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Analysis of Discrete SIR Models and Local Stability

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Abstract:
In this paper, discrete Susceptible-Infectious-Removed (SIR) models are considered. The dynamical behaviors of the models are investigated by obtaining equilibrium points are obtained and stability of the equilibrium points and the Jacobian. The Time Plot and Phase Portraits are obtained for different sets of parameter values from the model. Also numerical simulations are performed and they exhibit rich dynamics of the discrete model.

1. Introduction

Epidemic models described by ordinary differential equations have played important role in analyzing the spread and control of infectious diseases. In 1760, Bernoulli used mathematical models for smallpox. The deterministic S-I-R compartmental model was introduced by W.O. Kermack and A.G. McKendrick and has been a tool in mathematical epidemiology [6]. The models are compartmental because the population is divided into groups representing discrete disease states. In a Susceptible-Infectious-Recovered (SIR) model, infected individuals recover from disease into a life-long recovered state; they are never again susceptible to disease. A detailed history of mathematical epidemiology and basics of SIR epidemic models may be found in several monographs and classical books [1, 4, 7, 8]. The Antonine Plague, 165180 AD, was an ancient pandemic brought back to the Roman Empire by troops returning from campaigns in the Near-East. The epidemic invaded the Roman Empire, claimed the lives of two Roman emperors. Influenza or flu is a viral infection that is transmitted through the air and causes respiratory problems in humans and other animals. It occurs seasonally and results in an average of 30,000 deaths in the U. S. annually. The SARS epidemic of 2002 received attention of many authors and epidemic models were used to predict the spread of the disease. H1N1 flu, often called "swine flu," caused a worldwide pandemic in 2009. The outbreak began in Mexico. The virus spread worldwide. Ebola virus disease is a severe, often fatal illness, with a case fatality rate of up to 90%. The first Ebola outbreak took place in 1976 in Congo. Since August 2014, it is spreading mainly in Guinea, Sierra Leone, and Liberia. Asper WHO, more than 10,477 people have died and more than 25,263 have been infected with Ebola virus as on 31st March 2015, [2, 11]. The ongoing Zika virus (ZIKV) epidemic poses a major global public health emergency. It is known that ZIKV is spread by Aedes mosquitoes, recent studies show that ZIKV can also be transmitted via sexual contact, [3, 5].

2. Reproduction Number

In epidemiology, the reproduction number (R_0) is very important, because the stability of the proposed model is associated with reproduction number. R_0 determines whether there is an epidemic or not. If $R_0 < 1$, the infection dies out, while if $R_0 > 1$, there is an epidemic. The basic reproduction number R_0 is defined as the average number of secondary cases arising from an average primary case in an entirely susceptible population. The value $R_0 = 1$ of the basic reproduction number does not, in general, identify the threshold. The basic reproduction number R_0 is often available; it can be used to estimate the model parameters, [1,7]. Reproduction number for Ebola transmission is 5.15 [9]. For Zika virus, the estimated basic reproduction number is 2.055.

3. Equilibrium Points and Stability

Difference equations can be used in modeling discrete dynamical systems. A dynamical system at equilibrium does not change over time. x^* is an equilibrium point of the system $x(n+1) = f(x(n))$, if $x^* = f\left(x^*\right)$, x^* is also called fixed point or steady state. The linear difference equation $x(n+1) = ax(n) + b$ possess exactly one fixed point $x^* = \dfrac{b}{1-a}$. In 2 – D case, a discrete system can be expressed as $x(n+1) = f(x(n), y(n)); y(n+1) = g(x(n), y(n))$ and the stationary states satisfy $x^* = f(x^*, y^*); y^* = g(x^*, y^*)$. If λ_1 and λ_2 are the Eigen values of the Jacobian [8]

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

evaluated at the equilibrium points (x^*, y^*), then we have

- $|\lambda_{1,2}| < 1 \Rightarrow (x^*, y^*)$ is locally stable.
- $|\lambda_j| < 1$ for j = {1,2} $\Rightarrow (x^*, y^*)$ is unstable.

