THE INTERNATIONAL JOURNAL OF
SCIENCE & TECHNOLEDGE

Generalized Dual Fibonacci Sequence

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Abstract:
In this paper, we investigate the generalized dual Fibonacci sequence and the dual Fibonacci numbers. Furthermore, wegive recurrence relations, vectors, the golden ratio and Binet's formula for the generalized dual Fibonacci sequence.

Keywords: *Dual number, Fibonacci number, dual Fibonacci number, generalized Fibonacci sequence, generalized dual Fibonacci sequence.*

1. Introductıon

For the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots, F_n, \ldots$$

defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \ (n \geq 3),$$

with $F_1 = F_2 = 1$, it is well known (Daniel Bernoulli, 1732) that the n-th term of the Fibonacci sequence (F_n). Some recent generalizations have produced a variety of new and extended results (Berzsenyi, 1975; Horadam, 1965; Iyer, 1969; Walton 1974).

The generalized Fibonacci sequence defined by

$$H_n = H_{n-1} + H_{n-2}, \ (n \geq 3), \tag{1.1}$$

with $H_1 = p, H_2 = p + q$ where p, q are arbitrary integers (Horadam, 1961). That is, the generalized Fibonacci sequence is

$$p, \ p+q, \ 2p+q, \ 3p+2q, \ 5p+3q, 8p+5q, \ldots, (p-q)F_n + qF_{n+1}, \ldots. \tag{1.2}$$

Using the equations (1.1) and (1.2), we get

$$H_{n+1} = q\,F_n + p\,F_{n+1},$$
$$H_{n+2} = p\,F_n + (p+q)\,F_{n+1} \tag{1.3}$$

For the generalized Fibonacci sequence, it was obtained the following properties:

$$H_{n-1}^2 + H_n^2 = (2p-q)H_{2n-1} - e\,F_{2n-1}, \tag{1.4}$$

$$H_{n+1}^2 - H_{n-1}^2 = (2p-q)H_{2n} - e\,F_{2n}, \tag{1.5}$$

$$H_{n-1}\,H_{n+1} - H_n^2 = (-1)^n e, \tag{1.6}$$

$$H_{n+r} = H_{n-1}F_r + H_n F_{r+1} \quad (n \geq 3), \tag{1.7}$$

$$H_{n+1-r}\,H_{n+1+r} - H_{n+1}^2 = (-1)^{n-r} e\ F_r^2, \tag{1.8}$$

$$H_{n+1}^2 + e\ F_n^2 = p\,H_{2n+1}, \tag{1.9}$$

$$H_n\,H_{n+1+r} - H_{n-s}\,H_{n+r+s+1} = (-1)^{n+s} e\ F_s F_{r+s+1}, \tag{1.10}$$

$$[2H_{n+1}H_{n+2}]^2 + [H_n H_{n+3}]^2 = [2H_{n+1}H_{n+2} + H_n^2]^2, \tag{1.11}$$

$$\frac{H_{n+r} + (-1)^r H_{n-r}}{H_n} = F_{r+1} + (-1)^r F_{r-1}. \tag{1.12}$$

where $e = p^2 - pq - q^2$.

Also, for $p = 1, q = 0$, it was obtained the following well-known results:

$$F_{n-1}^2 + F_n^2 = F_{2n-1}, \quad \text{(Catalan)}, \tag{1.13}$$

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n, \quad \text{(Simpson or Cassini)}, \tag{1.14}$$

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \quad \text{(Lucas)}. \tag{1.15}$$

Furthermore, the n-th term C_n of the Complex Fibonacci sequence (C_n) defined by

$$C_n = F_n + i F_{n+1}, \quad i^2 = -1$$

and the generalized complex Fibonacci sequence defined by

$$\boldsymbol{C}_n = H_n + i H_{n+1},$$

(Horadam, 1961, 1963).

