THE INTERNATIONAL JOURNAL OF SCIENCE & TECHNOLEDGE

On Selecting an Optimal α in summing α-convex Series

Dr. Hamsapriye
Professor, Center for Incubation, Innovation, Research and Consultancy,
Jyothy Institute of Technology, Thathaguni, Off Kanakapura Road, Bangalore, Karnataka, India

Abstract:
Summing Leibnitz alternating series using α-convexity conditions imposed on the terms of theseries is a new concept. The art of choosing an optimal α for accelerating the summation procedure, based on these α-onvexity conditions is discussed and analyzed in the existingliterature. For a particular class of series this choice of an optimal α readily gives the exactsum and this is analyzed in the light of optimization theory. The aim of this note is to relatethe concept of choosing the optimal α for summing this particular class of alternating series to that of the linear programming techniques. The α-convexity imposed on the terms of theseries are exploited which determine a range of α and this range also is the convex region fordetermining the optimal α. For the sake of clarity, several numerical examples are worked out.

Keywords: *α-Convex series, α-Convexity conditions, Partially α-Convex Series, Generalized α-Convexity Conditions*

1. Introduction

There are numerous methods in the literature for summing the Leibnitz alternating series of the type

$$\sum_{n=0}^{\infty}(-1)^n a_n, \qquad a_n > 0, a_n \to 0 \tag{1}$$

In Chakrabarti & Hamsapriye (1996) and Hamsapriye (2000), the α-convexity conditions imposed on theterms of the series (1) give rise to various correction terms, which accelerate the summation procedure. Initially, the kth partial sum

$$S_k = a_0 - a_1 + a_2 - \cdots + (-1)^k a_k \tag{2}$$

is considered as an approximation to the sum of the series, with the remainder or the error term given to be $R_k = (-1)^{k+1}a_{k+1} + (-1)^{k+1}a_{k+2} + (-1)^{k+3}a_{k+3} + \cdots$. The procedure for obtaining the corrected partial sums and the corrected remainders can be utilized repeatedly, whenever additional α convexity conditions are satisfied by the terms of the series Chakrabarti & Hamsapriye (1996). On imposing α-convexity conditions the series will retain the structure of being a Leibnitz series. This further helps in estimating the error at every step of correction. In the first stage, α convexity conditions are given to be

$$\left.\begin{array}{l}(1-\alpha)a_{n+1} - a_{n+2} + \alpha a_{n+3} \geq 0\\ \alpha\, a_{n+2} - a_{n+3} + (1-\alpha)a_{n+4} \geq 0\end{array}\right\}, \quad \forall\, n > 0 \; and \; 0 < \alpha < 1 \tag{3}$$

and the correction term added to the kth partial, along with the corresponding remainder termare given to be Chakrabarti & Hamsapriye (1996):

$$\begin{array}{l}(-1)^n \alpha\, a_{k+1}\\ |R_k| \leq (1-\alpha)(a_{k+1} - a_k)\end{array} \tag{4}$$

We can also derive higher order correction and remainder terms, provided the terms of the series satisfy the higher order α-convexity conditions. The details of the explanation of these higher orderα-convexity conditions, higher order correction terms and corresponding remainder terms are described in Chakrabarti & Hamsapriye (1996) and Hamsapriye (2000). There are classes of series which do not satisfy (3) uniformly for all values of n. That is, the summation procedure depends on the number of terms (even or odd) chosen in the partial sumS_k. Such series are not α-convex Chakrabarti & Hamsapriye (1996) and in this case there exists different ranges of α for even and odd values of n. These series are therefore called as partially α-convex series. For example, the class of series of the form

$$S_k^{\pm} = \sum_{n=0}^{\infty} \frac{a \pm (-1)^n b}{c^{n+1}} \tag{5}$$

with the conditions

$$(i)\ a, b > 0; (ii)\ a > b; (iii)\ c > 1; (iv)\ c(a - b > (a + b) \tag{6}$$

is Leibnitz series. The terms of this series fail to satisfy the inequalities in (3) for all n, whenever $\left|\frac{a}{b}\right| < \left(\frac{c+1}{c-1}\right)^2$ holds good. But the series can still be effectively summed-up by imposing what are called the *partial α-convexity conditions*Chakrabarti & Hamsapriye (1996). It can be verified that the exact sums of the two series are$S^+ = \frac{a}{c+1} + \frac{b}{c-1}$and $S^- = \frac{a}{c+1} - \frac{b}{c-1}$. In fact, in the first correction itself the exact sum of the series is obtained by choosing the optimal αChakrabarti & Hamsapriye (1996). This fact has been proved theoretically in Hamsapriye (2000). The subject matter of this paper is to investigate the reason for obtaining the exact sum of the series, in the light of linear programming.

