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Equations of Motion, Symmetric Matrix Operators and Determinants of a Multi-Pendula System Oscillating in a Plane

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Abstract:
An n-tupple chain pendulum system constrained to move in a plane was studied within the framework of a generalized coordinates by using the abridged Lagrangian formalism with a view of developing its equations of motion. This study drew numerous explorations resulting in an exhibition of natural mathematical concepts that had been hitherto either unclear or unknown. The Lagrangian is developed and further solved with ease using matrix algebra resulting in the general Energy equations, Energy symmetric matrix operators, Energy eigen functions and their respective coefficients. As the number of mass units in an n-pendula system increases, more terms are added to the kinetic and potential energy equations, making the motion of the system more dependent on initial conditions. It is generally observed that the angular acceleration for any mass is influenced by the masses and angles of the immediate neighbours.

Keywords: *Lagrangian, Multi-pendula systems, Equations of motion*

1. Introduction

Equations of motion long established complicated approaches in their formulations and solutions. After the inception of Newtonian mechanics these equations were derived from the forces at play. To make this simpler, Lagrange formulated a method that was based on the kinetic and potential energies. For unclear reasons other than the complexity this formalism was unpopular hence the concept remained unexplained for a long time. Spiegel [1969] worked using the method for up to two masses suspended in series.The masses and their lengths of separation were equal in magnitude. The equations found were given by:

$2\ddot{\theta}_1 + \ddot{\theta}_2 = -2\theta_1 \frac{g}{l}$(1.1)

and

$\ddot{\theta}_1 + \ddot{\theta}_2 = -\theta_2 \frac{g}{l}$(1.2)

In latertimes Chow [1995] handled a simple pendulum and a double pendulum, with uneven quantities of mass and length, using the Lagrange method.

The two equations he found are:

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 = -(m_1 + m_2)l_1\theta_1 g \qquad (1.3)$$

and

$$m_2 l_1 l_2 \ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 = -m_2 l_2 \theta_2 g \qquad (1.4)$$

There was no effort made to work beyond a double pendulum by the Lagrangian formalism. In the work by Jones and Kush [2006] for the examination of chaos variation in multiple pendulum systems with different amounts of energy, no idea of determination of the equations of motion was done. They however noted that as the number of links (or masses in our case) increases; there was notable increase in divergence with inherent implications of heightened chaotic nature of the system hence the complexity. Thus, with increased terms and unknowns, the motion of the system will be majorly characterized by the initial conditions which concur with our findings for pendulum cascade. They were specific that with varying masses and lengths, it is more difficult when more masses are added to the system, thus admitting that creating higher n-pendula systems is tedious work which made them stop building systems after the quintuple pendulum. This is obviously not enough to identify an exact mathematical relationship between the order of chaos and the number of links.

There was an extensive study of coupled systems as a new quantitative system save for the theoretical challenges involved in the investigations of such systems. This is in reference to the work of Hedrich [1999,2007]. He worked on vibrations of a multi-pendulum system inter-coupled by standard light elements and different properties were explored in which the obtained analytical solutions were numerically analyzed. Acheson et al [1993,2005] made annotations on counter intuitive phenomenon of a driven inverted chain consisting of N linked pendulums balanced on top of one another. It was realized that the amplitudes diminish with the number of pendulums involved in the chain.

Furuta [1984] and Spong [2001] proposed and studied numerous extensions of pendulum systems which include various categories of elastic and multi-body pendulum models. Nedic [2008] found that the initial conditions in the agreement protocol could be manipulated to produce results that satisfy linear constraints and the same can apply to the control of a distributed network of linearized pendulums on a line topology. Joot [2010] noted that by introducing additional mass in a system, the interacting coupling terms increase thus complicating the kinetic energy specifications noting that calculating the energy explicitly for a general n-pendula system is likely thought to be too pedantic for even the most punishing instructor to inflict on students as a problem.

A rigorous study of a multi-pendula system has been carried out. Diminutive research may have been done on these systems but not for more than three mass units. This work covers all the aspects of linearly suspended mass units oscillating in a plane. The approach is pegged on the derivations of equations of motion using the energy in the systems. The traditional concept of force has been avoided. Apart from its utility as a timing device, the pendulum is used as an apparatus in learning of science in school. Its used as a model for the study of linear oscillators. In recent times, exciting research activities were noticeable for the study of the robotic marionettes. Unlike Alan B and Jindi S (1995) who found that selective partial-state feedback control approach was successful in the non-linear unstable and interactive double pendulum system and they projected that it could be applied to the triple pendulum systems for both swing-up control and non-linear pendulum position control.

