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The Radon-Nikodym Theorem and its Extension to Signed Measures

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Abstract:
Given two measures v and μ on a measurable space (X, $\mathcal{A}$), a natural question that comes out is that if one can represent v in terms of μ via some linear operator. The Radon-Nikodym Theorem states that it is possible, under some hypothesis, to find a representation via the integral operator, that, given a measurable space (X, $\mathcal{A}$), if v a σ finite measure which is absolutely continuous with respect to a σ finite measure μ on(X), then there is a non-negative measurable function f on X such that $v(E) = \int_E f\, d\mu$ for any measurable set E.

Keywords: *Signed measure, Measurable space, Positive set, Negative set, Hahn-Decomposition, Orthogonal measures.*

Definition
Let v_1 and v_2 be two non-negative measures such that at least one of them is finite, $v_1 \perp v_2$ and $v = v_1 - v_2$then the pair (v_1 , v_2) is called Jorden-Decomposition for v. The measure v_1 is called the positive variation and v_2 is called negative variation of v and are denoted by v^+ and v^- respectively.

Remark: $v = v^+ - v^-$

Definition: Let v be any signed measure then we define $|v| = v^+ + v^-$ the measure $|v|$ is called the total variation of v.

Note: It should be noted that $|v|$(E)and $|v(E)|$ are not the same quantities.
More precisely $|v|(E) = v^+(E) + v^-(E)$
and $|v(E)| = |v^+(E) - v^-(E)|$

Definition
Let v_1 and v_2 be two signed measures and suppose E be a null set for v_1, whenever it is a null set for v_2 then we say that v_1 is absolutely continuous with respect to v_2 and we write $v_1 \ll v_2$.

Example: Let N be any measurable set then the following statements are equivalent.
1. N is a null set for v.
2. N is a null set for v^+ and v^-.
3. N is a null set for $|v|$.

Sol: Let N be a null set for v,
Let (P, X-P) be a Hahn-Decomposition for v. Let E ⊂ N be measurable, then
$v^+(E) = v(E \cap P) = 0$ [As E∩ P ⊂ N]
⇒ N is a null set for v^+.
$Similaraly$ N is a null set for v^-. Hence (1) ⇒ (2)
Now suppose $that$ N is a null set for for v^+ and v^- both.
Let Let E ⊂ N be measurable, then $v^+(E) = 0$ and $v^-(E) = 0$
⇒ $v^+(E) + v^-(E) = 0$ [N is a null set for v^+ & v^-]
⇒ $|v|(E) = 0$ ⇒ N is a null set for $|v|$. Hence (2) ⇒ (3).
Further let N is a null set for $|v|$. Let E ⊂ N be measurable, then $|v|(E) = 0$
⇒ $v^+(E) + v^-(E) = 0$ ⇒ $v^+(E) = 0$ and $v^-(E) = 0$ ⇒ $v^+(E) - v^-(E) = 0$

$\Rightarrow v(E) = 0 \Rightarrow$ N is a null set for v.
Shows that (3) ⇒ (1). Hence proved.

Example: If $v_1 \perp v_2$ and $v_1 \ll v_2$ then $v_1 \equiv 0$.
As $v_1 \perp v_2$ ⇒ there exist a measurable set S such that v_1 is supported on S and v_2 is supported on X-S.
Let E⊂ X be measurable then E = (E∩ S) ∪ (E ∩ $X - S$) ⇒ v_1(E)
= v_1(E ∩ S) + v_1(E ∩ $X - S$) = v_1(E ∩ S)   [As v_1 is supported on S]
Also v_2(E ∩ S) = 0   [As v_2 is supported on $X - S$]
⇒ v_1(E ∩ S) = 0   [As $v_1 \ll v_2$]
⇒ v_1(E) = 0   ∀ E ⊂ X be measurable
Hence v_1= 0

Lemma (1) Suppose for each *member* α of a set D of real numbers, there is a given set B_α. Let for $x \in$ X the set Dx={ $\alpha \in$ D / $x \in B_\alpha$}
Define f(x)= Inf Dx. Then for any real number t
{f < t} = $\bigcup_{\alpha<t} B_\alpha$, {f ≤ t} = $\bigcap_{\beta>t}$ { $\bigcup_{\alpha<\beta} B_\alpha$ }