In section 2, we have explained the direct connection between choosing the optimal α of the first correction that leads to the exact sum of the series and its relation with the linear programming method in one variable. Few numerical examples are worked in section 3, based on first correction, which are also extended to higher correction levels.

2. First Correction and Linear Programming

We first rewrite the series in (5) as

$$\left.\begin{aligned} S &= S_k + R_k \\ &= \overset{\alpha}{\tilde{S}_k} + \overset{\alpha}{\tilde{R}_k} \end{aligned}\right\}, \quad \text{for } 0 < \alpha < 1 \tag{7}$$

where the first corrected sum $\overset{\alpha}{\tilde{S}_k}$ is defined as

$$\overset{\alpha}{\tilde{S}_k} = S_k + \alpha(-1)^n a_{k+1} \tag{8}$$

and the first error estimate $\overset{\alpha}{\tilde{R}_k}$ is given by

$$\overset{\alpha}{\tilde{R}_k} = R_k - \alpha(-1)^n a_{k+1} \tag{9}$$

Our objective function is $\overset{\alpha}{\tilde{S}_k}$which is linear in α and is also one dimension. The problem reduces to finding an appropriate α that optimizes the objective function, in the sense that $S - \overset{\alpha}{\tilde{S}_k}$ is zero, under the restrictions given by (3). Plotting the two inequalities in (3), results in an interval for α.

Since the problem is of one dimension, graphically the interval of α is the portion of the horizontal line bounded by these two inequalities, which may be called as the *convex interval*. A weak version of what is sometimes called the fundamental theorem of linear programming states that the extremal values of a linear function over a convex polygonal region are attained at corners of the convex region. Thus choosing α to be the right-end point of the interval we obtain the optimal value of the objective function or the exact sum of the series itself. The examples in the next section prove this fact. This can be extended to second, third corrections and these statements are supported by several examples. Since the remainder term contains (1 − α), choosing the left-end point of the convex interval results in a larger error.

3. Numerical Results

In this section we have worked out several examples for the sake of clarity. Although we have chosen the same examples as in Chakrabarti & Hamsapriye (1996), the theory is valid for any choice of the constants.

- Example 1.

We set $a = 1, b = 0$ and $c = 2$ in (5). Since $b = 0$ the $\pm$ sign is dropped and the series S is

$$S = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} \tag{10}$$

With $a_n = \frac{1}{2^n}$, the inequalities in (3) reduce to

$$\left.\begin{aligned} 2 - 3\alpha &\geq 0 \\ 3\alpha - 1 &\geq 0 \end{aligned}\right\} \tag{11}$$

On simplification we obtain $\frac{1}{3} \leq \alpha \leq \frac{2}{3}$. The objective function for the above series is

$$\overset{\alpha}{\tilde{S}_k} = S_k + \alpha(-1)^n \frac{1}{2^{k+1}} \tag{12}$$

Figure 1 shows the plot of the inequalities in (11) and the convex interval is marked in black. For $k = 0, S_0 = 1$ and with $\alpha = \frac{2}{3}$, the corrected sum coincides with the exact sum of the series. That is, $\overset{\alpha}{\tilde{S}}_0 = \frac{2}{3} = S$. For $k = 1, S_1 = \frac{1}{2}$and with $\alpha = \frac{2}{3}$, the corrected sum is again$\overset{\alpha}{\tilde{S}}_1 = \frac{2}{3} = S$. Thus in either case, whether k is odd or even, we get the sum of the series with first correction itself. It can be verified that for any k the value of $\overset{\alpha}{\tilde{S}}_k$ is unchanged.

