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Qualitative Analysis of Bifurcation: The First Border Steady-State Solution

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Abstract:
This paper examines the bifurcation analysis of two interacting ecological species in a qualitative manner. In the modelling of competition interaction between two ecological species, when the daily intrinsic growth rate a is varied while other model parameters are fixed, the fundamental qualitative changes of the borders steady-state solution $\left(\frac{sa}{b}, 0\right)$ are studied. This steady-state solution is stable if $a < \frac{2bN_{1e}}{s}$ and $d < eN_{1e}$ provided s is a positive parameter and d is one of the model parameters. It passes through a bifurcation point $a < \frac{2bN_{1e}}{s}$ and $d < eN_{1e}$ and is unstable if $a < \frac{2bN_{1e}}{s}$ and $d < eN_{1e}$.

Keywords: *Bifurcation, steady-state solution, stability, intrinsic growth rate, competition interaction, ecological species*

1. Introduction

The notion of a bifurcation analysis in ecological modelling is an important mathematical technique for understanding the fundamental changes in the qualitative behaviour of solutions due to a variation of a model parameter (Troost et al., 2007; Ford & Norton, 2009). Hence bifurcation analysis in ecological research is an active component of research. According to Ekaka-a (2009) and several cited authors by Ekaka-a (2009) and without loss of generality, we know that, for a system of non-linear first order differential equations, a steady-state solution can either be called stable if the signs of the eigenvalues are both negative and unstable if the signs of the eigenvalues are of opposite signs. However, the bifurcation values where a stable-steady state solution changes to an unstable steady-state solution remain to be an open problem in the context of this interdisciplinary research. These bifurcation values can provide some insights to ecologists.

In the sequel, we will present a few key novel results of our numerical bifurcation analysis.

This paper is organized into the following sections: Sections 2 and 3. It will tackle the notions of bifurcation and some simplifying modelling assumptions. Sections 4 and 5 will tackle the mathematical formulation and the characterization components of this proposed problem. Section 6 is the core component of this study which tackles the problem of constructing the bifurcation points and the fundamental changes in the qualitative behaviour of the steady-state solutions. The key results that one has achieved in this paper are quantitatively discussed in section 7 and summarized in section 8.

2. Bifurcation Analysis

If a model parameter is varied while other parameters are fixed, we can study the fundamental changes in the qualitative behaviour of steady-state solutions and hence find the bifurcation points where a stable steady-state solution changes to an unstable steady-state solution.

For example, after linearizing the interaction functions, which are continuous and partially differentiable in the neighbourhood of an arbitrary steady-state solution, we will aim to characterize the stability and instability behaviour of the steady-state solution qualitatively. In this respect, we can explore the standard mathematical technique of the changes in the signs of the eigenvalues to specify if a steady-state solution is either stable or unstable. A steady-state solution can sometimes be characterized as sitting on the cusp. A systematic calculation where steady-state changes from a stable node to a saddle can have an interesting application in the study of biological interaction, which is both attractive and cost-effective.

This numerical bifurcation analysis can be useful in ecological monitoring and stability. For other sophisticated bifurcation methods, see Ford and Norton (2009).

3. Modelling Assumptions

In this paper, our core assumptions will border on the linear Malthusian growth phenomenon, logistic population growth, and the law of mass action, which are central in the formulation of a system of first-order differential equations that describe the interspecific competition between two plant species in a Lotka-Voltera sense (Ekaka-a, 2009; Ford et al., 2010).

4. Mathematical Formulation

Recently, Ford et al. (2010) introduced a mathematical model of plant species interaction in a harsh climate. They consider whether interactions between the species change in character as the environment changes. The model is constructed based on the notion of a summer season when the plants grow, followed by a winter season when there is no growth but when the plants are subject to the effects of events such as winter storms, see also Ekaka-a (2009).

