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On Kadec Norms and Asplund Spaces

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Abstract:
This paper is focused on some properties that linked with renorming theory in connection with the Kadec norms. More important ramification of Kadec norms have been seen. A succinct introduction to Kadec- klee norms and Asplund spaces is given and provide the fascinating connection between w-w- Kadec –Klee norms and Asplund spaces. We study nearly about the problem whether the existence of a KK* renorming of X* implies that X is an Asplund space? which has apparently been posed by Godefroy, [i, ii].*

Keywords: *Kadec –klee norm, sequentially w*- Kadec Klee, continuous convex function, Frechet & Gateaux derivatives*

1. Introduction

Asplund spaces are an interesting and useful class of Banach spaces. They have the vast area of study and occupy a central place in a Banach space theory. Although a number of equivalent characterization of Asplund spaces has grown to what is new significance number. The Asplund spaces are quite rightly embed into the general theory of convex functions on infinite dimensional spaces [iii, xvii]. Moreover, in the separable Banach space, there is a tight connection between renormability of the space and Asplund property. Asplund spaces are the venue where continuous convex functions have a dense set of points of Frechet differentiability [xxiv].

An Asplund space is a Banach space, on which every continuous convex real- valued function is automatically Frechet differentiable on a dense $G_\delta$subset of X,i. e. a dense countable intersection of open subsets of X. Such a space mirrors the differentiability properties of continuous convex functions on Euclidean space. The differentiability properties of convex functions on Banach spaces have been studied for many years. The study of this properties on infinite dimensional spaces has continued on and off for over fifty years. The list of pioneering works in this area include the papers by Asplund [iv], Kurzweil [v], Lindenstrauss [vi], Leach and Whitfield [vii], Namioka and Phelps [viii, ix], Preiss [x, xi] and Stegall [xii, xiii, xiv] established the significance of this class of spaces by proving that a Banach space X is an Asplund space if and only if its dual X* has the Radon–Nikodym property [xv]. There are several classes of Asplund spaces was first introduced by Edger Asplund 1968 [xvi, xvii, xx], who was interested in the Frechet differentiability properties of Lipschitz functions on a Banach spaces. Originally these spaces were called “Strong differentiability spaces" by him. Latter it was renamed in his honor in [viii], and now known as Asplund space. He began the serious investigation of them. In 1933 S. Mazur [iii] showed that every separable Banach space is weak Asplund, and then there has been continuous interest in the study of weak Asplund spaces.

Ekeland and Lebourg established the following result: If X has an equivalent Frechet differentiable norm then X is Asplund but converse does not hold [xviii, xxxv]. Briefly we say that a class of spaces which has turned out to be of special importance is the class of strong differentiability spaces, now called Asplund spaces. Since the time of Asplund a considerable volume of literature has been written on Asplund spaces and it was one of the major achievements of functional analysis in the late 70's (in fact, in) when the class of Asplund spaces was characterized as those Banach spaces whose dual spaces possess the Radon-Nikodym property. Subsequent to this many other characterizations as Asplund spaces have been discovered [xix]. The most useful characterization is that a Banach space X is an Asplund space if and only if every separable subspace has a separable dual. For example, for any set Γ every separable subspace of $c_0(\Gamma)$ has separable dual, so $c_0(\Gamma)$ is an Asplund space [xxxiii]. A significant property of Asplund spaces, and with application in optimization theory, was established by D. Preiss [xii], who showed that every locally Lipschitz function on an open subset of an Asplund space is Fréchet differentiable at the points of a dense subset of its domain. In the context of Frechet differentiability we have to assume in existence results that the dual space X* is separable, i.e., Xis what is called an Asplund space. It is known, for example, that if X is separable with X* non separable, there is an equivalent norm $\|\cdot\|$ on X so that the convex continuous function $f:X \to \mathbb{R}$ defined by $f(x) = \| x \|$ is nowhere Frechet differentiable.