Also, for moduli of the complex Fibonacci number C_n, writing $|C_n|^2 = c_n$, we find

$$c_n = F_n^2 + F_{n+1}^2 = F_{2n+1}, \tag{1.16}$$

$$c_n - c_{n-1} = F_{n+1}^2 - F_{n-1}^2 = F_{2n} \tag{1.17}$$

i.e. terms $F_1, F_2, F_3, F_4, \ldots, F_{2n}, F_{2n+1}, \ldots$ of the classical Fibonacci sequence are expressible as $c_0, c_1 - c_0, c_1, c_2 - c_1, \ldots, c_n - c_{n-1}, c_n, \ldots$ respectively.

2. Generalızed Dual Fıbonaccı Sequence

In this section, we will define the generalized dual Fibonacci sequence denoted by D_n.

The n-th term of a generalized dual Fibonacci number defined by

$$\mathrm{D}_n = H_n + \varepsilon H_{n+1}. \tag{2.1}$$

where $\varepsilon^2 = 0,\ \varepsilon \neq 0$ and it's called dual[1] unit (Ercan & Yüce, 2011).

Using the equation (1.3) and (2.1), we write

$$\mathrm{D}_n = (p - q + \varepsilon q) F_n + (q + \varepsilon p) F_{n+1} \tag{2.2}$$

Thus, elements of the generalized dual Fibonacci sequence is

$$(\mathrm{D_n}): \quad p + \varepsilon(p+q),\ (p+q) + \varepsilon(2p+q), \ldots, (p - q + \varepsilon q) F_n + (q + \varepsilon p) F_{n+1}, \ldots \tag{2.3}$$

From the equations (2.1) and (2.2), we get the following properties for the generalized dual Fibonacci sequence:

$$\mathrm{D}_{n-1}^2 + \mathrm{D}_n^2 = (2p - q) + \varepsilon(p + 2q)\mathrm{D}_{2n-1} - e(1+\varepsilon) F_{2n-1}, \tag{2.4}$$

$$\mathrm{D}_{n+1}^2 + \mathrm{D}_n^2 = (2p - q) + \varepsilon(p + 2q)\mathrm{D}_{2n+1} - e(1+\varepsilon) F_{2n+1}, \tag{2.5}$$

$$\mathrm{D}_{n+1}^2 - \mathrm{D}_{n-1}^2 = (2p - q) + \varepsilon(p + 2q)\mathrm{D}_{2n} - e(1+\varepsilon) F_{2n}, \tag{2.6}$$

$$\mathrm{D}_{n-1}\mathrm{D}_{n+1} - \mathrm{D}_n^2 = (-1)^n e(1+\varepsilon), \tag{2.7}$$

$$\mathrm{D}_n \mathrm{D}_{n+r+1} - \mathrm{D}_{n-s}\mathrm{D}_{n+r+s+1} = (-1)^{n+s} e(1+\varepsilon) F_s F_{r+s+1}, \tag{2.8}$$

$$\mathrm{D}_{n+1-r}\mathrm{D}_{n+1+r} - \mathrm{D}_{n+1}^2 = (-1)^{n-r} e(1+\varepsilon) F_r^2, \tag{2.9}$$

$$\mathrm{D}_{n+1}^2 + e(1+\varepsilon)\, F_n^2 = \left[p+\varepsilon(p+q)\right]\mathrm{D}_{2n+1}, \tag{2.10}$$

$$\mathrm{D}_n\mathrm{D}_m + \mathrm{D}_{n+1}\mathrm{D}_{m+1} = (2p-q)+\varepsilon(p+2q)\mathrm{D}_{n+m+1} - e(1+\varepsilon)\, F_{n+m+1}, \tag{2.11}$$

$$\mathrm{D}_m\mathrm{D}_{n+1} - \mathrm{D}_{m+1}\mathrm{D}_n = e\,(-1)^n\mathrm{D}_{m-n} + e\,\varepsilon(-1)^{n+1}\, F_{n-m-1}, \tag{2.12}$$

$$\left[2\mathrm{D}_{n+1}\mathrm{D}_{n+2}\right]^2 + \left[\mathrm{D}_n\mathrm{D}_{n+3}\right]^2 = \left[2\mathrm{D}_{n+1}\mathrm{D}_{n+2} + \mathrm{D}_n^2\right]^2, \tag{2.13}$$

$$\frac{\mathrm{D}_{n+r} + (-1)^r\mathrm{D}_{n-r}}{\mathrm{D}_n} = L_r = F_{r+1} + (-1)^r F_{r-1}, \tag{2.14}$$

$$\mathrm{D}_{-n} = (-1)^{n+1}\mathrm{D}_n + (q+\varepsilon\, p)L_{-n},\ \ L_n \textit{ Lucas number} \tag{2.15}$$

where $e = p^2 - pq - q^2$.