The model of competition has the following form:

$$\frac{dy}{dt} = \alpha_1 y(t)\big(1 - \beta_1 y(t) - \gamma_1 z(t)\big) \qquad (4.1)$$
$$\frac{dz}{dt} = \alpha_2 z(t)\big(1 - \beta_2 z(t) - \gamma_2 y(t)\big) \qquad (4.2)$$

Here y and z denote the population of two plant species at time t. Here the non-negative constants $\alpha_i,\ \beta_i,\ \gamma_i,\ i = 1, 2$ are given, respectively, as the intrinsic growth rate, the intra-species competitive parameter, and the inter-species competitive parameter, respectively. These model equations have four steady-states.

$$y = 0, \quad z = 0$$
$$y = 0, \quad z = \frac{1}{\beta_2},$$
$$y = \frac{1}{\beta_1}, \quad z = 0$$
$$y = \frac{\beta_2 - \gamma_1}{\beta_1\beta_2 - \gamma_1\gamma_2}, \quad z = \frac{\beta_1 - \lambda_2}{\beta_1\beta_2 - \gamma_1\gamma_2}$$

They discussed how to choose the parameter value $\alpha_i,\ \beta_i,\ \gamma_i,\ i = 1, 2$ such that the model is reasonable. They noticed that although the variation in $\alpha_i,\ \beta_i,\ \gamma_i,\ i = 1, 2$ between the species is quite small, the behaviour of two such close species is much different over a growing season of several years length. The population of one species may die away and would become extinct over a growing season of several years length. They pointed out that small environmental perturbations could have quite devastating and unexpected results for ecosystems. Some steady-states are stable. For the purpose of this paper, we will consider a simplified version of the above model equations (Ekaka-a, 2009), such as

$$\frac{dN_1}{dt} = N_1(a - bN_1 - cN_2), \qquad (4.3)$$
$$\frac{dN_2}{dt} = N_2(d - eN_1 - fN_2), \qquad (4.4)$$

Where the initial conditions are $N_1(0) = N_{10} > 0$ and $N_2(0) = N_{20} > 0$.

For the purpose of this bifurcation analysis, we will consider the following system of first-order non-linear ordinary differential equations.

$$\frac{dN_1}{dt} = N_1(sa - bN_1 - cN_2), \qquad (4.5)$$
$$\frac{dN_2}{dt} = N_2(d - eN_1 - fN_2), \qquad (4.6)$$

Similarly, the initial conditions are $N_1(0) = N_{10} > 0$, $N_2(0) = N_{20} > 0$, and $s > 0$.

5. Characterization of Steady-State Solutions

If the rates of change are equated to zero, and the interaction functions are solved analytically, we will obtain the four steady-state solutions, namely(0,0), $\left(\frac{sa}{b}, 0\right)$, $\left(0, \frac{d}{f}\right)$, and $(N_{1e},\ N_{2e})$

Where $N_{1e} = \frac{asf - cd}{bf - ce}$ and $N_{2e} = \frac{bd - eas}{bf - ce}$ provided $a > \frac{cd}{f}$, $sa < \frac{bd}{e}$, $bf > ce$.

By using a standard mathematical technique of linearization at an arbitrary steady-state solution(N_{1e}, N_{2e}), we will consider two interaction functions $F(N_{1e}, N_{2e})$ and $G(N_{1e}, N_{2e})$, which are assumed to be partially differentiable and continuous at this arbitrary steady-state solution(N_{1e}, N_{2e}).

In our context, the mathematical structures of these two functions are

$$F(N_{1e}, N_{2e}) = asN_{1e} - bN_{1e}^2 - cN_{1e}N_{2e}, \qquad (5.1)$$
$$G(N_{1e}, N_{2e}) = dN_{2e} - eN_{1e}N_{2e} - fN_{2e}^2 \qquad (5.2)$$

To determine the stability property of each steady-state solution, we differentiated these two functions partially with respect to N_{1e} and N_{2e} and obtained the following Jacobian coefficients, such as

$$J_{11} = \frac{\partial F}{dN_{1e}} = as - 2bN_{1e} - cN_{2e}, \qquad (5.3)$$
$$J_{12} = \frac{\partial F}{dN_{2e}} = -cN_{1e}, \qquad (5.4)$$
$$J_{21} = \frac{\partial G}{dN_{1e}} = -eN_{2e}, \qquad (5.5)$$
$$J_{22} = \frac{\partial G}{dN_{2e}} = d - eN_{1e} - 2fN_{2e} \qquad (5.6)$$

Upon evaluating these values of partial derivatives at each steady-state solution, we can set up a Jacobian matrix from which two eigenvalues can be calculated. For example, at the steady-state$\left(\frac{sa}{b}, 0\right)$, the two eigenvalues which are unique to those model parameters are $\lambda_1 = as - 2bN_{1e}$ and $\lambda_2 = d - eN_{1e}$. Here, the point (N_{1e}, N_{2e}) is an arbitrary steady-state solution of the non-linear system of first-order ordinary differential equations, which have been defined in the previous section.

