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Combined Fourier Sine Integral Transform and Fractional Sumudu Transform Method (FSSTM) for Solving Time –Fractional Energy and Stokes´ First Equation

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Abstract:
In this paper, we obtain the solution of a time fractional energy equation and Stokes´ first equation by using Fourier Sine Sumudu Transform Method (FSSTM). This method is based on Fourier Sine Integral transform and Fractional Sumudu transform for solving the Stokes` first equation and energy equation with rational derivative, where the fractional derivative is defined in the Caputo sense of order* $0<\alpha\le 1$. *The solution of classical problems for both Stokes` first equation and energy equation has been obtained as limiting case.

Keywords: *Stokes` first equation, energy equation, fractional derivatives, sumudu transform, fourier sine integral transform, caputo fractional derivative, generalized mittag-leffler function*

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. In recent years, considerable amount of research in fractional calculus was published in engineering and mathematical physics literature. Indeed, recent advances of fractional calculus are dominated by modern examples of applications in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and electrostatics, fluid flow, steady state, heat conduction and many other topics in both pure and applied mathematics [1].

In this paper, a new approach is proposed to use Fourier Sine Sumudu Transform Method (FSSTM), which is combined between Fourier Sine Integral Transform and Fractional Sumudu Transform to derive the exact solution of the Stokes` first equation and energy equation. Exact solution of these equations will be investigated. The Fourier sine transform and fractional Sumudu transform are used for getting exact solution for these equations. The fractional terms in energy equation and Stokes´ equation are considered as Caputo fractional derivative [2].

2. Basic Definitions

- Definition 1: The Caputo fractional derivative [3] of order $n-1<\alpha\le n$

is defined as:

$$D^{\alpha} f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt \qquad (1)$$

- Definition 2: In early 90's [4] Watauga introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of functions:

$$A=\left\{f(t):\exists M\tau_1,\tau_2>0,\left|\ f(t)\ \right|<Me^{|t|/\tau_j}, if\ t\in(-1)^j\times[0,\infty)\right\}$$

By the following formula:

$$G(u) = S[f(t);u] = \int_0^\infty f(ut)e^{-t}dt, u \in (-\tau_{1,}, \tau_2). \quad (2)$$

For further details, we will present some special properties of the Sumudu transform are as follows:

2.1.1 $S[1] = 1$; (3)

2.1.2 $S[\frac{t^n}{\Gamma(n+1)}] = u^n; n > 0,$ (4)

2.1.3 $S[e^{at}] = \frac{1}{1-au}$; (5)

2.1.4 $S[af(t) + bg(t)] = aS[f(t)] + bS[g(t)]$ (6)

Other properties of the Sumudu transform can be found in [5].

- Definition 3: The Sumudu transform of the Caputo fractional derivative is defined as follows [6]:

$$S[D_t^\alpha f(t)] = u^{-\alpha} S[f(t)] - \sum_{k=0}^{n-1} u^{-\alpha+k} f^{(k)}(0+), (n-1 < \alpha \le n) \text{ where } G(u) = S[f(t)]. \quad (7)$$

For further details and properties of fractional Sumudu transform; we will establish the following results which are directly applicable in the analysis of
fractional energy equation and Stokes´ First equation:

(I) $$S^{-1}[\frac{k!}{u^{\alpha k+\beta-1}(\frac{1}{u^\alpha} \mp c)^{k+1}}] = t^{\alpha k+\beta-1} E_{\alpha,\beta}^k (\pm ct^\alpha), \quad (8)$$

(II) $$S^{-1}[\frac{u^{-\alpha} + au^{-\beta}}{u^{-\alpha} + au^{-\beta} + b}] = \sum_{k=0}^\infty (-b)^k t^{\alpha k} E_{\alpha-\beta,\alpha k+1}^k (-at^{\alpha-\beta}), \quad (9)$$

(III) $$S^{-1}[u^{\gamma-1}(1 \mp wu^\alpha)^{-k}] = t^{\gamma-1} E_{\alpha,\gamma}^k (\pm wt^\alpha). \quad (10)$$

where $S^{-1}[.]$ denote the inverse Sumudu transform and $E_{\alpha,\beta}^k(\pm ut^\beta)$ is generalized Mittage-leffler function [7].

