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More on Separation Axioms

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Abstract:
In this paper we study R0 space with the help of semi-preopen sets introduced by Andrijevic/ [1].
Semi-prekernel of a set is defined and its several properties are investigated. Semi-prekernel of a set is used as a tool to characterize the new space termed as sp- R0 space.
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1. Introduction

In 1943, N. Shanin [5] introduced R_0 separation axiom. Maheswari [4] et al. introduced $(R_0)_s$ space with the aid of semi-open sets while Caldas et al. [2] defined R_0 space utilizing preopen sets. The purpose of this paper is to introduce sp-R_0 space using semi-preopen sets introduced by Andrejevic/ [1]. In Section 2 of this paper some known definitions and results necessary for the presentation of the subject are given. Section 3 and Section 4 deal with the definitions and characterization along with some basic properties of semi-pre kernel (briefly sp- ker.) and sp- R_0 space respectively.

2. Preliminaries

Throughout the paper (X, τ) or X always denotes a non trivial topological space.

- Definition 2. 1. $A \subset X$ is called a semi-preopen set (briefly s. p. o. set) [1] iff $A \subset Cl(Int(Cl(A)))$. The family of all s. p. o. sets are denoted by SPO(X). For each $x \in X$, the family of all s. p. o. sets containing x is denoted by SPO (X, x).
- Definition 2. 2. The complement of a s. p. o. set is called semi-preclosed [1]. Equivalently a set F is semi-preclosed [1] iff $Int(Cl(Int(A))) \subset F$. The family of all semi-preclosed sets is denoted by SPF (X).
- Definition 2. 3. The semi-preclosure [1] of $A \subset X$ is denoted by spcl (A) and is defined by spcl (A) = $\cap$ {B: B is semi-preclosed and $B \supset A$}.
- Definition 2. 4. A topological space X is said to be sp- T_1 [3] iff for every pair of points $x,y \in X$ such that $x \neq y$, there exists a $U \in SPO(X,x)$ not containing y and a $V \in SPO(X,y)$ not containing x.
- Lemma 2. 1[1]. Let A be a subset of X, then spcl (A) = $A \cup Int(Cl(Int(A)))$.
- Lemma2. 2[3]. Every one pointic set in a topological space is either semi-preclosed or open.
- Theorem 2. 1[3]. A topological space X is sp-T_1 iff every one pointic set is semi-preclosed.

3. Semi-Pre Kernel

- Definition 3. 1. Let (X, τ) be a topological space and $A \subset X$. Then the semi-pre Kernel of A (briefly sp-Ker (A)) is defined to be the set sp-Ker (A) = $\cap$ {U: $U \in$ SPO (X), $A \subset U$}.
- Remark 3. 1. Ghosh [3] defined semi-pre Kernel of a point $x \in X$ as sp-Ker ({x})=$\cap$ {$y \in X$: $x \in$ spcl ({y})}.
- Theorem 3. 1. Let x, $y \in X$. Then $y \in$ sp-Ker({x}) iff $x \in$ spcl ({y}).
- Proof. Necessity. Let $y \in$ sp-Ker ({x})$\Rightarrow$ $y \in \cap$ {U : $U \in$ SPO (X),{x}$\subset$U}$\Rightarrow$ {y}$\cap$U $\neq \phi$ for every $U \in$ SPO (X, x). Therefore, $x \in$ spcl ({y}).
- Sufficiency. Suppose $x \in$ spcl ({y}). Then {y} $\cap$ U $\neq \phi$ for every $U \in$ SPO (X, x)$\Rightarrow$ $y \in \cap$ {U : $U \in$ SPO (X), {x} $\subset$ U} $\Rightarrow$ $y \in$ sp-Ker ({x}).
- Remark 3. 2. From the above lemma, one now observes that for a one pointic set, Ghosh's definition coincides with Definition 3. 1.
- Theorem 3. 2. Let X be a topological space and $A \subset X$. Then sp-Ker (A) = {$x \in X$: spcl ({x}) $\cap$ A $\neq \phi$}.