Mohamad et al [2014] studied the balance control of humanoid robots against external perturbations. The balancing actions are very demanding in terms of torque and power requirements for ankle joints. This is more so after strong and sudden impacts. An optimal control problem is formulated for the linearized inverted pendulum model to reduce the peak power requirements during ankle balancing strategy. This optimal control is calculated numerically and approximated by optimal compliance regulator whose ability is evaluated against other optimal methods

Maziar and Andre [2015]in Germany studied about the leg function and the ground reaction force in legged locomotion. They used the leg force modulated complaint hip in a new model for postural control in walking which employs the leg force feedback to adjust the hip compliance. This method gives a stable and robust walking in simulations and imitates human-like kinetic behavior. This gives the hip torque-angle relation for different walking speeds. The approach may physically implement the virtual pendulum concept in human animal locomotion.

Studies have been done by Prichani et al [2010,2012]and Sakwa et al [2012] about the planner motion of up to $n$masses linearly connected. We derived several equations of motion for unequal mass units and at unequal distances apart. We conclude from our work that the angular accelerations for each was dependent on the angular displacements of the nearest masses. In our later studies (2017), we worked on the Functions of Multi-Pendula Systems in Spatial Motion. We found that the angular accelerations are directly proportional to the sum of the products of azimuth velocity and the respective zenith displacement

2. Theoretical Derivations

For any number of suspended masses in an (x, y, o) plane, moving at small angles of displacement, the equations can be constructed as hereunder:

The position coordinate for the kth particleis designated as,

$(x_k, y_k) = \left(\sum_{i=1}^{k} l_i \sin\theta_i, \sum_{i=1}^{k} l_i \cos\theta_i\right)$(2.1)

Consequently its velocity is , $v_k = \sum_{i=1}^{k}\left(l_i^2\dot{\theta}_i^2\right) + 2\sum_{\substack{i=1\\ i\neq j}}^{k}\left\{l_i l_j \dot{\theta}_i\dot{\theta}_j \cos\left(\theta_i - \theta_j\right)\right\}$ (2.2)

The total kinetic energy in the system with n masses is derived as

$T = \sum_{k=f(i,j)}^{n} m_k \left\{\frac{1}{2}\sum_{i=1}^{n}\left(l_i^2\dot{\theta}_i^2\right) + \sum_{\substack{i=1\\ i\neq j}}^{n}\left(l_i l_j \dot{\theta}_i\dot{\theta}_j \cos\left(\theta_i - \theta_j\right)\right)\right\}$(2.3)

While the total potential energy is $V = \sum_{k=1}^{n} m_k \left\{\sum_{i=1}^{n} l_i - \sum_{i=1}^{k}\left(l_i \cos\theta_i\right)\right\}g$(2.4)

Using equation (2.3) and (2.4) we computed the Lagrangian of the system to be

$L = \sum_{k=f(i,j)}^{n} m_k \left\{\frac{1}{2}\sum_{i=1}^{n}\left(l_i^2\dot{\theta}_i^2\right) + \sum_{\substack{i=1\\ i\neq j}}^{n}\left(l_i l_j \dot{\theta}_i\dot{\theta}_j \cos\left(\theta_i - \theta_j\right)\right)\right\} - \sum_{k=1}^{n} m_k \left\{\sum_{i=1}^{n} l_i - \sum_{i=1}^{k}\left(l_i \cos\theta_i\right)\right\}g$(2.5)

which when subjected to the Lagrangian equation of motion$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0$ (2.6)

results to the following Lagrangian equations of motion for n unequal masses and lengths;

$\sum_{k=i}^{n} m_k\left(l_i^2\dot{\theta}_i\right) + \sum_{k=f(i,j)}^{n} m_k\left\{\sum_{(i\neq j)=1}^{n} l_i l_j \ddot{\theta}_j\right\} = -l_i\theta_i\left(\sum_{k=i}^{n} m_k\right)g$(2.7)

Where f: the greater of (i, j).

The combinations could be, for instance,

$i = 1, j = (2,3,4); i = 2, j = (1,3,4); i = 3, j = (1,2,4); and\ i = 4, j = (1,2,3).$

Hence using equation (2.7) we constructed the equations of motion as hereunder.