- Definition 4: The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [8]:

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)} \quad (11)$$

A generalization of the Mittag-Leffler function ([9], [10])

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)}, \ \alpha \in C, R(\alpha) > 0 \quad (12)$$

was introduced by Wiman [10] in the general form:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}, \ \alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0 \quad (13)$$

The main results of these functions are available in the handbook of Erdelyi, Magnus, Oberhettinger, and Tricomi [11] and the monographs By Dzherbashyan ([12], [13]). Prabhakar [14] introduced a generalization of (13) in the form*:*

$$E_{\alpha,\beta}^k(z) = \sum_{n=0}^\infty \frac{(k)_n z^k}{n!\Gamma(\alpha n+\beta)}, \qquad \alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0, R(k) > 0 \quad (14)$$

where $(k)_n$ is the Pochhammer's symbol (Rainville [15]) is defined by the equations:

$$(k)_n = k(k+1)(k+2)(k+3)........(k+n-1), n \ge 1 \quad (15)$$

which is generalization of factorial function? If k is neither zero nor a negative integer,

then we can define $(k)_n = \frac{\Gamma(k+n)}{\Gamma(k)}$ (16)

The *ordinary binomial expression* (Rainville [15]), defined as*:*

$$(1-z)^{-k} = \sum_{n=0}^\infty \frac{(k)_n z^n}{n!}. \quad (17)$$

- Definition 5: The Fourier sine integral transform [16, 17, 18] of the function

($f(x)$) is defined as:

$$F_e(f(x)) = F(\zeta) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\zeta x) f(x)dx \qquad (18)$$

3. Fractional Stokes' First Equation

Consider the Fractional Stokes' first Equation for a heated flat plate in dimensionless form as follows:-

$$\frac{\partial U(x,t)}{\partial t} = (1+\eta D_t^\alpha)\frac{\partial^2 U(x,t)}{\partial x^2}, 0<\alpha\le 1\ , \qquad (19)$$

$$U(x,0) = b_0(x), x>0\,, \qquad (20)$$

$$\frac{\partial U(x,0)}{\partial t} = b_1(x), x>0\,, \qquad (21)$$

$$U(0,t) = 1, x>0\,, \qquad (22)$$

$$U(x,t), \frac{\partial U(x,t)}{\partial x} \to 0\ for x \to \infty\,, \qquad (23)$$

3.1. The Solution

Making use of the Fourier sine integral trans form of (19), (20) and boundary conditions (21), (22), then we get:

$$\frac{dU(\zeta,t)}{dt} = -\zeta^2(1+\eta D_t^\alpha)U(\zeta,t) + \sqrt{\frac{2}{\pi}\zeta}\ , \qquad (24)$$

$$U(\zeta,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\zeta x) b_0(x)dx = B_0(\zeta) \qquad (25)$$

Hence Sumudu transform of Eq.(25) is;

$$\frac{G(\zeta,u)}{u} - \frac{U(\zeta,0)}{u} = -\zeta^2 G(\zeta,u) + \sqrt{\frac{2}{\pi}\zeta} - \eta\zeta^2[\frac{G(\zeta,u)}{u^\alpha} - \frac{U(\zeta,0)}{u^\alpha}]\ , \qquad (26)$$

$$\frac{G(\zeta,u)}{u} + \zeta^2 G(\zeta,u) + \eta\zeta^2\frac{G(\zeta,u)}{u^\alpha} = \frac{U(\zeta,0)}{u} + \sqrt{\frac{2}{\pi}}\zeta + \eta\zeta^2\frac{U(\zeta.,0)}{u^\alpha},$$

$$G(\zeta,u)[u^{\alpha-1} + \zeta^2 u^\alpha + \eta\zeta^2] = u^\alpha\sqrt{\frac{2}{\pi}}\zeta + [\eta\zeta^2 + u^{\alpha-1}]B_0(\zeta)$$

$$\therefore G(\zeta,u) = \frac{u^\alpha\sqrt{\frac{2}{\pi}}}{[u^{\alpha-1} + \zeta^2 u^\alpha + \eta\zeta^2]}\zeta + \frac{[\eta\zeta^2 + u^{\alpha-1}]}{[u^{\alpha-1} + \zeta^2 u^\alpha + \eta\zeta^2]}B_0(\zeta) \qquad (27)$$