→ Proof. Let $x \in$ sp-Ker (A) and suppose spcl ({x}) $\cap$ A = ϕ. Clearly $x \in$ spcl ({x}) $\Rightarrow$ $x \notin X$ – spcl ({x}). We observe that $A \subset X$ – spcl ({x}) $\in$ SPO (X), but X – spcl ({x}) does not contain x. This contradicts our assumption that $x \in$ sp-Ker (A).

Thus spcl ({x}) ∩ A ≠ ϕ ⇒ sp-Ker (A) ⊂ {x ∈ X : spcl ({x}) ∩ A ≠ ϕ}. To prove the reverse inclusion let x ∈ X be such that spcl ({x}) ∩ A ≠ ϕ. If possible let x ∉ sp-Ker (A). Then there exists a U_0 ∈ SPO (X) with A ⊂ U_0 such that x ∉ U_0. Again since spcl ({x}) ∩ A ≠ ϕ there exists a y ∈ A such that y ∈ spcl ({x}). By Theorem 3. 1 , x ∈ sp-Ker ({y}). Thus x ∈ ∩{U:U∈SPO(X),{y}⊂U}⊂∩{U:U∈SPO(X),A⊂U} ⇒ a contradiction to the above assertion that x∉ U_0 whence {x∈X: spcl ({x})∩A≠ϕ} ⊂ sp-Ker (A). Hence sp-Ker (A) = {x ∈ X : spcl ({x}) ∩ A ≠ ϕ}.

4. Semi-Pre R_0 Spaces

- Definition 4. 1. A topological space (X, τ) is said to be a semi-pre R_0 (briefly sp-R_0) space if spcl ({x}) ⊂ U for every U ∈ SPO (X, x).
- Theorem 4. 1. A topological space (X, τ) is sp-R_0 iff for each U ∈ SPO (X, x)
  Int (Cl (Int ({x}))) ⊂ U.
- → Proof. Let U∈SPO(X,x)⇒spcl({x}) ⊂U. Then Lemma 2. 1 yields{x}∪Int (Cl (Int ({x}))) ⊂ U ⇒ Int (Cl (Int ({x}))) ⊂ U. Conversely, let the given condition holds. Also let x ∈ X and U ∈ SPO (X, x). Then Int (Cl (Int ({x}))) ⊂ U ⇒ {x} ∪ Int (Cl (Int ({x}))) ⊂ U. Again, by Lemma 2. 1 one obtains spcl ({x}) ⊂ U.
- Theorem 4. 2. A topological space is sp-R_0 iff it is sp-T_1.
- → Proof. Let X be a sp-R_0 space and x ∈ X. Then, by Lemma 2. 2, {x} ∈ SPO (X) or {x} ∈ SPF (X). If {x} ∈ SPO (X) then spcl ({x}) ⊂ {x} ⇒ spcl ({x}) = {x}⇒ {x} ∈ SPF (X).

Thus in any case every one pointic set in X is semi- preclosed. Hence by Theorem 2. 1, X is sp-T_1. Conversely, let X be sp-T_1 and U∈ SPO(X, x). The sp-T_1-ness of X gives {x}∈ SPF (X) whence one deduces spcl ({x}) ⊂ {x} ⊂ U ⇒ X is sp-R_0.

- Theorem 4. 3. For a topological space (X, τ) the following statements are equivalent :
  - (i) (X, τ) is sp-R_0.
  - (ii) For any F ∈ SPF(X), x ∉ F there exists a U ∈ SPO(X) such that x ∉ U and F ⊂ U.
  - (iii) If F ∈ SPF (X) and x ∉ F then F ∩ spcl ({x}) = ϕ.
  - (iv) For any two points x and y of X
    spcl ({x}) ≠ spcl ({y}) ⇒ spcl ({x}) ∩ spcl ({y}) = ϕ.
- → Proof. (i) ⇒ (ii). Let (X, τ) be sp-R_0, F ∈ SPF (X) and x ∉ F. Then X – F ∈ SPO (X, x)

⇒ spcl ({x}) ⊂ X – F ⇒ X – spcl ({x}) ⊃ F. Set U = X – spcl ({x}). Clearly U ∈ SPO (X), x ∉ U and F ⊂ U.