Equivalently, $|\lambda_{1,2}| < 1$ and the steady state (x^*, y^*) is stable, if $2 > 1 + \det A > |\text{Trace } A|$, [10].

4. Discrete Sir Model and Stability Analysis

We shall investigate the Susceptible-Infectious-Removed (SIR) model as given below:

$$\begin{aligned} S(n+1) &= b - \beta S(n)I(n) + (1-\mu)S(n) \\ I(n+1) &= \beta S(n)I(n) + [1-\alpha]I(n) \\ R(n+1) &= \alpha I(n) + \beta R(n) \end{aligned} \quad (1)$$

where $b, \beta, \mu, \gamma, \alpha > 0$ The system (1) has two equilibria $E_0 = \begin{pmatrix} \frac{b}{\mu} & 0 & 0 \end{pmatrix}$ and $E_1 = \begin{bmatrix} \frac{\alpha}{\beta} & \frac{b}{\alpha} - \frac{\mu}{\beta} & \frac{b\beta - \alpha\mu}{\beta(1-\beta)} \end{bmatrix}$ with $R_0 = \frac{\beta b}{\alpha\mu}$. The local stability analysis of the model can be carried out by computing the Jacobian matrix corresponding to each equilibrium point. The Jacobian Matrix J for the system (1) is

$$J(S,I,R) = \begin{bmatrix} 1-\mu-\beta I & -\beta S & 0 \\ \beta I & 1-\alpha+\beta S & 0 \\ 0 & \alpha & \beta \end{bmatrix} \quad (2)$$

From (2), Jacobian matrix for the Equilibrium point E_0 is given by

$$J(E_0) = \begin{bmatrix} 1-\mu & \frac{-b\beta}{\mu} & 0 \\ 0 & 1-\alpha+\frac{b\beta}{\mu} & 0 \\ 0 & \alpha & \beta \end{bmatrix}$$

$\text{Trace } J(E_0) = 2 + \beta - \alpha - \mu + \frac{b\beta}{\mu}$ and $\text{Det } J(E_0) = \beta\left[(1-\mu)(1-\alpha) - \beta b\left(1 - \frac{1}{\mu}\right)\right]$.The Eigen values of the matrix $J(E_0)$ are $\lambda_1 = 1-\mu$, $\lambda_2 = \beta$ and $\lambda_3 = 1-\alpha+\frac{b\beta}{\mu}$. The equilibrium point E_0 is Stable, when $R_0 < 1$. From (2), Jacobian matrix for E_1 is given by

$$J(E_1) = \begin{bmatrix} 1-\frac{b\beta}{\alpha} & -\alpha & 0 \\ \frac{b\beta}{\alpha}-\mu & 1 & 0 \\ 0 & \alpha & \beta \end{bmatrix}$$

Trace $J(E_1) = 2+\beta-\frac{b\beta}{\alpha}$ and Det $J(E_1) = b\beta^2\left[1-\frac{1}{\alpha}\right]-\beta[\alpha\mu-1]$. The Eigen values of the matrix J(E_1) are $\lambda_1 = \beta$ and $\lambda_{2,3} = 1-\frac{b\beta}{2\alpha}\pm\frac{1}{2\alpha}\sqrt{b^2\beta^2 - 4\alpha^2(b\beta-\mu\alpha)}$. The equilibrium point E_1 is Stable, when $R_0 > 1$.

4.1. Numerical Simulation of the Model

In this section, we present some numerical simulations to verify our theoretical results. Numerical study of non linear discrete dynamical systems gives an insight in to dynamical characteristics. We also present the time plots of $S(n); I(n); R(n)$ for the system (1) with corresponding phase portraits. Dynamic behavior of the system (1) about the equilibrium points under different sets of parameter values is presented.