Taking the inverse Sumudu transform of Eq.(27)we have:

$$U(\zeta,t) = \sqrt{\frac{2}{\pi}}\zeta S^{-1}[\frac{1}{[u^{-1} + \eta\zeta^2 u^{-\alpha} + \zeta^2]}] + S^{-1}[\frac{[\eta\zeta^2 u^{-\alpha} + u^{-1}]}{[u^{-1} + \eta\zeta^2 u^{-\alpha} + \zeta^2]}]B_0(\zeta)$$

But

$$S^{-1}[\frac{u^{-\alpha} + au^{-\beta}}{u^{-\alpha} + au^{-\beta} + b}] = \sum_{k=0}^\infty (-b)^k t^{\alpha k} E_{\alpha-\beta,\alpha k+1}^k(-at^{\alpha-\beta}),$$

$$U(\zeta,t) = \sqrt{\frac{2}{\pi}}\zeta S^{-1}[\frac{1}{[u^{-1} + \eta\zeta^2 u^{-\alpha} + \zeta^2]}] + B_0(\zeta)[\sum_{n=0}^\infty (-1)^k \zeta^{2k} t^k E_{1-\alpha,k+1}^k(-\eta\zeta^2 t^{1-\alpha})] \qquad (28)$$

And

$$\frac{1}{u^{-1} + \eta\zeta^2 u^{-\alpha} + \zeta^2} = \frac{1}{(u^{-1} + \eta\zeta^2 u^{-\alpha})(1 + \frac{\zeta^2}{u^{-1} + \eta\zeta^2 u^{-\alpha}})} = \frac{1}{(u^{-1} + \eta\zeta^2 u^{-\alpha})}[1 + \frac{\zeta^2}{u^{-1} + \eta\zeta^2 u^{-\alpha}}]^{-1}$$

$$\therefore \frac{1}{u^{-1} + \eta\zeta^2 u^{-\alpha} + \zeta^2} = \frac{1}{(u^{-1} + \eta\zeta^2 u^{-\alpha})}\sum_{k=0}^\infty (-1)^k (\frac{\zeta^2}{u^{-1} + \eta\zeta^2 u^{-\alpha}})^k = \sum_{k=0}^\infty (-1)^k \zeta^{2k}\frac{1}{(u^{-1} + \eta\zeta^2 u^{-\alpha})^{k+1}}$$ By using the following relation:

$S^{-1}[u^{\gamma-1}(1\mp wu^{\alpha})^{-k}]=t^{\gamma-1}E^{k}_{\alpha,\gamma}(\pm wt^{\alpha})$

We get

$$S^{-1}[\frac{1}{u^{-1}+\eta\zeta^{2}u^{-\alpha}+\zeta^{2}}]=\sum_{k=0}^{\infty}(-1)^{k}\zeta^{2k}S^{-1}[u^{(k+2)-1}(1+\eta\zeta^{2}u^{1-\alpha})^{-(k+1)}]$$
$$=\sum_{k=0}^{\infty}(-1)^{k}\zeta^{2k}t^{k+1}E^{k+1}_{1-\alpha,k+2}(-\eta\zeta^{2}t^{1-\alpha}) \quad (29)$$

By substituting from (28) into (29) we get:-

$$U(\zeta,t)=\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}(-1)^{k}\zeta^{2k+1}t^{k+1}E^{k+1}_{1-\alpha,k+2}(-\eta\zeta^{2}t^{1-\alpha})d\tau+B_{0}(\zeta)\sum_{k=0}^{\infty}(-1)^{k}\zeta^{2k}t^{k}E^{k}_{1-\alpha,k+1}(-\eta\zeta^{2}t^{1-\alpha}) \quad (30)$$