6. Numerical Bifurcation Analysis

To study the fundamental changes in the qualitative behaviour of the border steady-state solution$\left(\frac{sa}{b}, 0\right)$, which is due to a variation of the intrinsic growth rate of the first population when other model parameters are fixed, we will tackle this problem analytically by using nine different scenarios which are listed below.

(1) If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $a < \frac{2bN_{1e}}{s}$ and $d > eN_{1e}$. This first observation indicates that the steady-state solution $\left(\frac{sa}{b}, 0\right)$ is stable.

(2) If $\lambda_1 < 0$ and $\lambda_2 = 0$, then $a < \frac{2bN_{1e}}{s}$ and $d = eN_{1e}$. This second observation indicates that the steady-state solution$\left(\frac{sa}{b}, 0\right)$, is sitting on the cusp.

(3) If $\lambda_1 = 0$ and $\lambda_2 < 0$, then $a < \frac{2bN_{1e}}{s}$ and $d > eN_{1e}$. This third observation indicates that the steady-state solution$\left(\frac{sa}{b}, 0\right)$, is sitting on the cusp.

(4) If $\lambda_1 = 0$ and $\lambda_2 = 0$, then $a < \frac{2bN_{1e}}{s}$ and $d = eN_{1e}$. This fourth observation indicates a loss of stability for the steady-state solution$\left(\frac{sa}{b}, 0\right)$.

(5) If $\lambda_1 = 0$ and $\lambda_2 > 0$, then $a < \frac{2bN_{1e}}{s}$ and $d < eN_{1e}$. This fifth observation indicates that the steady-state solution$\left(\frac{sa}{b}, 0\right)$, is sitting on the cusp.

(6) If $\lambda_1 > 0$ and $\lambda_2 = 0$, then $a < \frac{2bN_{1e}}{s}$ and $d = eN_{1e}$. This sixth observation indicates that the steady-state solution$\left(\frac{sa}{b}, 0\right)$, is sitting on the cusp.

(7) If $\lambda_1 < 0$ and $\lambda_2 > 0$, then $a < \frac{2bN_{1e}}{s}$ and $d > eN_{1e}$. This seventh observation indicates that the steady-state solution$\left(\frac{sa}{b}, 0\right)$, is unstable. Here, the instability criteria of the steady-state solution$\left(\frac{sa}{b}, 0\right)$, are the partial opposite of the stability criteria of the steady-state solution$\left(\frac{sa}{b}, 0\right)$.

(8) If $\lambda_1 > 0$ and $\lambda_2 < 0$, then $a < \frac{2bN_{1e}}{s}$ and $d > eN_{1e}$. This eighth observation indicates that the steady-state solution$\left(\frac{sa}{b}, 0\right)$, is unstable. Similarly, the instability criteria of the steady-state solution$\left(\frac{sa}{b}, 0\right)$, is the partial opposite of the stability criteria of the steady-state solution$\left(\frac{sa}{b}, 0\right)$.

(9) If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $a < \frac{2bN_{1e}}{s}$ and $d < eN_{1e}$. This ninth observation indicates that the steady-state solution $\left(\frac{sa}{b}, 0\right)$ is unstable. We can see clearly that the instability criteria of the steady-state solution$\left(\frac{sa}{b}, 0\right)$, are the complete opposite of the stability criteria of the steady-state solution$\left(\frac{sa}{b}, 0\right)$.

7. Discussion of Results

In this present analysis, the steady-state solution$\left(\frac{sa}{b}, 0\right)$, is a stable node for $a < \frac{2bN_{1e}}{s}$ and $d < eN_{1e}$. It is saddle for $a > \frac{2bN_{1e}}{s}$ and $d > eN_{1e}$. Therefore, the steady-state solution will change from a stable node to a saddle-node as it persists through the bifurcation point $a < \frac{2bN_{1e}}{s}$ and $d = eN_{1e}$.

8. Concluding Remarks

In this study, we have found that the steady-state solution $\left(\frac{sa}{b}, 0\right)$, is stable when the growth rate of the first population is strictly less than a multiple of the steady-state co-ordinate of the N_1 solution trajectory and when the growth rate of the second population is strictly less than multiple of the steady-state co-ordinate of the N_1 solution trajectory. This steady-state solution is similarly unstable when the growth rate of the first population and the growth rate of the second population are strictly greater than a multiple of the steady-state co-ordinate of the N_1 solution trajectory. A bifurcation point $a < \frac{2bN_{1e}}{s}$ and $d = eN_{1e}$ has been determined, which is capable of providing further insights into biosciences research.

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