In 1965, Kadec -Klee property (KK) properties was introduced by Kadec Klee. He was important contributor to the development of renorming theory. This property was first studied by J. Radon [xxi] and subsequently by F. Riesz [xxii, xxiii] who showed that the classical $L_p(\Gamma)$ spaces, $1 < p < \infty$, have the Kadec-Klee property. Although the space $L_1[0,1]$(with Lebesgue measure) fails to have the Kadec-Klee property, Riesz showed that each of the L p - spaces, $1 \leq p < \infty$, has the property that each sequence on the unit sphere that converges almost everywhere converges also in norm. Later, V. L. Smulian showed that every uniformly convex Banach space has the Kadec-Klee property. On the other hand, S.L. Troyanski showed that the sequence space (l_1) admits no norm with the Kadec-Klee property.

The Kadec-klee norms, play (explicitly or implicitly) an absolutely key role in the geometry of Banach space theory and its applications. So these are very nice geometrical property in a Banach spaces. This discussion is significant in its own right. Kadec norms are themselves interesting and common. A special case of normed space with Kadec norms is particularly important and we sketch the theory of most interesting Kadec norms.

1.1. Examples of Asplund Spaces

The set of real numbers ($\mathbb{R}$), reflexive spaces and the space $C_0(\Gamma)$ for any set Γ, Separable, WCG and Vasak spaces, Banach spaces whose duals are separable, Subspaces of C(K), where K is a scattered Compact Hausdorff Space, Spaces with Frechet differential norms,spaces whose duals have certain properties are Asplund spaces, while the spaces C ([0,1]), $l_1(\mathbb{N})$ and $l_\infty(\mathbb{N})$ are not Asplund spaces. Indeed, l_1 is not Asplund because its norm is not Frechet differentiable at any point [xvii, xxiv, x].

Notations.(X, ‖· ‖) is the real Banach space with norm ‖·‖; S(X) is the unit sphere in X; (X*,‖·‖*) is the dual space of X; S(X*) is the unit sphere of X*; B(X) is the unit ball of X; B(X*) is the unit ball of X*;→*;* ⇀*;* ⇀ are the strong, weak, w*-convergence of sequences respectively.

2. Definitions

2.1*A norm ‖·‖ on a Banach space X has Kadec –Klee Property (KK in short) property if any sequence $\{x_n\}$ inX, $x_n \rightharpoonup x$to some x ∈X*and‖ x_n ‖→ ‖ x‖*implies* ‖x_n - x ‖ → 0as n→∞.

2. 2*A norm‖,‖*on a Banach space X*has*dual Kadec –Klee Property (KK**) if anysequence$\{f_n\}$in X*, $f_n \rightharpoonup f$to some f ∈ X* and*‖f_n ‖→‖*f* ‖*implies* ‖f_n - *f* ‖ → *0*.

*i.e.*we will say a norm on *X* * is *weak*-Kadec*if the norm and weak*-topologies agree on its unit sphere (this is sometimes called the *weak*-Kadec–Klee property* [xiv].

2.3A norm ‖. ‖ is said to be w* - Kadec if ‖x_α - *x*‖ →0 whenever $x_\alpha \overset{w*}{\to} x$and ‖$x_\alpha$ ‖→‖ *x* ‖*,* if it holds forsequences then ‖.‖ is calledsequentially w*-Kadec –Klee[xvii].

2.4We say that a Banach space X is sequentially compact if every sequence in X has a convergent subsequence which converges to some point of X.

2.5.A dual Banach space X* is called weak *-Asplund if every w* -lower semi-continuous convex function F: X* → $\mathbb{R}$is Frechet differentiable on denseG_δ -subset ofX*.

2.6Dual weak Kadec-Klee property (w KK*) if the w*-convergent and weak convergent sequences coincide on the dual unit sphere S(X*).

2.7Let X be a Banach space. We say that a function f: X → $\mathbb{R}$is Gateaux differentiable at x ∈X if there exists a continuous linear functional Tsuch that

$T(h) = \lim_{t\to 0}\frac{f(x + th) - f(x)}{t}$for all h ∈X.

In this case, the linear functional Tis called the Gateaux derivative off at *x* ∈X.If the limit above is approached uniformly with respect to all h ∈B(X)-the closed unit ball in X, then T is said to be Frechet differentiable at *x* ∈X and Tis called the Frechet derivative off at *x*[x].

2.8A Banach space X has the Grothendieck property if in the dual space X*,the w*-convergent and w-convergent sequences coincide [xxxii].