Proof: To show {f < t} = $\bigcup_{\alpha<t} B_\alpha$, Let $x \in$ {f < t}, then f(x) < t ⇒ Inf Dx < t ⇒ there exist $\beta \in Dx$ s.t. $\beta < t \Rightarrow x \in B_\beta$ and $\beta < t \Rightarrow x \in \bigcup_{\alpha<t} B_\alpha$
⇒ {f < t} ⊂ $\bigcup_{\alpha<t} B_\alpha$ …………………(1)
Now let ⇒ $y \in \bigcup_{\alpha<t} B_\alpha$, this means $y \in \bigcup_{\gamma<t} B_\gamma$ and γ < t ⇒ $\gamma \in Dy, \gamma < t$
⇒ Inf $Dy \leq \gamma$ *and* $\gamma < t$
⇒ f(y) < γ and $\gamma < t$ ⇒ f(y) < t ⇒ $y \in$ {f < t}
⇒ $\bigcup_{\alpha<t} B_\alpha$ ⊂{f < t} ………………….(2)
Shows that {f < t } = $\bigcup_{\alpha<t} B_\alpha$.
Now to show {f ≤ t } = $\bigcap_{\beta>t}$ { $\bigcup_{\alpha<\beta} B_\alpha$ }
Define C_β = $\bigcup_{\alpha<\beta} D_\alpha$ Thus to show {f ≤ t } = $\bigcap_{\beta>t}$ {C_β}
Let any z ∈ {f ≤ t } and β > t . Then f(z) ≤ t and t < β ⇒ Inf Dz < β
⇒ There exists $\alpha \epsilon Dz$ s. t. $\alpha < \beta$
⇒ $z\epsilon D\alpha$ and $\alpha < \beta \Rightarrow z\epsilon C\beta \ \forall \beta > t \Rightarrow z\epsilon \bigcap_{\beta>t} C\beta$
⇒ {f ≤ t} ⊂ $\bigcap_{\beta>t} C\beta$ …………..(1)
Now take any w $\epsilon \bigcap_{\beta>t} C\beta \Rightarrow w \epsilon C\beta$ and $\beta > t \Rightarrow \beta\epsilon Dw$ and β > t
⇒ Inf $Dw \leq \beta, \beta$ > t
⇒ f(w) ≤ β, β > t ⇒ f(w) ≤ t ⇒ $w \epsilon$ { f ≥ t } ⇒ $\bigcap_{\beta>t} C\beta$ ⊂ { f ≥ t } …………..(2)
From (1) and (2) it shows that {f ≤ t } = $\bigcap_{\beta>t}$ { $\bigcup_{\alpha<\beta} B_\alpha$ }.

Corollary (1): If $B_\alpha < B_\beta$ whenever $\alpha < \beta$ then f ≤ α on B_α and f ≥ α on X-B_α.
Let x ∈ B_α then $\alpha \in D_x \Rightarrow inf D_x \leq \alpha \Rightarrow$ f(x) ≤ α, shows that f(x) ≤ α on B_α
Let f(y) < α ⇒ Inf $D_y < \alpha$ ⇒ there exist $\beta \in D_y$ s. t. $\beta < \alpha$
⇒ y ∈ B_β & $\beta < \alpha$ ⇒ y ∈ B_β and then $B_\beta \subset B_\alpha$ ⇒ y ∈ B_α
Hence y ∈ X − B_α then f(y) ≥ α
Shows that f ≥ α on X − B_α.

Corollary (2): If D is a subset of non negative real numbers then f is non negative.

Corollary (3): If D is countable and B_α is a measurable set for each α then f is a measurable function.

Note: Let D be any countable sub set of non negative real numbers. Suppose for each $\alpha \in$ D there is given a measurable set B_α s.t. B_α is contained in B_β whenever $\alpha < \beta$ then there exist a non negative measurable function f on X s. t. f ≥ α on B_α and f ≤ α on X − B_α.

Lemma (2): Suppose D is a countable set of non negative real numbers and for each α in D there is given a measurable set B_α of X such that μ (B_α- B_β) = 0 for $\alpha < \beta$ then there exist a non negative measurable function f on X such that f $\leq \alpha$ a. e. on B_α and f $\geq \alpha$ a. e. on X- B_α.

Proof: Define C = $\bigcup_{\alpha<\beta}(B_\alpha - B_\beta)$, then C is a countable union of null sets hence C is a null set. Define $B_\alpha^* = B_\alpha - C$ and $B_\beta^* = B_\beta - C$
Let $\alpha < \beta$ then $B_\alpha^* - B_\beta^* = (B_\alpha - B_\beta) - C = \phi$ [From the definition of C]
$\Rightarrow B_\alpha^* \subset B_\beta^*$. Hence by lemma (1) we can find a non negative measurable function f
On X s. t. f $\leq \alpha$ on B_α^* and f $\geq \alpha$ on X- B_α^*
As B_α and B_α^* differs only by null sets it follows that f $\leq \alpha$ a. e. on B_α and f $\geq \alpha$ a. e. on X- B_α.

Preposition: Let v be any signed measure on X and N be any measurable set, then the following statements are equivalent.

1. N is a null set for v.
2. N is a null set for $|v|$.
3. N is a null set for v^+ and v^-.

Definition
Let μ and v be two signed measures. Suppose $v(E) = 0$ whenever $\mu(E) = 0$ then we say that v *is absolutely continuous w.r.t.* μ and is denoted by $v \ll \mu$.