Where

$$E^{k}_{\alpha,k}(z)=\frac{d^{k}}{dz^{k}}E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(k)_{n}z^{n}}{n!\Gamma(n\alpha+\beta)}$$

Now considering the inverse Fourier sine integral transform of Eq. (30) we get:

$$U(x,t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\sin(\zeta x)[\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}(-1)^{k}\zeta^{2k+1}t^{k+1}E^{k+1}_{1-\alpha,k+2}(-\eta\zeta^{2}t^{1-\alpha})+$$
$$B_{0}(\zeta)\sum_{k=0}^{\infty}(-1)^{k}\zeta^{2k}t^{k}E^{k}_{1-\alpha,k+1}(-\eta\zeta^{2}t^{1-\alpha})]d\zeta \quad (31)$$

This is the exact solution of (19).

3.2. Special Case

When $B_{0}(\zeta)=0,\alpha=1$, then Eq. (31) becomes**:**

$$U(x,t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\sin(\zeta x)[\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}(-1)^{k}\zeta^{2k+1}t^{k+1}E^{k+1}_{0,k+2}(-\eta\zeta^{2})d\zeta \quad (32)$$

where $E^{k+1}_{1-\alpha,k+2}(-\eta\zeta^{2}t^{1-\alpha})$ is Mittag-Leffler function that has the following generalized form:

$$E^{k}_{\alpha,k}(z)=\sum_{n=0}^{\infty}\frac{(k)_{n}z^{n}}{n!\Gamma(n\alpha+\beta)}$$

Since the Pochhammer symbol $(k)_{n}$ satisfies the following identities:

$$\binom{-k}{n}=\frac{(-1)^{n}}{n!}(k)_{n} \text{ and } \sum_{n=0}^{\infty}\binom{-k}{n}(\eta\zeta^{2})=(1+\eta\zeta^{2})^{-k}$$

Then

$$E^{k+1}_{0,k+2}(-\eta\zeta^{2}t^{1-\alpha})=\sum_{n=0}^{\infty}\frac{(k+1)_{n}(-\eta\zeta^{2})^{n}}{n!\Gamma(k+2)},$$
$$E^{k+1}_{0,k+2}(-\eta\zeta^{2}t^{1-\alpha})=\sum_{=0}^{\infty}\frac{(-1)^{n}(k+1)_{n}(\eta\zeta^{2})^{n}}{n!(k+1)!}=\sum_{k=0}^{\infty}\binom{-(k+1)}{n}(\eta\zeta^{2})^{n}\frac{1}{(k+1)!}$$
$$\therefore E^{k+1}_{0,k+2}(-\eta\zeta^{2}t^{1-\alpha})=\frac{1}{(k+1)!}(1+\eta\zeta^{2})^{-(k+1)} \quad (33)$$

By substituting from (33) into (32) we have:

$$U(x,t)=-\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin(\zeta x)}{\zeta}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\zeta^{2(k+1)}t^{k+1}}{(1+\eta\zeta^{2})^{(k+1)}(k+1)!}d\zeta\ ,$$
$$U(x,t)=-\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin(\zeta x)}{\zeta}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}(\frac{-\zeta^{2}}{(1+\eta\zeta^{2})}t)^{(k+1)}d\zeta\ ,$$
$$U(x,t)=-\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin(\zeta x)}{\zeta}[(1+\sum_{k=0}^{\infty}\frac{1}{(k+1)!}(\frac{-\zeta^{2}}{(1+\eta\zeta^{2})}t)^{(k+1)})-1]d\zeta\ ,$$

$$U(x,t)=-\frac{2}{\pi}\int_0^\infty \frac{\sin(\zeta x)}{\zeta}[(\exp(\frac{-\zeta^2}{1+\eta\zeta^2}t)-1]d\zeta$$

But

$$\int_0^\infty \frac{\sin(\zeta x)}{\zeta}d\zeta=\frac{\pi}{2}\text{[19]}.$$

Then

$$U(x,t)=1-\frac{2}{\pi}\int_0^\infty \frac{\sin(\zeta x)}{\zeta}[(\exp(\frac{-\zeta^2}{1+\eta\zeta^2}t)d\zeta \tag{34}$$

which is the result obtained by (Fetacau et al [20]) and (Salim et al [21]).