Example 1. *Choose the parameter* $b = 0.8; \beta = 0.87; \mu = 0.859; \alpha = 0.5$ *with initial Conditions* $(S, I, R) = (0.6, 0.4, 0.2)$. *Here* $R_0 = \frac{\beta b}{\alpha\mu} = 0.7488 < 1$, *so the equilibrium point* E_0 *is globally stable, see fig - 1(A).*

Example 2. *Choose the parameter* $b = 0.099; \beta = 0.8; \mu = 0.03; \alpha = 0.6$, *with initial Conditions* $(S, I, R) = (0.6, 0.4, 0.2)$. *Here* $R_0 = \frac{\beta b}{\alpha\mu} = 4.4 > 1$, *so the equilibrium point* E_1 *is globally stable, see fig - 1(B).*

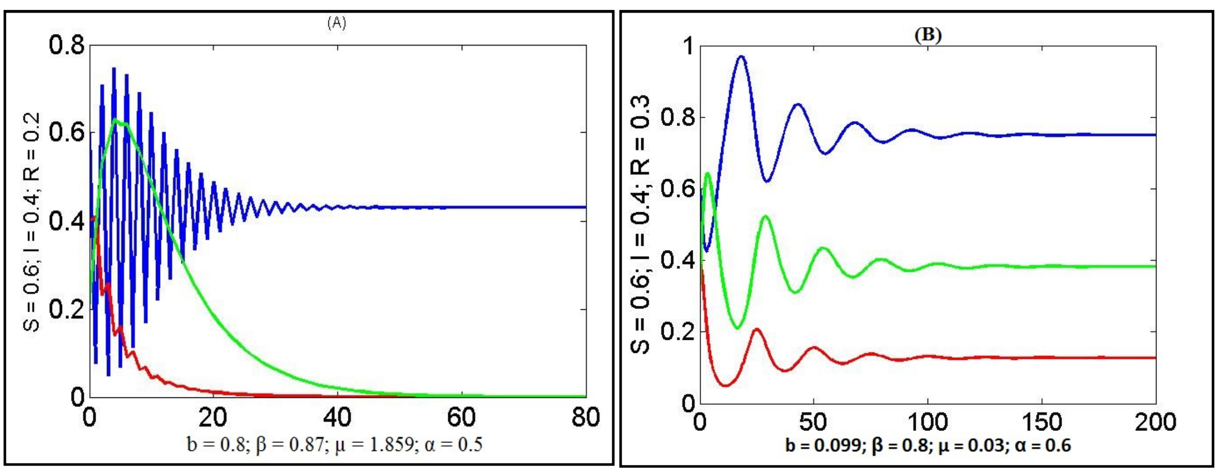

Figure 1: Time plot for the Axial and Interior fixed point of the system (1) with $R_0 < 1$ *and* $R_0 > 1$.