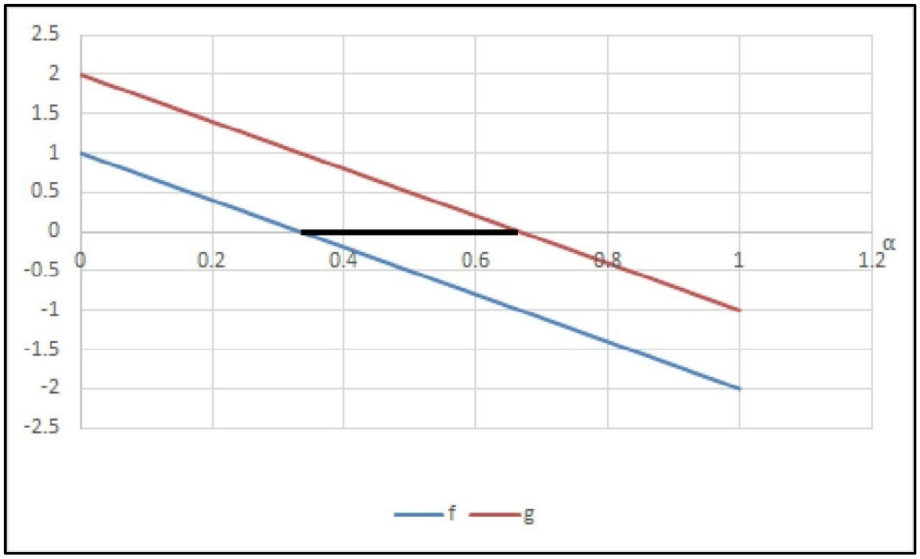

Figure 1: Feasible range of α for Example 1

3.1. Analysis and Reasoning

The question is why $\alpha = \frac{2}{3}$ is an optimal value? This is best understood through the conceptsof linear programming/optimization theory. The two inequalities are plotted in Figure 1 and the feasible region is marked in black. Figure 1 shows that $\alpha = \frac{1}{3}$ and $\alpha = \frac{2}{3}$ are the two extreme pointsand that $\alpha = \frac{2}{3}$ is the optimal point. At these points the inequalities cut the horizontal axis, giving us arange of α. Here we may recall the fundamental theorem of linear programming stated earlier and therefore $\alpha = \frac{2}{3}$ is the optimal choice, since this reduces the error.

- Example 2.

We next consider the example

$$S_k^- = \sum_{n=0}^{\infty} \frac{4-(-1)^n}{5^{n+1}} \qquad (13)$$

Using the constraints (3) on the terms of the series (13), we finally arrive at

$$\left.\begin{array}{l} 25\,(1-\alpha)(4-(-1)^{n+1}) - 5\,(4-(-1)^{n+2}) + \alpha\,(4-(-1)^{n+3}) \geq 0 \\ 25\,\alpha\,(4-(-1)^{n+2}) - 5\,(4-(-1)^{n+3}) + (1-\alpha)(4-(-1)^{n+4}) \geq 0 \end{array}\right\} \qquad (14)$$

k	S_k	a_{k+1}	$\overset{\alpha}{\tilde{S}}_k$
$k = 1$	$\frac{2}{5}$	$\frac{3}{125}$	$\frac{5}{12}$
$k = 3$	$\frac{52}{125}$	$\frac{3}{3125}$	$\frac{5}{12}$

Table 1: $\overset{\alpha}{\tilde{S}}_k = S^-$for different values of k and $\alpha = \frac{25}{36}$

It is to be noted that the constants $a = 4, b = 1$ and $c = 5$ of this series satisfy the condition$\left|\frac{a}{b}\right| > \left(\frac{c+1}{c-1}\right)^2$. Hence, for this series we cannot fix a common α for all n. We thus have to fixdifferent α depending on whether n is odd (the partial sum consists of even number of terms)or even (the partial sum consists of odd number of terms). We discuss these cases separately.

Case I: If n is odd then we obtain the range α to be$\frac{1}{12} \leq \alpha \leq \frac{25}{36}$. The table below showsthat the corrected partial sum$\overset{\alpha}{\tilde{S}_k}$ is equal to the exact sum S^-, for any odd k.