- We now face the following problem

If the norm of a Banach space X has a dual norm that is sequentially w*- Kadec-Klee, is the dual norm w*-Kadec –Klee? Equivalently, if a Banach space X admits a norm whose dual norm sequentially weak *-Kadec- klee, is X Asplund?[i, ii].

- This problem will be solved with additional conditions on X and X*.

In our note we are going to address the still weaker property; dual weak Kadec- Klee (wKK*) property ofX*. It follows that for every renorming of a given Grothendieck space X, its dual X* has the wKK*. Hence the property may not imply that X is Asplund, in general (e.g. ℓ∞).P. Hajek and J. Talponen showed in [xxxii] that if X is a Banach space which is a weak Asplund space and has the w* -w -Kadets Klee property, then X is already an Asplund space.

3. Some Results [24]

→ *Theorem 3.1*[xxvii]. Let X be a Banach space. Then TFAE:

(i) The dual norm on X* is w*- Kadec Klee.

(ii) For every subspace Y⊂ X, the dual norm on Y* is w*- Kadec –Klee.

(iii) For every separable subspace Y⊂ X, the dual norm on Y* is sequentially w*- Kadec –Klee.

- Proof:

(i)⇒ (ii): Assume that the dual norm on X* is w*- Kadec Klee. We have to show that for every subspace Y⊂ X, the dual normon Y* is w*- Kadec –Klee.Suppose that Y ⊂ X and dual norm on Y* is not w*-Kadec -Klee. Then there isa weakly compact set K, and ϵ > 0and a sequence$g_n \to g$w*-weakly such that
$\| g_n \| = \| g \| =1$ and$\| g_n - g \| = \sup_K | g_n - g | >$ ϵfor all n.
Let f_n be norm preserving extension of g_n. As we know, the unit dualball B(X*) is w*- compact (by Alaoglu'stheorem). So by sequentially w*- compactness, there is a subsequence $\{f_{nk}\}$ of the sequence $\{ f_n \}$: $f_{nk} \to f$ w* -weakly for some f∈ B(X*). And by Hahn Banach theorem we have f/Y = g and ‖ f ‖ =‖ g ‖=1 ⇒ ‖ f ‖ = 1.It follows that
$\| f_{n k} - f \| = \|f_{n k}-f\| = \sup_K | f_{n k} - f |>$ ϵfor all k.
This shows that the dual norm‖.‖ on X* is notw*- Kadec-Klee. Contradicting the hypothesis. So dual norm ‖.‖ on Y* must be w*- Kadec –Klee.
(ii) ⇒ (iii): As Y is separable then every bounded sequence in Y* has a w*-convergent subsequence. so Y* is sequentially compact. By hypothesis dual norm ‖.‖ on Y* is w*- Kadec –Klee, So it is sequentially w*- Kadec-Klee

→ *Theorem3.2*[xxvii]For a Banach space X, the following are equivalent:
(i)The dual norm on X* is w*- Kadec – Klee.
(ii)X is Asplund and the dual norm on X* is sequentially w*- Kadec-Klee.
(iii)B(X*) is w*-sequentially compact and the dual norm on X * is sequentiallyw*- Kadec –Klee.

- Proof:

(i) ⇒ (ii): Assume that the dual norm on X* is w*- Kadec- Klee. We have to show that X is Asplund and the dual norm on X* is sequentially w*- Kadec-Klee.
Let Y be separable subspace of X. Supose that ‖.‖* is a dual w*- Kadec-Klee norm on X*. Let $\{f_n\}$be w*- dense sequence in B(Y*). For fixed f ∈ S(Y*), choose $f_i \xrightarrow{w*} f$. Because the dual norm on Y* is sequentially w*- Kadec –Klee,$f_i \to$ f(∵by theorem1). ThusY* is separable. So by definition of Asplund space X is Asplund.
(ii) ⇒(iii):Assume that X is Asplund and the dual norm on X* is sequentially w*- Kadec-Klee. Since Asplund spaces have w*-sequentially compact balls[5. p 230] so, by assumption, the dual norm on X* is sequentially w*- Kadec-Klee. □
(iii) ⇒ (i): Suppose the unit ball B(X*) of a dual space X* is w*-sequentially compact and the dual norm on X * is sequentially w*- Kadec –Klee. Then by corollary (1)the dual norm on X* is w*- Kadec – Klee. □

→ *Theorem 3.3*Suppose the unit ball B(X*) of a dual space X* is w*- sequentially compact, if the dual norm on X * issequentially w*- Kadec –Klee, then it isw*- Kadec –Klee.

