_0\right) i = 1, 2, \ldots, n+1$

Then the maximum spacing estimates of θ_0 is defined as a value that maximizes the logarithm of the geometric mean of the sample spacing.

$$\max S_n(\theta_0) = \ln\left(\sqrt[n+1]{\prod_{i=1}^{n+1} D_i}\right) \quad (2.21)$$

$$= \ln\left(\prod_{i=1}^{n+1} D_i\right)^{1/n+1} = \frac{1}{n+1}\ln\left(\prod_{i=1}^{n+1} D_i\right)$$

$$= \frac{1}{n+1}\sum_{i=1}^{n+1} \ln D_i = \frac{1}{n+1}\sum_{i=1}^{n+1}\left(F\left(x_{(i)},\ \theta_0\right) - F\left(x_{(i-1)},\ \theta_0\right)\right) \quad (2.22)$$

Forthe Weibull distribution we have,$= \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\left(1 - e^{-\left(\frac{x_i-\gamma}{\theta}\right)^\beta}\right) - \left(1 - e^{-\left(\frac{x_{i-1}-\gamma}{\theta}\right)^\beta}\right)\right)$ (2.23)

The estimators (β, θ *and* γ) are gotten by maximizing the above equation. Since this equation is not differentiable, we used the Newton Raphson method for maximization.

2.8. The Modified Method of Moments

This modification of the method of moment was introduced by Cohen and Whitten (1982).

The modification is based on the assumption that; $\boldsymbol{E[F(x_r)] = F(x_r)}$.

Where $E(.)$ is the usual expectation, x_r is the r^{th} order statistic in a random sample of size n and $F(x_r)$ is the associated value of the cumulative distribution.

Cohen and Whitten also stated that $\boldsymbol{E[F(x_r)] = \frac{r}{n+1}}$

Therefore it follows that $\boldsymbol{\frac{r}{n+1} = 1 - e^{-\left(\frac{x-\gamma}{\theta}\right)^\beta}}$ (2.24)

Following Cohen and Whitten (1982), we set r = 1. The first order statistic contains more information about γ than any other order statistics and often more information than all the other order statistics combined.

When r = 1

$$\frac{1}{n+1} = 1 - e^{-\left(\frac{x_1-\gamma}{\theta}\right)^\beta}$$

$$\frac{n}{n+1} = e^{-\left(\frac{x_1-\gamma}{\theta}\right)^\beta}$$

taking logarithm of both sides gives; $ln\left(\frac{n}{n+1}\right) = ln\left(e^{-\left(\frac{x_1-\gamma}{\theta}\right)^\beta}\right) = ln\left(\frac{n}{n+1}\right) = -\left(\frac{x_1-\gamma}{\theta}\right)^\beta$

$\therefore -ln\left(\frac{n}{n+1}\right) = \left(\frac{x_1-\gamma}{\theta}\right)^\beta$ (2.25)

Solving equations 3.25, 3.26 and 3.32 to eliminate θ and γ;

3. Simulation and Comparison

For the purpose of comparing the performance of the methods, the root mean square error (RMSE) was used in this work as a measure of accuracy of each method in estimating each of the three parameters. Also, the Euclidean norm was used to measure the total accuracy of each method in estimating the three parameters.

The RMSE of an estimator is calculated as;

$$RMSE = \sqrt{Var(x) + bias^2}$$

The Euclidean norm, or Euclidean length or magnitude of a vector measures the length of the vector. It is given by;

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

For each method, a vector was formed using the root mean square error of the estimates of the three parameters, and then the distance will be calculated. Intuitively, since the vectors have errors (RMSE) as elements, it follows that the method that has a vector with shorter length is better than a method with a lengthier vector.

The values of the shape parameter for simulation were chosen based on the three regions of the bath-tub curve (as in reliability testing).