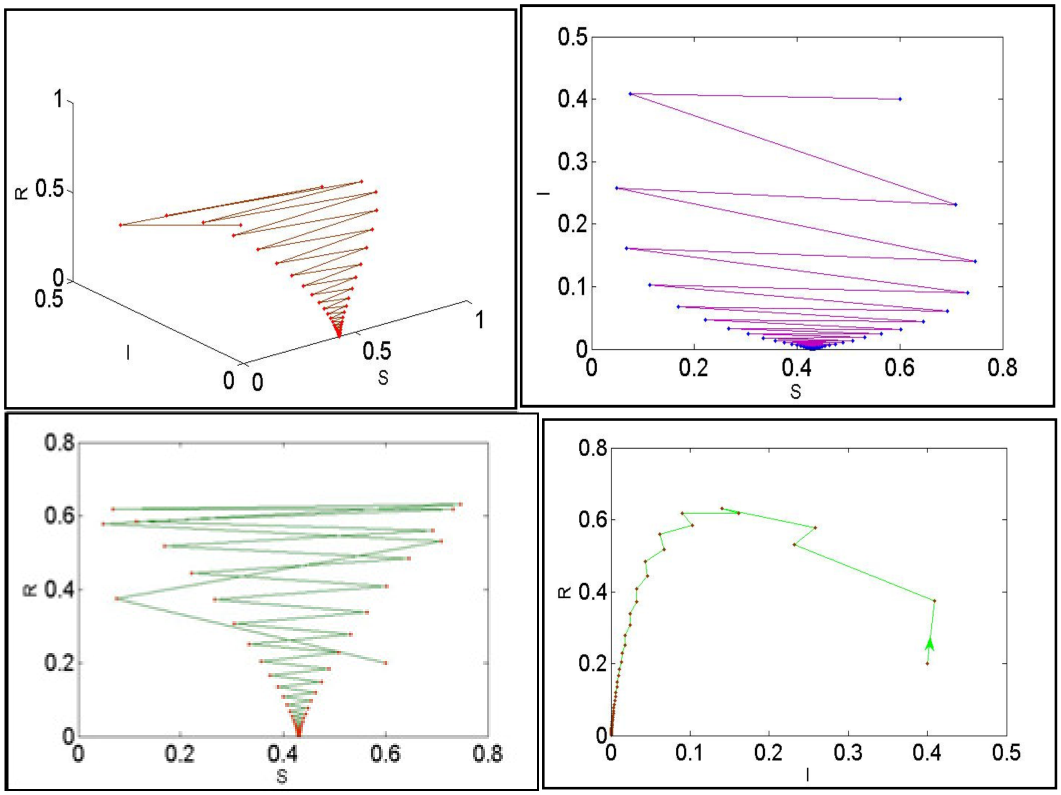

Figure 2: Phase Portrait of the Axial fixed point for the system (1)

Example 3. In this example, $b=0.099; \beta=0.8; \mu=0.099; \alpha=0.8,$ for $R_0=1,$ $b=0.099;$ $\beta=0.8;$ $\mu=0.099;$ $\alpha=0.4,$ for $R_0=2,$ $b=0.099; \beta=0.8; \mu=0.099; \alpha=0.2666,$ for $R_0=3,$ $b=0.099;$ $\beta=0.8; \mu=0.099; \alpha=0.2,$ for $R_0=4,$ $b=0.099; \beta=0.8; \mu=0.099; \alpha=0.16,$ for $R_0=5.$ Figure – 4; describes the dynamical behavior of the model for different values $1 \le R_0 \le 5.$