(ii) ⇒ (iii). Let F ∈ SPF (X), x ∉ F. By (ii) there exists U ∈ SPO (X) such that F ⊂ U and x ∉ U. Now U ∩ {x} = ϕ ⇒ U ∩ spcl ({x}) = ϕ ⇒ F ∩ spcl ({x}) = ϕ.

(iii) ⇒ (iv). Suppose spcl ({x}) ≠ spcl ({y}) for x, y ∈ X. Obviously, x ≠ y. Now distinct semi-pre-closures indicates that there exists a z ∈ spcl ({x}) but z ∉ spcl ({y}) or
z ∉ spcl ({x}) but z ∈ spcl ({y}). To fix our ideas let z ∈ spcl ({x}) and z ∉ spcl ({y}).

Since z ∉ spcl ({y}) there exists a V ∈ SPO (X, z) such that {y}∩V = ϕ ⇒ y ∉ V. Again z ∈spcl ({x}) yields that x ∈ V. Therefore,V∈SPO (X, x) and hence x ∉ spcl ({y}). If we write F = spcl ({y}), then x ∉ F. Now using (iii) we deduce spcl ({x}) ∩ spcl ({y}) = ϕ.

(iv) ⇒ (i). Suppose (iv) holds. Let x ∈ V ∈ SPO (X). Also let y ∉ V. This, then, assures that x ≠ y and x ∉ spcl ({y}). Thus spcl ({x}) ≠ spcl ({y}). By (iv), spcl ({x}) ∩ spcl ({y}) = ϕ for each y ∈ X – V. Let y run over X – V. Then from above
spcl ({x}) ∩ (∪spcl ({y})) = ϕ. Since X–V is a semi-preclosed set containing {y},
X – V = $\bigcup_{y\in X-V}$ spcl ({y}). So from above spcl ({x}) ∩ (X – V) = ϕ⇒ spcl ({x}) ⊂ V.

Hence (X, τ) is sp-R_0.

- Lemma 4. 1. Let X be a topological space. Then for any two points x, y ∈ X
  sp-Ker ({x}) ≠ sp-Ker ({y}) iff spcl ({x}) ≠ spcl ({y}).
- → Proof. Suppose sp-Ker ({x}) ≠ sp-Ker ({y}). This then guarantees the existence of a point z such that either z ∈ sp-Ker ({x}) but z ∉ sp-Ker ({y}) or z ∉ sp-Ker ({x}) but z ∈ sp-Ker ({y}). To fix our ideas let z ∈ sp-Ker({x}). Since z ∈sp-Ker({x}), by Theorem3. 1, one obtains x ∈ spcl ({z}). Again z ∉ sp-Ker ({y}) ensures the emptiness of the set {y}∩ spcl ({z}). Now x ∈ spcl ({z}) ⇒ spcl ({x}) ⊂ spcl ({z}). Thus {y}∩ spcl ({x}) = ϕ ⇒ spcl ({x}) ≠ spcl ({y}). Next let spcl ({x})≠spcl({y}). Then there exists a point z ∈ X such that z ∈ spcl ({x}) but z ∉ spcl ({y}). This indicates that there exists a U ∈ SPO (X, z) such that U ∩ {y} = ϕ. Now z ∈ spcl ({x}) ⇒{x} ∩ U ≠ ϕ ⇒ x ∈ U⇒ x ∉ spcl ({y}) ⇒ y ∉ sp-Ker ({x})⇒ sp-Ker ({x}) ≠sp-Ker ({y}).
- Example 4. 1. Let X = {a, b, c} be the set with the topology τ = {ϕ, X, {a}} . Then SPO (X) = {ϕ, X, {a}, {a, b}, {a, c}}. Clearly X is not sp-R_0. For a, b ∈ X we note that

sp-Ker ({a}) = {a} and sp-Ker ({b}) = {a, b} i. e. sp-Ker({a})∩sp-Ker ({b}) ≠ ϕ. Thus in an arbitrary topological space sp-Ker of any two points may not be disjoint. But the disjointness of sp-Ker of any two points characterises sp-R_0 space. In fact we have