Special Case- 1: From the generalized dual Fibonacci sequence (D_n) for $p=1,\ q=0$ in the equation (2.2), we obtain dual Fibonacci sequence (D_n) as follows:

$$(D_n):\quad 1+\varepsilon,\ 1+2\varepsilon,\ 2+3\varepsilon,\ 3+5\varepsilon, \ldots, F_n + \varepsilon F_{n+1}, \ldots$$

where given by Güven and Nurkan (2014).

Theorem 1. Let H_n and D_n are the generalized Fibonacci number and generalized dual Fibonacci number, respectively, then

$$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \frac{p\alpha + q}{q\alpha + (p-q)}$$

and

$$\lim_{n\to\infty} \frac{\mathrm{D}_{n+1}}{\mathrm{D}_n} = \frac{(p^2 - pq + q^2)\alpha + pq(1+\alpha) + q^2}{q^2\alpha^2 + 2q(p-q)\alpha + (p-q)^2}$$

where $\alpha = (1+\sqrt{5})/2 = 1.618033..$ is the golden ratio.

Proof . We have for the Fibonacci number F_n,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$$

where $\alpha = (1+\sqrt{5})/2 = 1.618033..$ is the golden ratio.

Then for the generalized Fibonacci number H_n, we obtain

$$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \lim_{n\to\infty} \frac{p\,F_{n+1} + qF_n}{qF_{n+1} + (p-q)F_n} = \frac{p\alpha + q}{q\alpha + (p-q)} \tag{2.16}$$

and

$$\lim_{n\to\infty} \frac{\mathrm{D}_{n+1}}{\mathrm{D}_n} = \lim_{n\to\infty} \frac{(p-q+\varepsilon q)F_{n+1} + (q+\varepsilon p)F_{n+2}}{(p-q+\varepsilon q)F_n + (q+\varepsilon p)F_{n+1}}$$

$$= \lim_{n\to\infty} \frac{[(p\,F_{n+1} + qF_n] + \varepsilon[(p+q)F_{n+1} + pF_n]}{[(p-q)F_n + qF_{n+1}] + \varepsilon[qF_n + pF_{n+1}]} \tag{2.17}$$

$$= \lim_{n\to\infty} \frac{[(p^2 - pq - q^2)F_nF_{n+1} + p\,q(F_{n+1}^2 + F_n^2) - q^2F_n^2]}{[(p-q)^2F_n^2 + 2q(p-q)F_nF_{n+1} + q^2F_{n+1}^2]}$$

$$+ \lim_{n\to\infty} \varepsilon\ \frac{[(-1)^{n+1}(p^2 - pq - q^2)]}{[(p-q)^2F_n^2 + 2q(p-q)F_nF_{n+1} + q^2F_{n+1}^2]}$$

$$= \frac{(p^2 - pq - q^2)\alpha + p\,q\,(1+\alpha^2) - q^2}{q^2\alpha^2 + 2q(p-q)\,\alpha + (p-q)^2}.$$

where $F_{n+2} = F_n + F_{n+1}$.

Special Case- 2: From $p = 1,\ q = 0$ in the equations (2.16) and (2.17), we obtain

$$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$$

which given by (Koshy, 2001) and

$$\lim_{n\to\infty} \frac{\mathrm{D}_{n+1}}{\mathrm{D}_n} = \lim_{n\to\infty} \frac{D_{n+1}}{D_n} = \alpha + 0 = \alpha .$$

Theorem 2. The Binet formula[2] for the generalized dual Fibonacci sequence is as follows;

$$\mathrm{D}_n = \frac{1}{\alpha - \beta}(\bar{\alpha}\ \alpha^n - \bar{\beta}\ \beta^n). \tag{2.18}$$