3.2. Analysis and Reasoning

If n is odd, then the first and the second constraints in (14) simplifies to

$$p(\alpha) = -120\alpha + 10 \geq 0 \tag{15}$$

and

$$q(\alpha) = -72\alpha + 50 \leq 0 \ (16)$$

Figure 2 displays the plot of these two functions and the feasible or the convex interval is marked in black color. In Figure 2, $\alpha = \frac{1}{12}$ and $\alpha = \frac{25}{36}$ are the two extreme points and the choice of $\alpha = \frac{25}{36}$gives the exact sum.

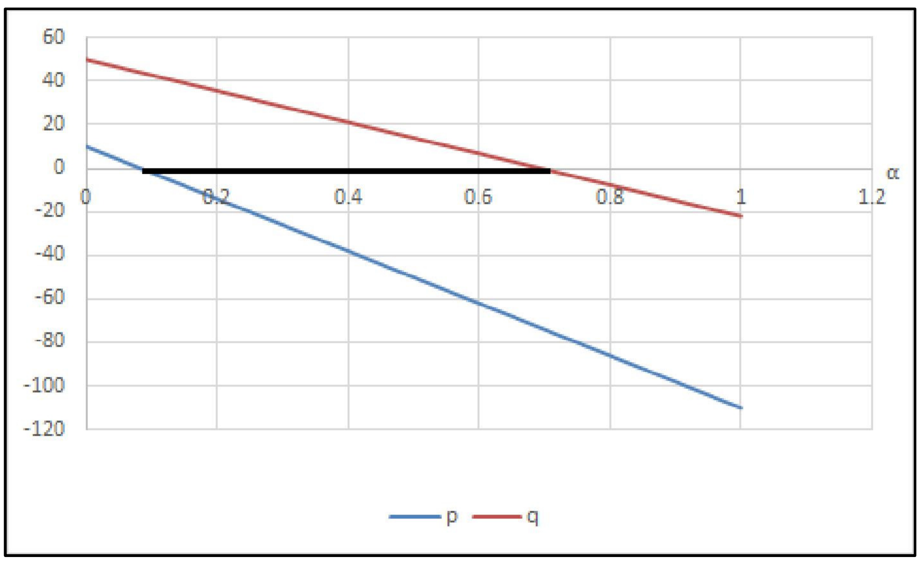

Figure 2: Feasible range of α for Example 2, for odd k

Case II: If n is even then then we obtain the range α to be$\frac{11}{36} \leq \alpha \leq \frac{11}{12}$.The table below shows that the corrected $\overset{\alpha}{\tilde{S}_k}$ is equal to the exact sum S^-, for any even k.

k	S_k	a_{k+1}	$\overset{\alpha}{\tilde{S}_k}$
$k = 2$	$\frac{53}{125}$	$-\frac{1}{125}$	$\frac{5}{12}$
$k = 4$	$\frac{1303}{3125}$	$-\frac{1}{3125}$	$\frac{5}{12}$

Table 2: $\overset{\alpha}{\tilde{S}_k} = S^-$for different values of k and$\alpha = \frac{11}{12}$

If n is even, then the first and the second constraints in (14) simplifies to

$$f(\alpha) = -72\alpha + 22 \geq 0 \tag{17}$$

and

$$g(\alpha) = -120\,\alpha + 110 \le 0 (18)$$

These functions are plotted as shown in Figure 3 and the feasible region is as usual marked in black color. In Figure 3, $\frac{11}{36}$and $\frac{11}{12}$ are the extreme points and that $\frac{11}{12}$ is the optimal choice.