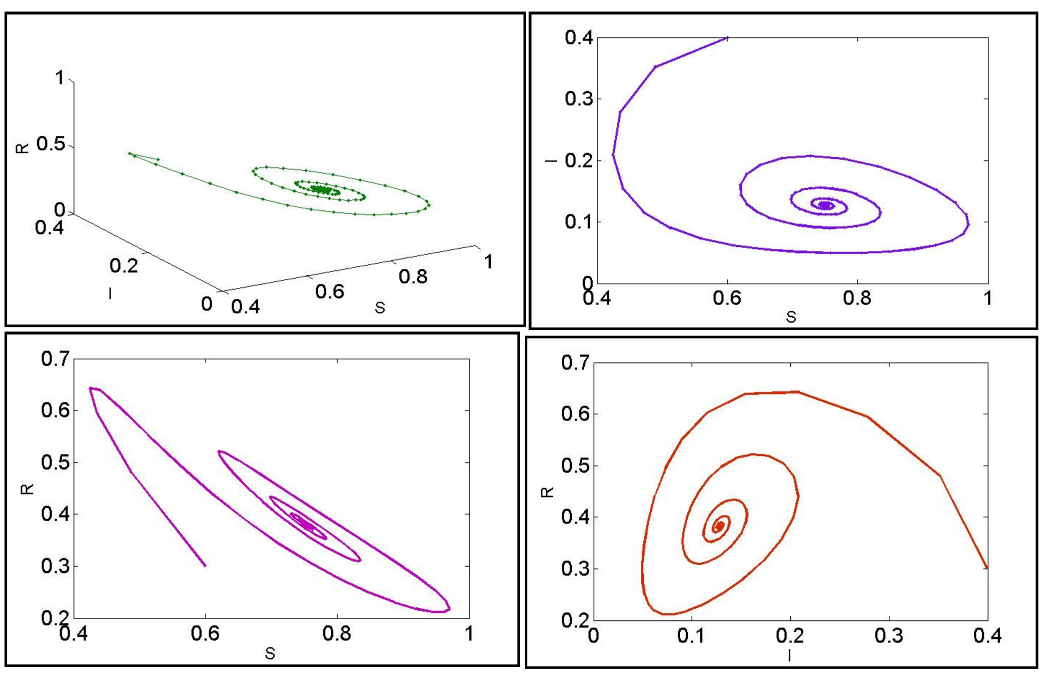

Figure 3: Phase Portrait of the Interior fixed point for the system (1)

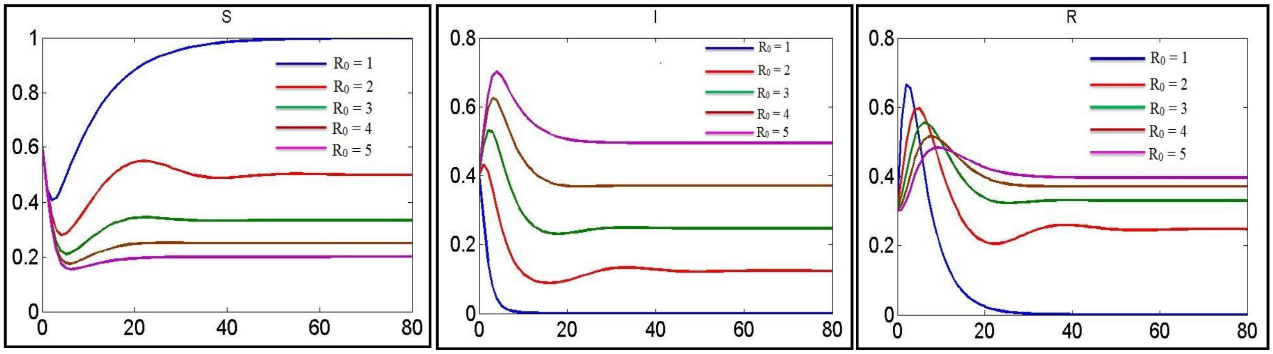

Figure 4: Variation of R_0 for the system (1)

5. Another Discrete Sir Model

In this section, we consider the following discrete SIR model for discussion of qualitative behavior.

$$\begin{aligned} S(n+1) &= b+(1-d)S(n)-\beta S(n)I(n) \\ I(n+1) &= (1-d)I(n)+\beta S(n)I(n)-gI(n) \\ R(n+1) &= (1-d)R(n)+gI(n) \end{aligned} \tag{3}$$

where $\beta, b, g, d > 0.$ Here we have, b is the recruitment rate, d is the death rate and g is the recovery rate. The system (3) has the two equilibria $E_0 = \left(\frac{b}{d}, 0, 0\right)$ and $E_1 = \left(\frac{d+g}{\beta}, \quad \frac{b}{d+g} - \frac{d}{\beta}, \quad \frac{gb}{d(d+g)} - \frac{g}{\beta}\right)$ provided $R_0 = \frac{b\beta}{d(d+g)}$.