Fact: If X is an Asplund space, then the unit ball of X* is w*-sequentially compact. This fact is proved in [17, prop 11].

→ *Theorem 3.4* [xv]If X* has Radon-Nikodym Property then X is an Asplund space.

→ *Theorem 3.5* [xvii]. Let X be a Banach space. The following assertions are equivalent.
(i) X is an Asplund space.
(ii) Each equivalent norm on X is Frechet differentiable at least at one point.
(iii) For each separable subspace Y ofX its dual Y* is separable.
(iv) X * has the Radon-Nikodym property.
(v) Each bounded nonempty subset of X * admits weak* slices of arbitrarily small norm-diameter.

→ *Theorem 3.6* [24,p. 223].Suppose *X* has an equivalent norm whose dual norm is weak*-Kadec. Show that *X* is an Asplund space.

- Proof:

We use the fact" ABanach space *X*is an Asplund space if and only each of its separable subspaces has a separable dual " [24, Corollary 6.6, 10].Consider the separable case first, and show *X* * separable.
Indeed, if *X* is separable, then *B(X *)* is weak*-separable, so we fix a countable set $\{ f_n\} \subset B(X^*)$ that is weak*-dense in *B(X *)*. Now fix $f \in S(X^*)$. Then find a subnet$f_{n_\alpha} \to f$.By the weak*-Kadec property, $\|f_{n_\alpha} - f \| \to 0$. Conclude *S(X*)*is separable, and hence *X* * is as well. In general, if *X* is not separable, consider a separable subspace *Y* of*X,* and show that the restricted norm on *Y* has weak*-Kadec dual norm (see e.g. [28, Proposition 1.4] for the straightforward details).Then *Y* * is separable, and so *X* is an Asplund space.
We remark that Raja [xxx] has shown the stronger result that if *X* * has an equivalent (dual) weak*-Kadec norm, then *X* * has an equivalent dual locally uniformly convex norm.□
Namioka and Phelps [iv] showed that if X* has the K* property then X is an Asplundspace. M. Raja strengthened this result by showing that X* has LUR norm (which implies automatically the K* property and also that X is an Asplundspace) iffit hasa wK* renorming [xxv, xxxiv].

→ *Theorem 3.7* [xxiv]Suppose a norm on *X* * has the weak*-Kadecproperty. Show that it is automatically a dual norm.

- Proof:

It suffices to show that the unit ball with respect to this norm is weak*-closed. Since the norm on X* with w* -KKP is a dual norm iffits unit dual ball B(X*) is weak * - closed [24,p.226]. Suppose $\| x_\alpha \| \leq 1$ and $x_\alpha \to x$ where $\| x \| = K$. It suffices to show that $K \leq 1$. So suppose it is not. Consider the line through $x_\alpha$and x. Since this line passes through the interior of the ball of radius K, it must intersect its sphere in two places. One of those points is x, call the otherx'_α(draw the picture). Now $x'_\alpha - x = \lambda_\alpha(x_\alpha - x)$ where$1 < \lambda_\alpha \leq 2K/(K-1)$. Then $(x'_\alpha - x) \to 0$ and so by the weak*-Kadec property $\| x'_\alpha - x \| \to 0$, butthen $\| x_\alpha - x \| \to 0$ and $\| x \| < K$, a contradiction.□

→ *Theorem 3.8.*[33, p. 23]. If the dual space X* of the Banach space X is separable, then X is an Asplund space.