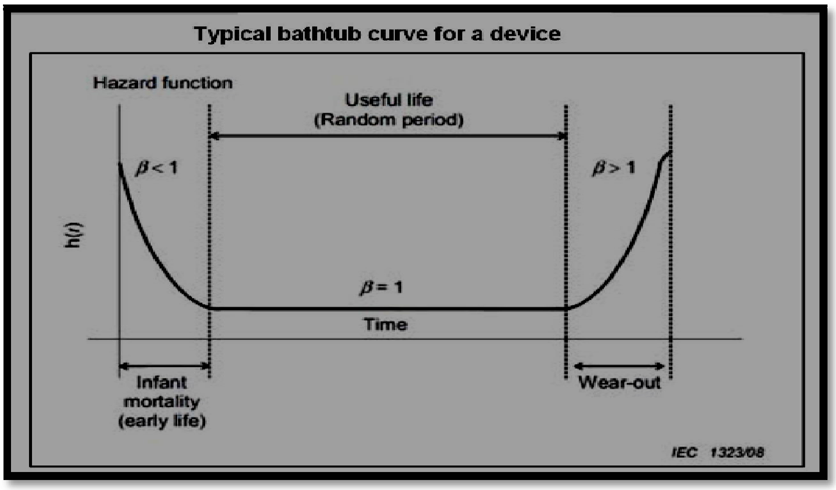

Figure 1: Bath-tub Curve
Source: www.reliawiki.com

Also, it is known that data gotten from reliability testing are always positively skewed. The Weibull distribution is positively skewed when $0.5 < \beta < 2.6$. Based on these, we chose 0.5, 1, 1.5 and 2.5 as true values for β to represent the three regions of the bath-tub curve ($\beta < 1, \beta = 1$ *and* $\beta > 1$ *respectively*). The scale and location parameters can vary depending on its unit of measurement, and since they are scaling parameters, we don't need to vary them. Therefore, we chose 100 and 10 as the values for the scale and location parameters respectively. We also varied the sample size (10, 20 and 50) to represent small, median and large sample sizes.4.0

4. Results

All computation such as generating random samples from the Weibull distribution, computation of estimates for the various methods were done using inbuilt and external packages in the R statistical software.

True β	True θ	True γ	Sample size n
0.5	100	10	10
0.5	100	10	20
0.5	100	10	50
1	100	10	10
1	100	10	20
1	100	10	50
1.5	100	10	10
1.5	100	10	20
1.5	100	10	50
2.5	100	10	10
2.5	100	10	20
2.5	100	10	50

Table 1: Categories of Data Simulation

Methods	Parameters	BIAS	RMSE
MAHDI AND GUPTA METHOD	β	0.1327512	0.1626008
	θ	18.42228	71.90500
	γ	-2.163484	5.309800
METHOD OF L-MOMENTS	β	-0.023047	0.3765046
	θ	-25.74027	74.389000
	γ	1.855000	10.155000
METHOD OF MOMENTS	β	0.215852	0.2498848
	θ	-98.28613	98.28782
	γ	-7.527209	7.537441
PERCENTILE METHOD	β	-1.389274	1.413455
	θ	54.71819	104.595
	γ	1.967292	4.552863
MAXIMUM PRODUCT OF SPACING	β	-0.0267443	0.1834215
	θ	40.2032	117.7352
	γ	0.8197365	5.394052
MODIFIED MOMENTS METHOD	β	0.2443309	0.414332
	θ	54.03477	126.5229
	γ	-3.619115	7.603042
GODA'S POLYNOMIAL METHOD	β	0.472447	1.084414
	θ	217.4522	796.6967
	γ	-6.567682	9.461734
MAXIMUM LIKELIHOOD METHOD	β		
	θ	NON-CONVERGENCE	NON-CONVERGENCE
	γ		

Table 2: Results for Sample Size=10, Beta=0.51

Methods	Parameters	BIAS	RMSE
MAHDI AND GUPTA METHOD	β	-0.00234304	0.1188928
	θ	-11.15802	52.17167
	γ	0.4591412	1.247136
METHOD OF L-MOMENTS	β	0.06593231	0.2091123
	θ	11.60552	74.67806
	γ	-2.217191	7.946221

MAXIMUM PRODUCT OF SPACING	β θ γ	-0.0301591 23.23571 0.1570346	0.1072908 79.07406 1.385503
Methods	**Parameters**	**BIAS**	**RMSE**
MODIFIED MOMENTS METHOD	β θ γ	0.1430128 35.90339 -1.179686	0.2456955 84.02007 2.683105
PERCENTILE METHOD	β θ γ	-1.247119 63.56168 0.4846851	1.253523 91.57962 1.134567
METHOD OF MOMENTS	β θ γ	0.1162601 38.14502 5.88574	0.1361887 95.55827 18.60958
GODA'S POLYNOMIAL METHOD	β θ γ	0.2244376 103.7502 -6.111638	0.4675652 361.5274 8.193022
MAXIMUM LIKELIHOOD METHOD	β θ γ	NON-CONVERGENCE	NON-CONVERGENCE