When $i = 1$, $\sum_{k=1}^{4} m_k l_1\ddot{\theta}_1 + \sum_{k=2}^{4} m_k l_2\ddot{\theta}_2 + \sum_{k=3}^{4} m_k l_3\ddot{\theta}_3 + m_4 l_4\ddot{\theta}_4 = -l_1\theta_1\sum_{k=1}^{4} m_k\, g$(2.8)

$i = 2, \sum_{k=2}^{4} m_k l_1\ddot{\theta}_1 + \sum_{k=2}^{4} m_k l_2\ddot{\theta}_2 + \sum_{k=3}^{4} m_k l_3\ddot{\theta}_3 + m_4 l_4\ddot{\theta}_4 = -l_2\theta_2\sum_{k=2}^{4} m_k\, g$(2.9)

$i = 3, \sum_{k=3}^{4} m_k l_1\ddot{\theta}_1 + \sum_{k=3}^{4} m_k l_2\ddot{\theta}_2 + \sum_{k=3}^{4} m_k l_3\ddot{\theta}_3 + m_4 l_4\ddot{\theta}_4 = -l_3\theta_3\sum_{k=3}^{4} m_k\, g$(2.10)

$i = 4, m_4 l_1\ddot{\theta}_1 + m_4 l_2\ddot{\theta}_2 + m_4 l_3\ddot{\theta}_3 + m_4 l_4\ddot{\theta}_4 = -l_4\theta_4 m_4 g$(2.11)

Suppose the mass units suspended are $m_1 = 6, m_2 = 4, m_3 = 3\ and\ m_4 = 2, where\ m_i \times 10^{-2}kg$ with the lengths of separation $l_i$and the angles of inclination to the vertical as θ_i: $i = (1,2,3,4)$. Substituting these values into equations (2.8-2.11) above yield:

$15l_1\ddot{\theta}_1 + 9l_2\ddot{\theta}_2 + 5l_3\ddot{\theta}_3 + 2l_4\ddot{\theta}_4 = -15\theta_1 g$ (2.12)

$9l_1\ddot{\theta}_1 + 9l_2\ddot{\theta}_2 + 5l_3\ddot{\theta}_3 + 2l_4\ddot{\theta}_4 = -9\theta_2 g$ (2.13)
$5l_1\ddot{\theta}_1 + 5l_2\ddot{\theta}_3 + 5l_3\ddot{\theta}_3 + 2l_4\ddot{\theta}_4 = -5\theta_3 g$(2.14)
$2l_1\ddot{\theta}_1 + 2l_2\ddot{\theta}_2 + 2l_3\ddot{\theta}_3 + 2l_4\ddot{\theta}_4 = -2\theta_1 g$ (2.15)
The application of the tri-diagonal symmetric inverse matrix operator gives

$$\begin{pmatrix} l_1\ddot{\theta}_1 \\ l_2\ddot{\theta}_2 \\ l_3\ddot{\theta}_3 \\ l_4\ddot{\theta}_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & -\frac{5}{12} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{7}{12} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & -\frac{5}{6} \end{pmatrix} \begin{pmatrix} 15\theta_1 \\ 9\theta_2 \\ 5\theta_3 \\ 2\theta_4 \end{pmatrix} g \quad (2.16)$$

After computation the resultant equations are:
$\ddot{\theta}_1 = \frac{1}{2}(3\theta_2 - 5\theta_1)\frac{g}{l_1}$(2.17)
$\ddot{\theta}_2 = \frac{5}{4}(2\theta_1 - 3\theta_2 + \theta_3)\frac{g}{l_2}$(2.18)
$\ddot{\theta}_3 = \frac{1}{12}(27\theta_2 - 35\theta_3 + 8\theta_4)\frac{g}{l_3}$(2.19)
$\ddot{\theta}_4 = \frac{5}{3}(\theta_3 - \theta_4)\frac{g}{l_4}$(2.20)

If the mass units suspended in (a) above were re-arranged such that $m_1 = 3, m_2 = 6, m_3 = 2$ and $m_4 = 4$, $where$ $m_i \times 10^{-2} kg$ with the lengths of separation $l_i$and the angles of inclination to the vertical as θ_i: $i = (1,2,3,4)$.Substituting these values into equations (2.8-2.11) above yield:
$15l_1\ddot{\theta}_1 + 12l_2\ddot{\theta}_2 + 6l_3\ddot{\theta}_3 + 4l_4\ddot{\theta}_4 = -15\theta_1 g$ (2.21)
$912\ddot{\theta}_1 + 12\ddot{\theta}_2 + 6l_3\ddot{\theta}_3 + 4l_4\ddot{\theta}_4 = -12\theta_2 g$ (2.22)
$6l_1\ddot{\theta}_1 + 6l_2\ddot{\theta}_3 + 6l_3\ddot{\theta}_3 + 4l_4\ddot{\theta}_4 = -6\theta_3 g$(2.23)
$4l_1\ddot{\theta}_1 + 4l_2\ddot{\theta}_2 + 4\ddot{\theta}_3 + 4l_4\ddot{\theta}_4 = -4\theta_1 g$ (2.24)
From this equation, the determinant of the symmetric matrix operator is