- Theorem 4. 4. A topological space (X, τ) is sp-R_0 iff x, y ∈ X and

sp-Ker ({x}) ≠ sp-Ker ({y}) ⇒ sp-Ker ({x}) ∩ sp-Ker ({y}) = ϕ.

→ Proof. Necessity. Let X be sp-R_0 and x, y ∈ X be such that sp-Ker ({x}) ≠ sp-Ker ({y}).
Lemma4. 1 yields spcl({x})≠spcl({y}). If possible suppose sp-Ker({x})∩sp-Ker({y})≠ ϕ. Let z∈sp-Ker({x})∩ sp-Ker ({y}). Since z ∈ sp-Ker ({x}) by Theorem 3. 1, x ∈spcl ({z}) ⇒ spcl ({x}) ∩ spcl ({z}) ≠ ϕ. The sp-R_0-ness of X and Theorem 4. 3 together indicate that spcl ({x}) = spcl ({z}). Similarly spcl ({z}) = spcl ({y}) ⇒ spcl ({x}) = spcl ({y}) ⇒ a contradiction. Hence sp-Ker ({x}) ∩ sp-Ker ({y}) = ϕ.

- Sufficiency. Let the given condition holds and x,y ∈X with spcl({x})≠spcl ({y}). Then by Lemma 4. 1 sp-Ker ({x}) ≠ sp-Ker ({y}) ⇒sp-Ker ({x}) ∩ sp-Ker ({y}) = ϕ. We assert that spcl({x})∩spcl ({y}) = ϕ. If possible suppose spcl ({x})∩spcl ({y})≠ϕ. This indicates that there exists a z ∈ spcl ({x}) and z ∈ spcl ({y})⇒ x ∈sp-Ker ({z}) and also y∈sp-Ker({z})⇒sp-Ker({x})∩sp-Ker({z})≠ϕ and sp-Ker ({y})∩sp-Ker ({z})≠ϕ. Then by hypothesis sp-Ker ({x}) = sp-Ker ({z}) = sp-Ker ({y}) ⇒ a contradiction to the foregoing. Therefore spcl ({x}) ∩ spcl ({y}) = ϕ. Hence by Theorem 4. 3, (X, τ) is sp-R_0.
- Theorem 4. 5. For a topological space (X, τ) the following statements are equivalent :
  (i) (X, τ) is sp-R_0 space.
  (ii) For any non-empty set A and any G ∈ SPO (X) with A ∩ G ≠ ϕ, there exists a F ∈ SPF (X) such that A ∩ F ≠ ϕ and F ⊂ G.
  (iii) G ∈ SPO (X) ⇒ G = ∪ {F : F ∈ SPF (X), F ⊂ G}.
  (iv) F ∈ SPF (X) ⇒ F = ∩ {G : G ∈ SPO (X), F ⊂ G}.
  (v) x ∈ X ⇒ spcl ({x}) ⊂ sp-Ker ({x}).