5.1. Stability: The Jacobian Matrix J for the system (3) is

$$J(S,I,R) = \begin{bmatrix} 1-d-\beta I & -\beta S & 0 \\ \beta I & 1-d+\beta S-g & 0 \\ 0 & g & 1-d \end{bmatrix} \tag{4}$$

Trace $J(S,I,R) = 3(1-d)+\beta(S-I)-g$ and

Det $J(S,I,R) = (1-d)[(1-d)(1-d-g)+\beta[(1-d)(S-I)+gI]]$. From (4), using the equilibrium point E_0, Jacobian matrix at E_0 is given by

$$J(E_0) = \begin{bmatrix} 1-d & \frac{-\beta b}{d} & 0 \\ 0 & 1-d+\frac{\beta b}{d}-g & 0 \\ 0 & g & 1-d \end{bmatrix}$$

Trace $J(S,I,R) = 3-3d-d+\frac{\beta b}{d}$ and Det $J(S,I,R) = (1-d^2)\left[1-d-g+\frac{\beta b}{d}\right]$. The Eigen values of the matrix $J(E_0)$ are $\lambda_{1,2} = 1-d$ and $\lambda_3 = 1-d-g+\frac{\beta b}{d}$. The equilibrium point E_0 is stable, when $R_0 < 1$. From(4), at the equilibrium point E_1, Jacobian matrix for E_1 is

$$J(E_1)=\begin{bmatrix}1-\dfrac{\beta b}{d+g} & -(d+g) & 0\\ \dfrac{\beta b}{d+g}-d & 1 & 0\\ 0 & g & 1-d\end{bmatrix}$$

We obtain Trace $J(S,I,R)=3-d-\dfrac{\beta b}{d+g}$ and

Det $J(S,I,R)=(1-d)[1-d(g+d)+ab]-\dfrac{\beta b(1-d)}{d+g}$. The Eigen values of the matrix $J(E_1)$ are $\lambda_1=1-d$ and $\lambda_{2,3}=1-\dfrac{\beta b}{2(d+g)}\pm\dfrac{1}{2(d+g)}\sqrt{\beta^2b^2+4(d+g)[d(d+g)-\beta b]}$. The equilibrium point E_1 is stable when $R_0>1$.

5.2. Numerical Examples: Numerical study of non linear discrete dynamical systems gives an insight in to dynamical characteristics. In this section, we present the time plots of $S(n);I(n);R(n)$ for the system (3). Dynamic behavior of the system (3) for disease free equilibrium and endemic equilibrium states are presented.

Example 4. *For the parameter values* $b=0.12;\beta=0.99;g=0.53;d=0.19$, *and the initial condition* $(S,I,R)=(0.7,0.5,0.2)$. *Here* $R_0=\dfrac{\beta b}{d(d+g)}=0.868<1$ *so the equilibrium point* E_0 *is stable, see fig - 5.*

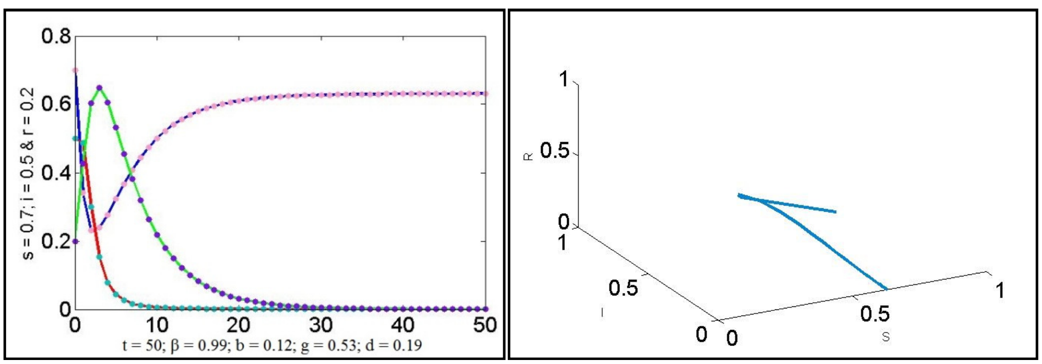

Figure 5: Time Series and Phase Portrait for the Axial fixed point of the system (3) with $R_0<1$