- Proof:

If f is continuous and convex on the open convex set D ⊂ X, then ∂f is monotone, so by theorem "Suppose that the Banach space X has separable dual and that T: X → X* is monotone, then there exists an angle-small set A ⊂ D(T) such that T is single-valued and norm-to-norm upper semicontinuous at each point of D(T) \ A". it is single-valued and norm-to-norm upper semicontinuous at the points of some dense $G_\delta$subset G of D. thus, any selection for ∂ f is continuous at the points of G, so by Proposition " If f is convex and continuous on the nonempty open convex subset D of X, then it is Gateaux [Frechet] differentiable at a point x∈D if and only if there is a selection ψfor the sub differential map∂f which is norm-to-weak* [norm-to-norm] continuous at x". So f is Frechet differentiable at the points of G. Thus by definition X is Asplund space. □

→ *Theorem 3.9*[**3,**10, p.25, 24,p.319, 26,28]. A Banach space X that carries an equivalent weakly uniform rotund (WUR) norm is an Asplund space.

- Proof:

We assume that X is separable and prove that X* is then separable. For n ∈ℕ, put
$V_n=\{f \in B(X^*): | f (x- y)| \leq 1/3 \text{ if } x, y \in B(X): \|x + y\| \geq 2 - 1 / n\}$.
As X is weakly uniformly rotund, we note that $B (X^*)=\cup_{n \in \mathbb{N}} V_n$. Since (B(X*),w*) is a metric compact space, for everyn∈ N there is acountable and weak* dense set$S_n$in V_n. We claim that
$\overline{\text{span}}^{\|.\|}(\cup_{n\in\mathbb{N}} S_n)= X^*$. Assume that this is not the case and find
F∈ S(X**) withF (f) = 0 for all f $\in\cup_{n\in\mathbb{N}}S_n$, and choose
$f_0 \in S(X^*)$: $F(f_0)>8/9$.
Let $n_0 \in N$ such that $f_0 \in V_{n_0}$. Let $\{x_\alpha\}$ be a net in B(X)which is weak* convergentto F, the w* -closure of B(X)in X** is B (X**) (Goldstine). We know that if $\{ f_n \}$ be a sequence in X*then $f_n \to f$ifand only if $\{f_n\}$ is bounded andthe set$\{x \in X: f_n(x) \to f(x) \}$ isdense in X [32, p.150]. We find $\alpha_0$such that
$\|x_\alpha + x_\beta \| > 2 - 1/n$for all$\alpha, \beta \geq \alpha_0$.
By definition ofV_{n_0}; it follows that there is α_0 such that $|f (x_\alpha) - f (x_\beta)| \leq 1/3$ for all $\alpha, \beta \geq \alpha_0$ and $f \in V_{n_0}$.
Since $\{x_\beta\}$ weak* converges to F it follows that $|f (x_{\alpha_0}) - F(f)| \leq 1/3$for all$f \in V_{n_0}$. Hence for $f \in S_{n_0}$
$|(f - f_0) (x_{\alpha_0}) | = | F(f) - F(f_0)+ F(f_0) - f_0(x_{\alpha_0}) + f (x_{\alpha_0}) - F(f) |$
$\geq |F(f) - F(f_0)| - |F(f_0) -f_0(x_{a_0})| - |f (x_{\alpha_0}) - F(f)|$
$>8/9- 1/3 - 1/3$
$= 2/9$.
Thus f_0 does not belong to the weak* closure of S_{n_0}, i.e. Contradicts the fact that $f_0 \in \overline{S_{n_0}}^{w^*}$. This contradiction concludes the proof.

- Remarks

1.[xxvii] *(a) If* X *has the Schur property, then every dual norm on* X* *is* w*-*Kadec.*
(b) There are spaces X *such that* X* *has a dual* w*-*Kadec norm, but* B(X*) *is not* w*-*sequentially compact.*
3. By Ekeland's variational principle to show that a Banach space with a Frechet differentiable bump function is an Asplund space [xxiv].
4. If every closed separable subspace ofX is Asplund, thenX is Asplund [xxix].
5. *Let X be a Banach space. Then, ifX* is WCG, the space X is Asplund* [xxix].

Final remarks. Regarding the above mentioned problem of Godefroy, the example [xxxi] of countably tight compact K without any convergent sequence provides a possible direction for searching a counter example C(K) space. Finally, let us mention that the following is not clear to us: if the dual norm on X* is sequentially w*-Kadec, then must it be w*-Kadec ? This, of course is true if the dual ball is w*-sequentially compact (Theorem 3.2). So this question is equivalent to: if the dual norm is sequentially w*-Kadec, is the dual ball w*-sequentially compact ? [xxvii].
However, the general case still remains open.

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