4. Solution of a Time- Fractional Energy Equation

We consider the non dimensional Energy Fractional Equation when the Fourier's law of heat conduction is considered as follows:

$$\frac{1}{\Pr}\frac{\partial^2 U(x,t)}{\partial x^2}+\lambda[\frac{\partial v(x,t)}{\partial x}]^2=\frac{\partial^\alpha U(x,t)}{\partial t^\alpha},0<\alpha\le 1 \tag{35}$$

$$U(x,0)=a_0(x)\,forx>0, \tag{36}$$

$$U(0,t)=1\,forx>0, \tag{37}$$

$$U(x,t),\frac{\partial U(x,t)}{\partial x}\to 0\,forx\to\infty. \tag{38}$$

Letting $f(x,t)=\lambda[\frac{\partial v(x,t)}{\partial x}]^2$,then Eq.(35)can be rewritten as:

$$\frac{1}{\Pr}\frac{\partial^2 U(x,t)}{\partial x^2}+f(x,t)=\frac{\partial^\alpha U(x,t)}{\partial t^\alpha}\quad,\ 0<\alpha\le 1 \tag{39}$$

where $r(x,t)$ is the radiant heating, which is neglected in this paper, c is

the specific heat and k is the conductivity which is assumed to be constant and $\Pr=\frac{c\mu}{k}$ and $\lambda=\frac{U^2}{cT_0}$ which is assumed to be constant.

4.1. The Solution

Applying Fourier integral sine transform to Eq. (39) and (36), we get**:**

$$\frac{1}{\Pr}[\sqrt{\frac{2}{\pi}}\zeta-\zeta^2U(\zeta,t)]+F(\zeta,t)=\frac{\partial^\alpha U(\zeta,t)}{\partial t^\alpha}\quad, \tag{40}$$

$$U(\zeta,0)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\zeta x)a_0(x)dx=A_0(\zeta) \tag{41}$$

Using initial condition (41) for getting fractional Sumudu transform of Eq. (40) as

$$\frac{1}{\Pr}\sqrt{\frac{2}{\pi}}\zeta+F(\zeta,u)=[\frac{G(\zeta,u)}{u^\alpha}-\frac{U(\zeta,0)}{u^\alpha}]+\frac{1}{\Pr}\zeta^2G(\zeta,u),U(\zeta,0)=A_0(\zeta),$$

$$\frac{u^\alpha}{\Pr}\sqrt{\frac{2}{\pi}}\zeta+u^\alpha F(\zeta,u)+A_0(\zeta)=[1+\frac{\zeta^2}{\Pr}u^\alpha]G(\zeta,u)$$

$$\therefore G(\zeta,u)=\sqrt{\frac{2}{\pi}}\frac{\zeta}{\Pr}\frac{u^\alpha}{[1+\frac{\zeta^2}{\Pr}u^\alpha]}+\frac{u^\alpha}{[1+\frac{\zeta^2}{\Pr}u^\alpha]}F(\zeta,u)+\frac{1}{[1+\frac{\zeta^2}{\Pr}u^\alpha]}A_0(\zeta) \tag{42}$$

where $F(\zeta,u)$ is Sumudu transform of $F(\zeta,t)$.

Taking the inverse Sumudu transform of Eq. (42) and using the relation**:**

$$S^{-1}[u^{\gamma-1}(1\mp wu^\alpha)^{-k}]=t^{\gamma-1}E^k_{\alpha,\gamma}(\pm wt^\alpha)$$

We have

$$U(\zeta,t)=\sqrt{\frac{2}{\pi}}\frac{\zeta}{\Pr}t^\alpha E_{\alpha,\alpha+1}(-\frac{\zeta^2}{\Pr}t^\alpha)+t^{\alpha-1}E_{\alpha,\alpha}(-\frac{\zeta^2}{\Pr}t^\alpha)F(\zeta,t)+E_{\alpha,1}(-\frac{\zeta^2}{\Pr}t^\alpha)A_0(\zeta) \tag{43}$$