$$\begin{vmatrix} 15 & 12 & 6 & 4 \\ 12 & 12 & 6 & 4 \\ 6 & 6 & 6 & 4 \\ 4 & 4 & 4 & 4 \end{vmatrix} \times 10^{-2} = (3 \times 10^{-2}).(6 \times 10^{-2}).(2 \times 10^{-2}).(4 \times 10^{-2}) \quad (2.25)$$

$$= \prod_{i=1,2,3,4} m_i \quad (2.26)$$

When the symmetric inverse matrix operator is applied, the resultant matrix equation is

$$\begin{pmatrix} l_1\ddot{\theta}_1 \\ l_2\ddot{\theta}_2 \\ l_3\ddot{\theta}_3 \\ l_4\ddot{\theta}_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} 15\theta_1 \\ 12\theta_2 \\ 6\theta_3 \\ 4\theta_4 \end{pmatrix} g \quad (2.27)$$

Solving this equationfor the angular accelerations the results are:

$\ddot{\theta}_1 = (4\theta_2 - 5\theta_1)\frac{g}{l_1}$(2.28)
$\ddot{\theta}_2 = (5\theta_1 - 6\theta_2 + \theta_3)\frac{g}{l_2}$(2.29)
$\ddot{\theta}_3 = (2\theta_2 - 4\theta_3 + 2\theta_4)\frac{g}{l_3}$(2.30)
$\ddot{\theta}_4 = 3(\theta_3 - \theta_4)\frac{g}{l_4}$ (2.31)

If the masses m_i : $m_{i+1} = m_{i+2} = m_{i+3} = 5 \times 10^{-2} kg$ and $m_{i+4} = m_{i+5} = 3 \times 10^{-2} kg$ when substituted in equation (2.7) the symmetric matrix operator is

$$\begin{pmatrix} 21 & 16 & 11 & 6 & 3 \\ 16 & 16 & 11 & 6 & 3 \\ 11 & 11 & 11 & 6 & 3 \\ 6 & 6 & 6 & 6 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} \times 10^{-2} \quad (2.32)$$

whose determinant is

$$\left|\begin{pmatrix} 21 & 16 & 11 & 6 & 3 \\ 16 & 16 & 11 & 6 & 3 \\ 11 & 11 & 11 & 6 & 3 \\ 6 & 6 & 6 & 6 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} \times 10^{-2}\right| \quad (2.33)$$

$$= (5\times 10^{-2}).(5\times 10^{-2}).(5\times 10^{-2}).(3\times 10^{-2}).(3\times 10^{-2}) \quad (2.34)$$

$$= \prod_{i=i+1,i+2,i+3.} m_i . \prod_{j=i+4,i+5.} m_j \quad (2.35)$$

But $m_{i+1} = m_{i+2} = m_{i+3}$ and $m_{i+4} = m_{i+5}$

Therefore for $n(m_i) = x$ and $n(m_j) = y$ the determinant for the symmetric matrix operator $= m_i^x . m_j^y$.

Consequently for n unequal mass units the determinant for the symmetric matrix operator is

$$\left|\sum_{k=i}^n m_k \quad \sum_{k=(i+1)}^n m_k \quad \sum_{k=(i+2)}^n m_k \quad \sum_{k=(i+3)}^n m_k \cdots \sum_{k=(n-2)}^n m_k \; m_{(n-1)} \quad m_n\right| = \prod_{k=i}^n m_k \quad (2.36)$$

It can be deduced that if all the n mass units are equal then the determinant for symmetric matrix operator $= m^n$

3. Discussion and Conclusion

The general equations of motion for an n-pendula system are established as well as the Lagrangian of the system. Various matrix operators from a variety of combinations of equal and unequal quantities of masses, lengths and angles are constructed. For the derivations of equations of motion, the respective inverse matrix operators are used to obtain the subsequent angular accelerations of the systems. Each inverse matrix operator is unique to its mother matrix. The application of these operators gives many determinants that are also computed to a pattern as explained in the text.

It might be interesting to advance this study in the determination of the Hamiltonian formulations for n suspended mass units oscillating in a plane or in space. This research may prove handy for robotic science, theoretical physics and engineering.

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