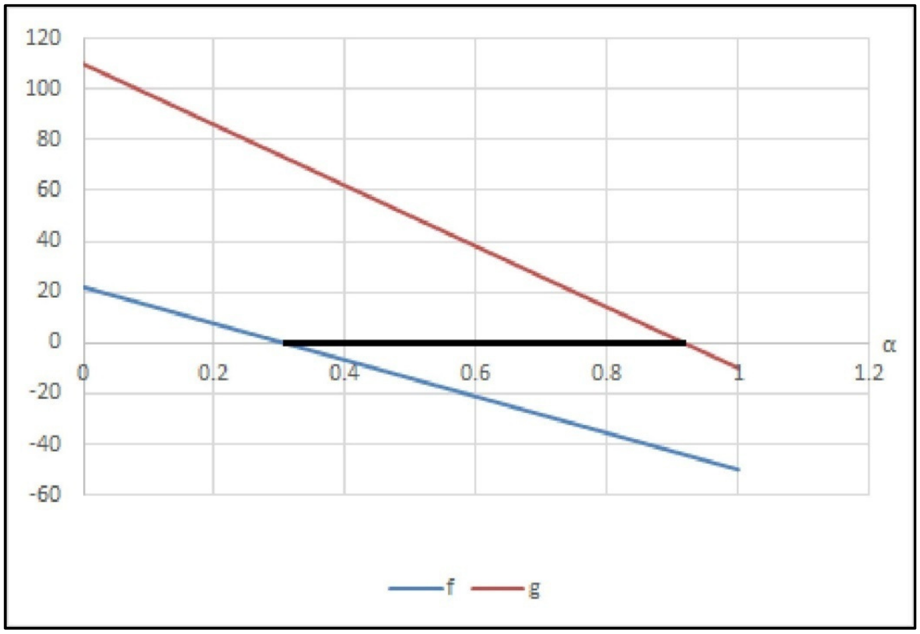

Figure 3: Feasible range of α for Example 2, for even k

- Example 3.
  We next consider the example

$$S_k^- = \sum_{n=0}^{\infty} \frac{13-(-1)^n.3}{2^{n+5}} \qquad (19)$$

Case I: If n is odd then we obtain the range α to be$\frac{1}{12} \le \alpha \le \frac{4}{15}$. The table below showsthat the corrected partial sum$\overset{\alpha}{\tilde{S}_k}$ is equal to the exact sum S^-, for any odd k.

k	S_k	a_{k+1}	$\overset{\alpha}{\tilde{S}_k}$
$k=1$	$\frac{1}{16}$	$\frac{5}{64}$	$\frac{1}{12}$
$k=3$	$\frac{5}{64}$	$\frac{5}{256}$	$\frac{1}{12}$

Table 3: $\overset{\alpha}{\tilde{S}_k} = S^-$for different values of k and $\alpha = \frac{4}{15}$

Figure 4 shows the plots of the functions $p(\alpha) = -12\,\alpha + 1 \ge 0$ and $q(\alpha) = -15\,\alpha + 4 \le 0$ and the two extreme points are $\frac{1}{12}$ and $\frac{4}{15}$ and the optimum choice is $\alpha = \frac{4}{15}$.

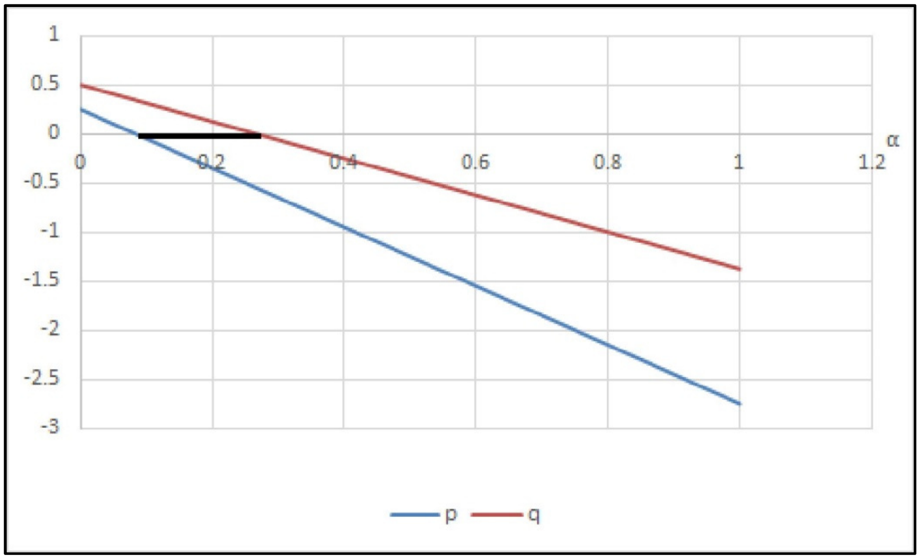

Figure 4: Feasible range of α for Example 3, for odd k

Case II: If n is even then then we obtain the range α to be $\frac{11}{15} \leq \alpha \leq \frac{11}{12}$.The table below shows that the corrected $\overset{\alpha}{\tilde{S}_k}$ is equal to the exact sum S^{-}, for any even k.