Applying the inverse Fourier integral sine transform to Esq. (43), we get

$$U(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\sin(\zeta x)}{\zeta}\frac{\zeta^2}{\Pr}t^\alpha E_{\alpha,\alpha+1}(-\frac{\zeta^2}{\Pr}t^\alpha)d\zeta+\sqrt{\frac{2}{\pi}}\int_0^\zeta \sin(\zeta x)t^{\alpha-1}E_{\alpha,\alpha}(-\frac{\zeta^2}{\Pr}t^\alpha)f(x,t)d\zeta+ \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\zeta x)E_{\alpha,1}(-\frac{\zeta^2}{\Pr}t^\alpha)A_0(\zeta)d\zeta \quad (44)$$

This is the exact solution of (35).

4.2. Special Case

When $A_0(\zeta)=0,\ f(x,t)=0,\alpha=1$, then Eq.(44) becomes:

$$U(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\sin(\zeta x)}{\zeta}\frac{\zeta^2}{\Pr}t^\alpha E_{\alpha,\alpha+1}(-\frac{\zeta^2}{\Pr}t^\alpha)d\zeta \quad (45)$$

But

$$t^\alpha\frac{\zeta^2}{\Pr}E_{\alpha,\alpha+1}(-\frac{\zeta^2}{\Pr}t^\alpha)=\frac{\zeta^2}{\Pr}t^\alpha\sum_{k=0}^\infty\frac{(-\frac{\zeta^2}{\Pr}t^\alpha)^k}{\Gamma(\alpha k+\alpha+1)}=-\sum_{k=0}^\infty\frac{(-\frac{\zeta^2}{\Pr}t)^{k+1}}{(k+1)!},\alpha=1,$$

$$\therefore \frac{\zeta^2}{\Pr}t^\alpha E_{\alpha,\alpha+1}(-\frac{\zeta^2}{\Pr}t)=-[\sum_{k=0}^\infty\frac{(-\frac{\zeta^2}{\Pr}t)^{k+1}}{(k+1)!}+1-1]=-[\exp(-\frac{\zeta^2}{\Pr}t)-1]=[1-\exp(-\frac{\zeta^2}{\Pr}t)]$$

Then Eq. (45) becomes:

$$U(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\sin(\zeta x)}{\zeta}[1-\exp(-\frac{\zeta^2}{\Pr}t)]d\zeta,$$

$$U(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\sin(\zeta x)}{\zeta}d\zeta-\frac{2}{\pi}\int_0^\infty\frac{\sin(\zeta x)}{\zeta}\exp(-\frac{\zeta^2}{\Pr}t)]d\zeta,$$

But

$$\int_0^\infty \frac{\sin(\zeta x)}{\zeta}d\zeta=\frac{\pi}{2}\,[19].$$

$$U(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\sin(\zeta x)}{\zeta}d\zeta-\frac{2}{\pi}\int_0^\infty\frac{\sin(\zeta x)}{\zeta}\exp(-\frac{\zeta^2}{\Pr}t)]d\zeta$$

$$U(x,t)=1-\frac{2}{\pi}\int_0^\infty\frac{\sin(\zeta x)}{\zeta}\exp(-\frac{\zeta^2}{\Pr}t)]d\zeta \quad (46)$$

$$U(x,t)=1-erf[\frac{x}{2\sqrt{\frac{t}{\Pr}}}] \quad (47)$$

which is the result obtained also by Fetacau and Corina [20].

5. Conclusion

In this paper, we have introduced a combination of the Fourier sine transform and the Sumudu transform method for time fractional problems. This combination builds a strong method called the Fourier Sine Sumudu Transform Method (FSSTM). This method has been successfully applied to solve Stokes' first problem and energy equation .The FSSTM is an analytical method and runs by using the initial and boundary conditions. The Caputo fractional derivative is considered in both Stokes' first problem and energy problem as time derivatives, where the order of the fractional derivative is considered as $0<\alpha\le 1$. Special cases have been considered in the case at $\alpha=1$. An important advantage of the new approach is its low computational load.

6. References

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