Example 5. *Considering* $b=0.1;\beta=1.299;g=0.5;d=0.1$, *and the initial condition* $(S,I,R)=(0.7,0.5,0.2)$. *Here* $R_0=\dfrac{\beta b}{d(d+g)}=2.1650>1$, *Hence the equilibrium point* E_1 *is stable, see fig – 6.*

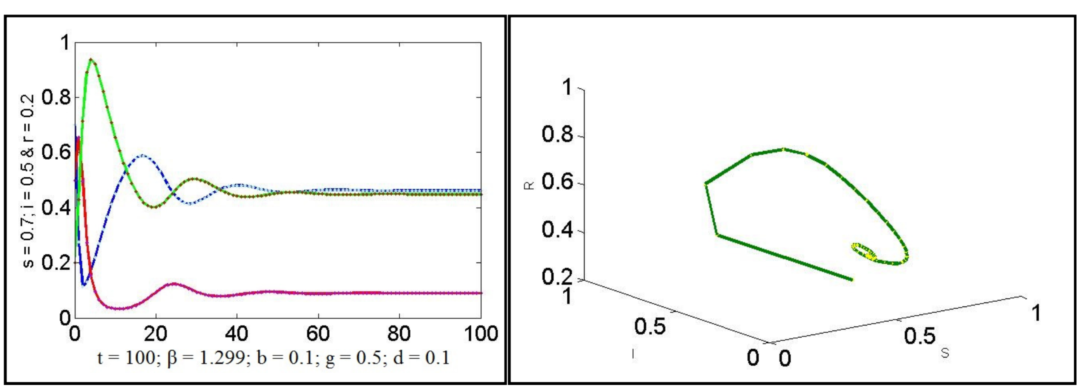

Figure 6: Time Series and Phase Portrait for the Interior fixed point of the system (3) with $R_0>1$.

Here $b=0.1;\beta=1.299;g=0.5;d=0.1886,$ *for* $R_0=1;\ b=0.1;\beta=1.299;g=0.5;d=0.107,$ *for* $R_0=2;$ $b=0.1;\beta=1.299;g=0.5;d=0.07527,$ *for* $R_0=3;\ b=0.1;\beta=1.299;g=0.5;d=0.05818,$ *for* $R_0=4;$ $b=0.1;\beta=1.299;g=0.5;d=0.047456,$ *for* $R_0=5.$ Figure - 7; describes the dynamical behavior of the model for different values $1\le R_0\le 5.$

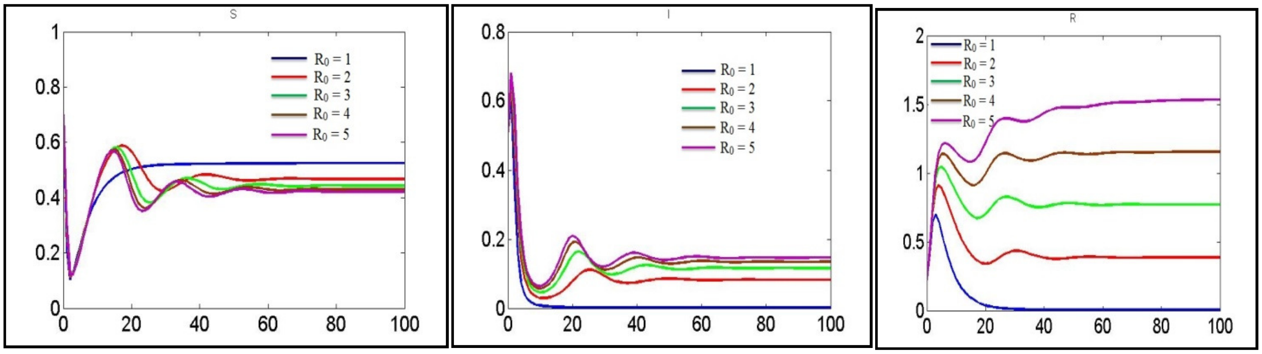

Figure 7: Variation of R_0 for the system (3)

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