k	S_k	a_{k+1}	$\overset{\alpha}{\tilde{S}_k}$
$k = 2$	$\frac{9}{64}$	$-\frac{1}{16}$	$\frac{1}{12}$
$k = 4$	$\frac{25}{256}$	$-\frac{1}{64}$	$\frac{1}{12}$

Table 4: $\overset{\alpha}{\tilde{S}_k} = S^{-}$for different values of k and $\alpha = \frac{11}{12}$

Figure 5 shows the plots of the functions $f(\alpha) = -15\,\alpha + 11 \geq 0$ and $g(\alpha) = -12\,\alpha + 11 \leq 0$ and the two extreme points are $\frac{11}{15}$ and $\frac{11}{12}$ and the optimum choice is $\alpha = \frac{11}{12}$.

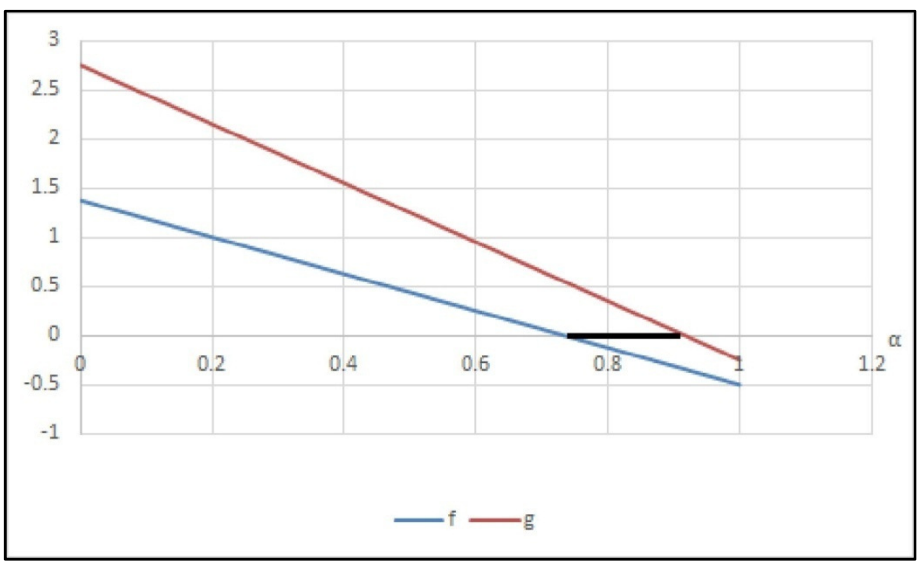

Figure 5: Feasible range of α for Example 3, for even k

4. Conclusions

Summation procedure of Leibnitz alternating series using α-convexity conditions, imposed on the terms of the series, has been re-considered and the basic idea of choosing an appropriate α has been explained in the light of optimization technique. Determining a range for α has been related to determining the convex region formed by the α-convexity constraints. Several numerical examples have been worked out in support of the statements. Although the same examples as in Chakrabarti & Hamsapriye (1996) and Hamsapriye (2000) have been considered, the results of this paper are not just restricted to them. We first impose the α convexity conditions on the class of series in (5) and obtain pair/s of functional inequalities. By plotting these inequations we obtain the region/s of their intersection with that of the horizontal axis. Depending on whether the series is α-convex or partially α-convex, we obtain a common range of α for all n or different intervals of α separately for n odd and n even. We choose the right end point of the convex interval as the optimal α and proceed for correcting the partial sum. The corrected partial sum coincides with the exact sum or the limit of the series. These ideas provide a new way of thinking for researchers and the student community and in particular to those interested in Mathematics Education. In other words, summing of α-convex series can be now understood in the light of optimization theory.

5. References

i. Chakrabarti & Hamsapriye (1996), On corrected summation of alternating series, Int.J. Math. Educ. Sci. Technol., 27, 565 - 582.
ii. Hamsapriye, (2000), α-convex convergent alternating series, Intl. Jl. Math. Edu. Sci.Technol., 31, 691 - 701.