Proof . If we use definition of the generalized dual Fibonacci sequence and substitute first equation in footnote, then we get

$$\begin{aligned} \mathrm{D}_n &= (p - q + \varepsilon q) F_n + (q + \varepsilon p) F_{n+1} \\ &= (p - q + \varepsilon q)(\frac{\alpha^n - \beta^n}{\alpha - \beta}) + (q + \varepsilon p)(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}) \\ &= \frac{1}{\alpha - \beta} \alpha^n \left[(p - q + \varepsilon q) + \alpha (q + \varepsilon p) \right] - \beta^n \left[(p - q + \varepsilon q) + \beta (q + \varepsilon p) \right] \\ &= \frac{\bar{\alpha}\, \alpha^n - \bar{\beta}\, \beta^n}{\alpha - \beta}. \end{aligned} \tag{2.19}$$

where $\bar{\alpha} = (p - q + \varepsilon q) + \alpha (q + \varepsilon p)$ and $\bar{\beta} = (p - q + \varepsilon q) + \beta (q + \varepsilon p)$.

3. Generalızed Dual Fıbonaccı Vectors

A generalized dual Fibonacci vector is defined by

$$\overrightarrow{\mathrm{D_n}} = (\mathrm{D}_n, \mathrm{D}_{n+1}, \mathrm{D}_{n+2})$$

Also, from equations (2.1) and (1.1), (1.3) it can be expressed as

$$\begin{aligned} \overrightarrow{\mathrm{D_n}} &= \vec{H}_n + \varepsilon \vec{H}_{n+1} \\ &= (p - q + \varepsilon q) \vec{F}_n + (q + \varepsilon p) \vec{F}_{n+1} \end{aligned} \tag{3.1}$$

where $\overrightarrow{H_n} = (H_n, H_{n+1}, H_{n+2})$ and $\overrightarrow{F_n} = (F_n, F_{n+1}, F_{n+2})$ are the generalized Fibonacci vector and the Fibonacci vector, respectively.

The product of $\overrightarrow{\mathrm{D_n}}$ and $\lambda \in R$ is given by

$$\lambda \overrightarrow{\mathrm{D}_n} = \lambda\ \vec{H}_n + \varepsilon \lambda\ \vec{H}_{n+1}$$

and

$\overrightarrow{\mathrm{D_n}}$ and $\overrightarrow{\mathrm{D_m}}$ are equal if and only if

$$H_n = H_m$$
$$H_{n+1} = H_{m+1}$$
$$H_{n+2} = H_{m+2}.$$

Some examples of the generalized dual Fibonacci vectors can be given easily as;

$$\begin{aligned}\overrightarrow{D_1} &= (D_1, D_2, D_3)\\ &= (H_1, H_2, H_3) + \varepsilon(H_2, H_3, H_4)\\ &= \left[(p + \varepsilon(p+q),\ (p+q) + \varepsilon(2p+q),\ (2p+q) + \varepsilon(3p+2q)\right]\end{aligned}$$

$$\begin{aligned}\overrightarrow{D_2} &= (H_2, H_3, H_4) + \varepsilon(H_3, H_4, H_5)\\ &= \left[(p+q+\varepsilon(2p+q),\ (2p+q) + \varepsilon(3p+2q),\ (3p+2q) + \varepsilon(5p+3q)\right]\end{aligned}$$

Theorem 3. Let $\vec{D}_n$ and $\vec{D}_m$ be two generalized dual Fibonacci vectors. The dot product of these two generalized dual Fibonacci vectors is given by

$$\begin{aligned}\left\langle \vec{D}_n, \vec{D}_m \right\rangle &= (p-q)^2 \left[(F_{n+m+3} + F_n F_m) + \varepsilon(2F_{n+m+4} + F_{n+1}F_m + F_n F_{m+1})\right]\\ &\quad + q(p-q)\begin{bmatrix}(2F_{n+m+2} + F_{n+2}F_{m+3} + F_{n+3}F_{m+2})\\ +\varepsilon(4F_{n+m+3} + F_{n+4}F_{m+2} + F_{n+2}F_{m+4} + 2F_{n+3}F_{m+3})\end{bmatrix}\\ &\quad + q^2\left[(F_{n+m+5} + F_{n+1}F_{m+1}) + \varepsilon(2F_{n+m+6} + F_{n+2}F_{m+1} + F_{n+1}F_{m+2})\right].\end{aligned} \tag{3.2}$$