Table 3: Results for Sample Size=20 and Beta=0.5

Methods	**Parameters**	**BIAS**	**RMSE**
MAHDI AND GUPTA METHOD	β θ γ	4.37837e-05 -5.792638 0.06195151	0.07617271 31.03455 0.1941643
MAXIMUM PRODUCT OF SPACING	β θ γ	-0.0200027 11.67251 0.0387571	0.06271845 38.5956 0.1683332
METHOD OF L-MOMENTS	β θ γ	0.02439475 2.624542 -1.420584	0.1095837 39.62587 5.558183
METHOD OF MOMENTS	β θ γ	0.05011312 9.045346 -6.94911	0.07442058 41.45837 19.73336
MODIFIED MOMENTS METHOD	β θ	0.07502497 19.23583	0.1424681 46.64336

Methods	Parameters	BIAS	RMSE
	γ	-0.7588571	1.399621
Methods	**Parameters**	**BIAS**	**RMSE**
PERCENTILE METHOD	β	-1.196053	1.198189
	θ	63.76469	76.60829
	γ	0.07693967	0.173176
GODA'S POLYNOMIAL METHOD	β	0.08950027	0.1849431
	θ	41.47789	124.5539
	γ	-4.831244	6.812224
MAXIMUM LIKELIHOOD METHOD	β	NON-CONVERGENCE	NON-CONVERGENCE
	θ		
	γ		

Table 4: Results for Sample Size=50 and Beta=0.5

Methods	Parameters	BIAS	RMSE
MAXIMUM PRODUCT OF SPACING	β	-0.0440639	0.437876
	θ	5.489691	32.0976
	γ	6.276413	15.06475
MAHDI AND GUPTA METHOD	β	-0.1689997	0.2847648
	θ	-23.16872	38.70501
	γ	10.29491	17.60728
PERCENTILE METHOD	β	-2.598237	2.643818
	θ	11.08241	35.97132
	γ	9.663415	14.21158
METHOD OF MOMENTS	β	0.3668697	0.448224
	θ	-31.32084	42.67084
	γ	24.46045	29.18214
MODIFIED MOMENTS METHOD	β	0.825923	1.313565
	θ	17.4023	58.94141
	γ	-2.457843	10.15621
METHOD OF L-MOMENTS	β	0.5012449	2.086022
	θ	25.06442	148.8132
	γ	0.3177051	12.79854
GODA'S POLYNOMIAL METHOD	β	0.5120562	1.977292
	θ	35.00642	151.9566
	γ	7.849979	27.45279

MAXIMUM LIKELIHOOD METHOD	β θ γ	NON-CONVERGENCE	NON-CONVERGENCE

Table 5: Results for Sample Size=10 and Beta=1

Methods	Parameter	BIAS	RMSE
MAHDI AND GUPTA METHOD	β	-0.08500607	0.1989304
	θ	-11.91453	25.06324
	γ	4.938966	8.725661
PERCENTILE METHOD	β	-2.35998	2.373444
	θ	14.1715	28.64229
	γ	4.905211	6.784815
MODIFIED MOMENTS METHOD	β	0.6613739	1.059958
	θ	5.554375	32.91854
	γ	-2.070715	6.851777
MAXIMUM PRODUCT OF SPACING	β	-0.0161961	0.437876
	θ	4.065683	32.0976
	γ	1.754199	15.06475
METHOD OF L-MOMENTS	β	0.1330612	0.562391
	θ	4.852756	41.89465
	γ	-0.1622821	9.785603
GODA'S POLYNOMIAL METHOD	β	0.1305346	0.5352034
	θ	4.717018	41.03389
	γ	-0.1522319	9.807257
METHOD OF MOMENTS	β	0.249682	0.2905533
	θ	-44.0607	47.35216
	γ	31.21525	33.20622
MAXIMUM LIKELIHOOD METHOD	β θ γ	NON-CONVERGENCE	NON-CONVERGENCE