→ Proof. (i) ⇒ (ii). Suppose A ≠ ϕ, G ∈ SPO (X) and A ∩ G ≠ ϕ. Let x ∈ A ∩ G. Put F = spcl ({x}). Then F ∈ SPF (X, x). Since X is sp-R_0 , x ∈ G ∈ SPO (X)⇒ spcl ({x}) ⊂ G ⇒ F ⊂ G. Finally x ∈ F and x ∈ A ensures that A ∩ F ≠ ϕ.
(ii) ⇒ (iii). Assume G ∈ SPO (X). Clearly ∪{F : F ∈ SPF (X), F ⊂ G} ⊂ G. To prove the reverse inclusion suppose x ∈ G. By hypothesis there exists a F ∈ SPF (X) such that
x ∈ F and F ⊂G, which, in its turn, imply that x ∈ F ⊂ ∪ {F : F ∈ SPF (X), F ⊂ G}. Hence G = ∪ {F : F ∈ SPF (X), F ⊂ G}. (1).
(iii) ⇒ (iv). Assume that (iii) holds. An application of De Morgan's Law to (1) yields the desired result.
(iv) ⇒ (v). Let x, y ∈ X with the property y ∉ sp-Ker ({x}). Then there exists a U ∈ SPO (X, x) such that y ∉ U. Hence {y} ∩ U = ϕ whence spcl ({y}) ∩ U = ϕ. Now, by (iv), we have spcl ({y}) = ∩ {G : G ∈ SPO (X), spcl ({y}) ⊂ G}. So, from above
∩{G : G ∈ SPO (X), spcl ({y}) ⊂ G} ∩ U = ϕ. This, then, ensures the existence of a G∈ SPO(X) such that spcl({y}) ⊂ G and x ∉ G. So {x}∩G = ϕ whence spcl({x})∩G = ϕ.
This gives y ∉ spcl ({x}). Consequently spcl ({x}) ⊂ sp-Ker ({x}). (v) ⇒ (ii). Let G ∈ SPO (X, x). By definition sp-Ker ({x}) ⊂ G. Also by (v)
spcl ({x}) ⊂ sp-Ker ({x}). So, x ∈ spcl ({x}) ⊂ sp-Ker ({x}) ⊂ G ⇒ spcl ({x}) ⊂ G.
Thus x ∈ G ⇒ spcl ({x}) ⊂ G ⇒ X is sp-R_0.

- Corollary 4. 1. (X, τ) is a sp-R_0 space iff spcl ({x}) = sp-Ker ({x}) for all x ∈ X.

→ Proof. Let (X, τ) be sp-R_0 and y ∈sp-Ker ({x}). Then Theorem 3. 1 gives x∈ spcl ({y}). Hence spcl ({x})∩spcl ({y}) ≠ ϕ. Therefore, by Theorem 4. 3, spcl ({x}) = spcl ({y})from which one concludes y∈spcl({x})⇒ sp-Ker ({y}) ⊂ spcl ({x}). The reverse inclusion because of sp-R_0-ness,directly follows from Theorem4. 5(v). Hence spcl({x})=sp-Ker({y}). Conversely,sp-R_0 – ness of X follows readily from (v) of Theorem 4. 5.

- Theorem 4. 6. For a topological space X, the following statements are equivalent :
  (i) X is sp-R_0.
  (ii) F ∈ SPF (X) then F = sp-Ker (F).
  (iii) F ∈ SPF (X) and x ∈ F, then sp-Ker ({x}) ⊂ F.
  (iv) If x ∈ X, then sp-Ker ({x}) ⊂ spcl ({x}).

→ Proof. (i) ⇒ (ii). This follows from Theorem 4. 5(iv) and the definition of sp-Ker.
(ii) ⇒ (iii). Let F ∈ SPF (X) and x ∈ F. Clearly {x} ⊂ F ⇒sp-Ker ({x}) ⊂ sp-Ker (F) = F.
(iii) ⇒ (iv). Suppose x ∈ X . Obviously spcl ({x}) ∈ SPF (X) and x ∈ spcl ({x}).
Therefore by (iii) sp-Ker ({x}) ⊂ spcl ({x}).
(iv) ⇒ (i). Let y ∈ X and x ∈ X be such that x ∈ spcl ({y}). Then, by Theorem 3. 1, y ∈ sp-Ker ({x}). By hypothesis sp-Ker ({x}) ⊂ spcl ({x}) which, in its turn, indicates that y ∈ spcl ({x}). This then implies, again by Theorem 3. 1, that x ∈ sp-Ker ({y}). From this one infers that spcl ({y}) ⊂ sp-Ker ({y}). Hence by Theorem 4. 5(v), X is sp-R_0.

5. References

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