Proof. The dot product of $\overrightarrow{D_n} = (D_n, D_{n+1}, D_{n+2})$ and $\overrightarrow{D_m} = (D_m, D_{m+1}, D_{m+2})$ defined by

$$\begin{aligned}\left\langle \vec{D}_n, \vec{D}_m \right\rangle &= D_n D_m + D_{n+1} D_{m+1} + D_{n+2} D_{m+2}\\ &= \left\langle H_n, H_m \right\rangle + \varepsilon(\left\langle H_{n+1}, H_m \right\rangle + \left\langle H_n, H_{m+1} \right\rangle)\end{aligned}$$

where $\overrightarrow{H_n} = (H_n, H_{n+1}, H_{n+2})$ is the generalized Fibonacci vector. Also, the equations (1.1), (1.2) and (1.3), we obtain

$$\begin{aligned}\left\langle \vec{H}_n, \vec{H}_m \right\rangle &= (p-q)^2\left[(F_{n+m+3} + F_n F_m)\right]\\ &\quad + q(p-q)\left[2F_{n+m+2} + F_{n+2}F_{m+3} + F_{n+3}F_{m+2}\right]\\ &\quad + q^2\left[F_{n+m+5} + F_{n+1}F_{m+1}\right]\end{aligned} \tag{3.3}$$

$$\begin{aligned}\left\langle \vec{H}_n, \vec{H}_{m+1} \right\rangle &= (p-q)^2\left[(F_{n+m+4} + F_n F_{m+1})\right]\\ &\quad + q(p-q)\left[2F_{n+m+3} + F_{n+2}F_{m+4} + F_{n+3}F_{m+2}\right]\\ &\quad + q^2\left[F_{n+m+6} + F_{n+1}F_{m+2}\right]\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\left\langle \vec{H}_{n+1}, \vec{H}_m \right\rangle &= (p-q)^2\left[(F_{n+m+4} + F_{n+1}F_m)\right]\\ &\quad + q(p-q)\left[2F_{n+m+3} + F_{n+3}F_{m+3} + F_{n+4}F_{m+2}\right]\\ &\quad + q^2\left[F_{n+m+6} + F_{n+2}F_{m+1}\right]\end{aligned} \tag{3.5}$$

Then from equation (3.3), (3.4) and (3.5), we have the equation (3.2).

Special Case- 2: For the dot product of generalized dual Fibonacci vectors $\overrightarrow{D_n}$ and $\overrightarrow{D_{n+1}}$, we get

$$\begin{aligned}\left\langle \vec{D}_n , \vec{D}_{n+1} \right\rangle &= D_n D_{n+1} + D_{n+1} D_{n+2} + D_{n+2} D_{n+3} \\ &= \left\langle H_n , H_{n+1} \right\rangle + \varepsilon (\left\langle H_n , H_{n+2} \right\rangle + \left\langle H_n , H_{n+1} \right\rangle) \\ &= (p-q)^2 \{ (F_{2n+4} + F_n F_{n+1}) + \varepsilon [(2F_{2n+5} + F_{n+1}^2 + F_n F_{n+2})] \} \\ &+ q(p-q) \{ (2F_{2n+5} + F_{n+1}^2 + F_n F_{n+2}) + \varepsilon [(4F_{2n+6} + 3F_{n+1} F_n + F_n F_{n+3})] \} \\ &+ q^2 \{ (F_{2n+6} + F_{n+1} F_{n+2}) + \varepsilon [(2F_{2n+7} + F_{n+2}^2 + F_{n+1} F_{n+3})] \}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\left\langle \vec{D}_n , \vec{D}_n \right\rangle &= D_n^2 + D_{n+1}^2 + D_{n+2}^2 \\ &= \left\langle H_n , H_n \right\rangle + 2\varepsilon \left\langle H_n , H_{n+1} \right\rangle \\ &= (p-q)^2 [(F_{2n+3} + F_n^2) + 2\varepsilon (F_{2n+4} + F_n F_{n+1})] \\ &+ q(p-q)[(2F_{2n+4} + 2F_n F_{n+1}) + 2\varepsilon (2F_{2n+5} + F_n F_{n+2})] \\ &+ q^2 [(F_{2n+5} + F_{n+1}^2) + 2\varepsilon (F_{2n+6} + F_{n+1} F_{n+2})]\end{aligned} \tag{3.7}$$