Table 6: Results for Sample Size=20 and Beta=1

Methods	Parameters	BIAS	RMSE
MAHDI AND GUPTA METHOD	β	-0.0327187	0.4812885
	θ	-4.030083	15.42769
	γ	1.97096	3.39117

MAXIMUM PRODUCT OF SPACING	β θ γ	-0.030265 1.979195 0.0293641	0.1357624 16.39277 2.363727
MODIFIED MOMENTS METHOD	β θ γ	0.1015814 8.168283 -4.894601	0.2163439 20.57716 7.771806

Methods	**Parameters**	**BIAS**	**RMSE**
METHOD OF L-MOMENTS	β θ γ	0.03443095 1.450009 -0.4822946	0.2054361 21.47622 7.096058
PERCENTILE METHOD	β θ γ	-2.236591 16.11249 1.975506	2.24113 24.47369 2.743128
GODA'S POLYNOMIAL METHOD	β θ γ	0.1305346 10.05517 2.422828	0.5352034 51.11268 13.46079
METHOD OF MOMENTS	β θ γ	0.1735489 -52.11863 36.78098	0.1882376 53.15337 37.51275
MAXIMUM LIKELIHOOD METHOD	β θ γ	NON-CONVERGENCE	NON-CONVERGENCE

Table 7: Results for Sample Size=50 and Beta=1

Methods	**Parameters**	**Bias**	**Rmse**
PERCENTILE METHOD	β θ γ	-3.796833 6.718831 19.17645	3.855535 24.54974 23.12697
MAHDI AND GUPTA METHOD	β θ γ	-0.4584609 -31.26084 19.34675	0.5364848 37.86138 23.54054
GODA'S POLYNOMIAL METHOD	β θ γ	0.9289596 35.13106 2.533689	3.3298 124.3454 15.97168

MAXIMUM LIKELIHOOD METHOD	β θ γ	0.7095411 18.15194 41.62018	3.965586 167.2714 164.8175
MAXIMUM PRODUCT OF SPACING	β θ γ	0.5571118 35.23447 43.43592	8.497256 203.1478 157.7633
Methods	**Parameters**	**Bias**	**Rmse**
METHOD OF MOMENTS	β θ γ	NO ROOT	NO ROOT
MODIFIED MOMENTS METHOD	β θ γ	NO ROOT	NO ROOT
METHOD OF L-MOMENTS	β θ γ	NO ROOT	NO ROOT

Table 8: Results for Sample Size=10 and Beta=1.5

Methods	Parameters	BIAS	RMSE
PERCENTILE METHOD	β θ γ	-3.469112 8.56358 12.17397	3.49189 19.08257 14.49892
MAHDI AND GUPTA METHOD	β θ γ	-0.2805753 -17.20969 12.10774	0.3622391 23.52643 14.44512
MAXIMUM PRODUCT OF SPACING	β θ γ	0.1232299 10.53943 9.415137	8.546578 119.491 57.81012
MAXIMUM LIKELIHOOD METHOD	β θ γ	0.0107799 -5.359939 10.65083	0.979813 36.87105 27.64767
MODIFIED MOMENTS METHOD	β θ γ	0.4452731 -40.0008 37.4892	2.433209 50.0004 38.0003

METHOD OF L-MOMENTS	β θ γ	0.3061825 12.37453 0.5281383	1.238395 52.99993 11.67473
METHOD OF MOMENTS	β θ γ	-0.6032349 -49.3039 36.7787	0.6448244 53.42191 40.31892
GODA'S POLYNOMIAL METHOD	β θ γ	0.2732907 11.15789 0.4844729	0.9382057 44.14521 11.65515

Table 9: Results for Sample Size=20 and Beta=1.5

Methods	Parameters	BIAS	RMSE
MAXIMUM LIKELIHOOD METHOD	β θ γ	-0.05796005 -5.63222 3.972861	0.2651848 14.09852 6.807028
MAHDI AND GUPTA METHOD	β θ γ	-0.1574609 -9.816728 6.803893	0.2237386 14.2841 8.214172
PERCENTILE METHOD	β θ γ	-3.309581 10.35368 6.845279	3.316965 15.80495 8.24641
GODA'S POLYNOMIAL METHOD	β θ γ	0.05491001 1.607711 0.2095117	0.366442 20.50919 8.785405
METHOD OF L-MOMENTS	β θ γ	0.05359229 1.515607 0.2912337	0.3683909 20.59285 8.828512
METHOD OF MOMENTS	β θ γ	0.4713041 25.24634 -7.237766	1.045212 53.46118 9.045932
MODIFIED MOMENTS METHOD	β θ γ	-0.6965429 -57.33639 41.39441	0.7087268 58.30003 42.10247
MAXIMUM PRODUCT OF SPACING	β θ γ	-0.03404346 1.349955 0.1827269	8.538009 92.45336 6.001584