Then for the norm of the generalized dual Fibonacci vector[3], we have, using identities of the Fibonacci numbers

$$\begin{aligned}F_n^2 + F_{n+1}^2 &= F_{2n+1} \\ F_{n+1}^2 - F_{n-1}^2 &= F_{2n+1} \\ F_n F_m + F_{n+1} F_{m+1} &= F_{n+m+1}\end{aligned}$$

(see, (Vajda, 1989)), we have

$$\begin{aligned}\left\| \overrightarrow{D_n} \right\| &= \sqrt{\left\langle \vec{D}_n , \vec{D}_n \right\rangle} = \sqrt{D_n^2 + D_{n+1}^2 + D_{n+2}^2} \\ &= \sqrt{(p-q)^2 (F_{2n+3} + F_n^2) + q(p-q)(2F_{2n+4} + 2F_n F_{n+1})} \\ &+ \sqrt{q^2 (F_{2n+5} + F_{n+1}^2)} \\ &+ \sqrt{2\varepsilon \left\{ (p-q)^2 (F_{2n+4} + F_n F_{n+1}) + q(p-q)(2F_{2n+5} + F_n F_{n+2}) \right\}} \\ &+ \sqrt{2\varepsilon \left\{ q^2 (F_{2n+6} + F_{n+1} F_{n+2}) \right\}}.\end{aligned} \tag{3.8}$$

Special Case- 3: For $p = 1,\ q = 0$, in the equations (3.2), (3.6) and (3.8), we have

$$\begin{aligned}\left\langle \vec{D}_n , \vec{D}_m \right\rangle &= (F_{n+m+3} + F_n F_m) + \varepsilon (2F_{n+m+4} + F_{n+1} F_m + F_n F_{m+1}) \\ \left\langle \vec{D}_n , \vec{D}_{n+1} \right\rangle &= (F_{2n+4} + F_n F_{n+1}) + \varepsilon (2F_{2n+5} + F_{n+1}^2 + F_n F_{n+2})\end{aligned}$$

and

$$\begin{aligned}\left\| D_n \right\| &= \sqrt{(F_{2n+3} + F_n^2) + 2\varepsilon (F_{2n+4} + F_n F_{n+1})} \\ &= (F_{2n+3} + F_n^2) + \varepsilon \frac{(F_{2n+4} + F_n F_{n+1})}{\sqrt{(F_{2n+3} + F_n^2)}}\end{aligned}$$

which given by (Güven & Nurkan, 2014).

Theorem 4. Let $\vec{D}_n$ and $\vec{D}_m$ be two generalized dual Fibonacci vectors. The cross product of $\vec{D}_n$ and $\vec{D}_m$ is given by

$$\vec{D}_n \times \vec{D}_m = (-1)^m F_{n-m}(1+\varepsilon)(p^2 - pq - q^2)((-i - j + k). \tag{3.9}$$

Proof. The cross product of $\vec{D}_n = \vec{H}_n + \varepsilon \vec{H}_{n+1}$ and $\vec{D}_m = \vec{H}_m + \varepsilon \vec{H}_{m+1}$ defined by

$$\vec{D}_n \times \vec{D}_m = (\vec{H}_n \times \vec{H}_m) + \varepsilon(\vec{H}_n \times \vec{H}_{m+1} + \vec{H}_{n+1} \times \vec{H}_m).$$

where $\vec{H}_n$ is the generalized Fibonacci vector and $\vec{H}_n \times \vec{H}_m$ is the cross product for the generalized Fibonacci vectors $\vec{H}_n$ and $\vec{H}_m$.

Now, we calculate the cross products $\vec{H}_n \times \vec{H}_m$, $\vec{H}_n \times \vec{H}_{m+1}$ and $\vec{H}_{n+1} \times \vec{H}_m$:
Using the property $F_m F_{n+1} - F_{m+1} F_n = (-1)^n F_{m-n}$, we get

$$\vec{H}_n \times \vec{H}_m = (-1)^m F_{n-m}(-i - j + k)((p^2 - pq - q^2) \tag{3.10}$$

$$\vec{H}_n \times \vec{H}_{m+1} = (-1)^{m+1} F_{n-m-1}(-i - j + k)(p^2 - pq - q^2) \tag{3.11}$$

and

$$\vec{H}_n \times \vec{H}_m = (-1)^m F_{n-m+1}(-i - j + k)(p^2 - pq - q^2) \tag{3.12}$$

Then from the equations (3.10), (3.11) and (3.12), we obtain the equation (3.9).

Special Case- 4: For $p = 1,\ q = 0$, in the equations (3.9), we have

$$\vec{D}_n \times \vec{D}_m = (-1)^m F_{n-m}(1+\varepsilon)(-i - j + k)$$

which given by (Güven & Nurkan, 2014).

Theorem 5. Let $\vec{D}_n$, $\vec{D}_m$ and $\vec{D}_k$ be the generalized dual Fibonacci vectors. The mixed product of these three vectors is

$$\left\langle \vec{D}_n \times \vec{D}_m, \vec{D}_k \right\rangle = 0 \tag{3.13}$$

Proof. Using the properties

$$\vec{D}_n \times \vec{D}_m = (\vec{H}_n \times \vec{H}_m) + \varepsilon(\vec{H}_n \times \vec{H}_{m+1} + \vec{H}_{n+1} \times \vec{H}_m)$$

and

$$\vec{D}_k = \vec{H}_{n-k} + \varepsilon \vec{H}_{k+1}$$

we can write,

$$\left\langle \vec{D}_n \times \vec{D}_m, \vec{D}_k \right\rangle = \left\langle H_n \times H_m, H_k \right\rangle + \varepsilon[\left\langle H_n \times H_m, H_{k+1} \right\rangle + \left\langle H_n \times H_{m+1}, H_k \right\rangle + \left\langle H_{n+1} \times H_m, H_{k+1} \right\rangle]$$

Then from equations (3.10), (3.11) and (3.12), we obtain

$$\left\langle (-i - j + k), \vec{H}_k \right\rangle = -H_k - H_{k+1} + H_{k+2} = 0,$$

$$\left\langle (-i - j + k), \vec{H}_{k+1} \right\rangle = -H_{k+1} - H_{k+2} + H_{k+3} = 0.$$

Thus, we have the equation (3.13).

4. Notes

[1] The dual numbers extended to the real numbers has the form $d = a + \varepsilon a^*$ where $a, a^* \in R$. The set $\mathrm{D} = \left\{ d = a + \varepsilon a^* \,\middle|\, a, a^* \in R , \varepsilon^2 = 0, \varepsilon \neq 0 \right\}$ is called dual number system (Ercan & Yüce, 2011).

The set $\mathrm{D}^3 = \left\{ \vec{d} = \vec{a} + \varepsilon \overrightarrow{a^*} \ \ where \ \ \vec{a}, \overrightarrow{a^*} \in R^3 \right\}$ is a module on the ring D which is called D- Module and the elements of D^3 are called dual vectors (Ercan & Yüce, 2011).

[2] Binet formula is the explicit formula to obtain the n-th Fibonacci and Lucas numbers. It is well known that for Fibonacci and Lucas numbers, Binet formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n$$

respectively, where $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, $\alpha \beta = -1$ and $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$ (Kalman & Mena, 2003; Rosen 1999).

[3] Norm of dual number as follows:

$$\left\| \vec{A} \right\| = \sqrt{a + \varepsilon\, a^*} = \sqrt{a} + \varepsilon\, a^* \,\frac{1}{2\sqrt{a}} \ , \qquad A = a + \varepsilon\, a^*$$ (Ercan & Yüce, 2011).

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