Table 10: Results for Sample Size=50 and Beta=1.5

Methods	Parameters	BIAS	RMSE
PERCENTILE METHOD	β	-5.188339	5.300268
	θ	5.860043	15.87704
	γ	35.59741	38.51351
MAHDI AND GUPTA METHOD	β	-1.231671	1.283226
	θ	-43.98572	41.46263
	γ	35.7258	38.80891
MAXIMUM PRODUCT OF SPACING	β	1.08509	1.224916
	θ	-4.97473	44.38471
	γ	25.40446	33.44971
Methods	**Parameters**	**BIAS**	**RMSE**
MAXIMUM LIKELIHOOD METHOD	β	0.935712	5.151017
	θ	11.47466	140.8101
	γ	65.26447	153.7486
GODA'S POLYNOMIAL METHOD	β	2.992037	7.289067
	θ	48.19675	255.5098
	γ	14.89093	29.15153
METHOD OF MOMENTS	β		
	θ	NO ROOT	NO ROOT
	γ		
METHOD OF L-MOMENTS	β	NO ROOT	NO ROOT
	θ		
	γ		

Table 11: Results for Sample Size=10 and Beta=2.5

Methods	Parameters	BIAS	RMSE
PERCENTILE METHOD	β	-4.757424	4.801904
	θ	7.411526	13.24821
	γ	27.00638	29.26563
MAHDI AND GUPTA METHOD	β	-0.9306541	0.9873057
	θ	-32.20311	35.10124
	γ	27.42744	29.91266
METHOD OF MOMENTS	β	-1.367543	1.439514
	θ	-56.18469	59.88441
	γ	45.68911	48.35342
GODA'S POLYNOMIAL METHOD	β	0.6378122	3.358856
	θ	-73.63471	75.82363
	γ	65.34878	67.54127
MAXIMUM LIKELIHOOD METHOD	β	0.5162815	3.919403
	θ	5.064054	101.0869

	γ	32.27247	107
MAXIMUM PRODUCT OF SPACING	β θ γ	0.9545239 36.00158 46.37472	5.320103 149.4277 147.4487
METHOD OF L-MOMENTS	β θ γ	NO ROOT	NO ROOT
MODIFIED MOMENTS METHOD	β θ γ	NO ROOT	NO ROOT

Table 12: Results for Sample Size=20 and Beta=2.5

Methods	Parameters	BIAS	RMSE
PERCENTILE METHOD	β θ γ	-4.483661 8.235697 18.81099	4.501284 11.01037 20.41387
MAHDI AND GUPTA METHOD	β θ γ	-0.6004832 -21.00493 1.970963	0.655485 23.08954 2.759584
MAXIMUM LIKELIHOOD METHOD	β θ γ	0.03464351 -3.083195 8.928148	0.8526618 23.71099 17.58165
MAXIMUM PRODUCT OF SPACING	β θ γ	0.1474586 7.695812 6.584551	0.8400696 26.60051 17.83054
GODA'S POLYNOMIAL METHOD	β θ γ	0.122791 6.149524 2.211636	0.9142861 29.39152 12.77273
METHOD OF L-MOMENTS	β θ γ	0.2517798 6.620358 1.813612	0.951136 29.60712 12.63197
METHOD OF MOMENTS	β θ γ	-1.370495 -50.18469 45.68911	1.409083 52.88441 41.35342
METHOD OF L-MOMENTS	β θ γ	0.2517798 6.620358 1.813612	0.951136 29.60712 12.63197

Table 13: Results for Sample Size=50 and Beta=2.5

5. Findings

From the results of the simulation experiments, the Teimouri and Gupta method is the best method of estimating the parameters of the three-parameter Weibull distribution. On the average, it produces very good estimates for small, medium and large sample sizes and across the different beta values.
The maximum product of spacing and the percentile method follow behind the Teimouri and Gupta method. Though the maximum product of spacing did not do well when $\beta = 2.5$.

The method of moment and the modified method of moments performed relatively below average and even produced too many 'no root' situations when $\beta = 2.5$. The method of L-moment also did not perform relatively well across the simulated samples. The polynomial didn't do well too.

No roots mean there were too many no situations were no root was found using a particular method. Non-convergence means that there were too many situations were a particular method did not converge.

5.1. Application to Real Data

Two real life data sets are presented and the methods of estimation are used to find estimates for each data set.

Bearing cage fracture times (hrs)	230, 334, 423, 990, 1009, 1510

Table 14: Data of Time to Failure of Airplane Glass
Source: Abernethy Et Al Weibull Handbook (AFWAL-TR-83-2079) Pg. 43

failure times locomotive controls (miles)	225, 375, 460, 485, 515, 530, 545, 575, 665, 680, 695, 765, 770, 785, 800, 815, 820, 830, 840, 915, 935, 1025, 1070, 1085, 1125, 1135, 1160, 1170, 1185, 1190, 1200, 1225, 1230, 1275, 1310, 1325, 1340

Table 15: Data of Failure Times Locomotive Controls
Source: Hahn G.J. and Shapiro S.S (1967); Statistical Models in Engineering

ESTIMATES			
	β	θ	γ
Teimouri and Gupta Method	0.6933112	406.6268	229.8333
Method of L-moments	1.519542	907.8518	747.9875
Modified Method of Moment	1.117227	581.4921	191.0501
Maximum Product of Spacing	0.8890775	688.6378	146.6762
Goda's Polynomial	1.522161	909.2661	747.9639
Method of Moments	NO ROOT	-	-
Percentile Method	1.971364	736.1991	230.052
Maximum Likelihood Method	0.5938066	442.4246550	230.000

Table 16 Result for Bearing Cage Fracture Times Data (Sample Size = 6)

ESTIMATES			
	$\boldsymbol{\beta}$	$\boldsymbol{\theta}$	$\boldsymbol{\gamma}$
Teimouri And Gupta Method	2.240967	755.2693	224.973
Method Of L-Moments	6.537696	1872.262	887.8253
Modified Method of Moment	No result	No result	No result
Maximum Product of Spacing	10.81984	3056.769	2026.893
Goda's Polynomial	5.832769	1691.457	888.5162
Percentile Method	3.367362	893.6662	225.54
Maximum Likelihood	Non-Convergence	-	-

Table 17: Result for Failure Times of Locomotive Controls (Sample Size = 37)

5.2. Conclusion

From the above findings, the Teimouri and Gupta method is the best method for estimating the parameters of the three-parameter Weibull distribution when applied to reliability data (i.e. when the distribution is positively skewed).
Sample size does not affect the choice of method, though the performances of the methods always improved with sample size.

6. References

i. Cheng, R. C. H. and Amin, N. A. K. (1979). Maximum Product of Spacing Estimation With Application To The Log-Normal Distribution, University of Wales IST, Methat Report 79.
ii. Cheng, R. C. H. and Amin, N. A. K. (1983). Estimating Parameters In Continuous Univariate Distributions With A Shifted Origin, Journal of the Royal Statistical Society, Series B (Methodological) 45(3), 394–403.
iii. Cohen, A. C and Whitten, B (1982). Modified Maximum Likelihood And Modified Moment
iv. Estimators For The Three-Parameter Weibull Distribution, Communications in Statistics
v. Theory and Methods 11.
vi. Dubey S.Y.D (1967). Normal and weibull distributions, naval research logistics quarterly, 14(1), 69-79 dio:10.1004nav.3800140107.
vii. Goda, Y., Kudaka, M and Kawai, H. (2010). Incorporation Of Weibull Distribution In L-
viii. Moments Method For Regional Frequency Analysis Of Peaks-Over-Threshold Wave Height
ix. Proceedings of the International Conference on Coastal Engineering,
x. http://journals.tdl.org/ICCE/article/viewFile/1154/pdf_42.
xi. Lloyd, N. S. (1967). Weibull Probability Paper, Industrial Quality Control 23, 452–453.
xii. Teimouri M. and Gupta A.K (2013). On the three-parameter weibull distribution shape parameter estimation, journal of data science 11, 403-414.