Remark: $v \ll \mu$ iff $|v| \ll |\mu|$.
Suppose $v \ll \mu$. Let $|\mu|(E) = 0$, this means E is a null set for $|\mu|$. $\Rightarrow$ E is a null set for μ hence it is a null set for $v \Rightarrow$ E is a null set for $|v| \Rightarrow |v| \ll |\mu|$.
Conversely assume that $|v| \ll |\mu|$. Let F be any null set for μ then F is null set for $|\mu|$
$\Rightarrow$ F is a null set for $|v|$ as $|v| \ll |\mu| \Rightarrow$ F is a null set for v. Hence $v \ll \mu$.

Radon Nikodym Theorem: Let $(X, \mathcal{A}, \mu)$ be a σ −finite measure space. Let v be any measure on $\mathcal{A}$ such that $v \ll \mu$ then there exist a non negative measurable function f on X such that $v(E) = \int_E f \, d\mu$ for every measurable set E. More over f is unique a.e. [μ].

Proof: First suppose that μ is finite.
Let D be the set of non negative rational numbers and for each $\alpha \in$ D define $v_\alpha = v - \alpha\mu$.
Then v_α is a signed measure.
Let (A_α, B_α) be Hahn-Decomposition for v_α where we have A_0 = A and $B_0 = \phi$ then
$B_\alpha - B_\beta = B_\alpha \cap B_\beta^c = B_\alpha \cap A_\beta \Rightarrow v_\alpha(B_\alpha - B_\beta) \leq 0$ [As B_α is a negative set for v_α]
and $v_\beta(B_\alpha - B_\beta) \geq 0$ [As A_β is a positive set for v_β]
$\Rightarrow v_\alpha(B_\alpha - B_\beta) \leq 0 \leq v_\beta(B_\alpha - B_\beta) \Rightarrow (v - \alpha\mu)(B_\alpha - B_\beta) \leq (v - \beta\mu)(B_\alpha - B_\beta)$
$\Rightarrow v(B_\alpha - B_\beta) - \alpha\mu(B_\alpha - B_\beta) \leq v(B_\alpha - B_\beta) - \beta\mu(B_\alpha - B_\beta)$
$\Rightarrow \beta\mu(B_\alpha - B_\beta) \leq \alpha\mu(B_\alpha - B_\beta) \Rightarrow \mu(B_\alpha - B_\beta) = 0 \; \forall \alpha < \beta$ …………..(1)
Hence by lemma (2) there exist non negative measurable function f on X such that for any real number t
$\{ f < t\} = \bigcup_{\alpha<t} B_\alpha$, $\{f \leq t\} = \bigcap_{\beta<t}\{\bigcup_{\alpha<\beta} B_\alpha\} \Rightarrow$ f $\leq \alpha$ on B_α a. e. [μ] , f $\geq \alpha$ on X-B_α a. e. [μ]
Also B_α is a negative set for v_β for all $\beta \leq \alpha$ and
A_α is a positive set for v_β for all $\beta \geq \alpha$ …………(2)
Define $A_\infty = \bigcap_\alpha A_\alpha$, let x $\in A_\infty$, then x $\in A_\alpha$ for all $\alpha \Rightarrow f(x) \geq \alpha, \forall\alpha \Rightarrow f(x) = \infty$
Shows that f = ∞ on A_∞ ……………(3)
Let E be any measurable sub set of A_∞ then E $\subset A_\alpha \; \forall\alpha \Rightarrow v_\alpha(E) \geq 0 \; \forall\alpha$
$\Rightarrow (v - \alpha\mu)(E) \geq 0 \; \forall\alpha \Rightarrow v(E) \geq \alpha\,\mu(E) \; \forall\alpha$
$\Rightarrow v(E) = \infty$ whenever $\mu(E) \neq 0$ ...……………(4)
Let $\beta \in$ D, F = $\{ f < \beta\}$ and S be any measurable sub set of F, we have F = $\bigcup_{\alpha<\beta} B_\alpha$
Since B_α is a negative set for v_β for every $\alpha < \beta$.
F is a negative set for $v_\beta \Rightarrow v_\beta(S) \leq 0 \Rightarrow (v - \beta\mu)(S) \leq 0$
$\Rightarrow v(S) \leq \beta\,\mu(S)$ ……………..(5)
In the same manner we can show that if T be any measurable sub set of $\{ f \geq \beta \}$ then
$v(T) \geq \beta\,\mu(T) \; \forall \beta \in D$ ……………….(6)
let E be any measurable set, Then Define $E_0 = E \cap A_\infty$ and suppose that $\mu(E_0) \geq 0$ then $v(E_0) = \infty$ [From (4)] Also f = ∞ on E_0 [From (3)]
This gives $\int_E f \, d\mu \geq \int_{E_0} f \, d\mu = \int_{E_0} \infty \, d\mu = \infty\,\mu(E_0) = \infty$

$\Rightarrow v$ (E) $\geq v(E_0) \Rightarrow v$ (E) = ∞ Therefore v (E) = $\int_E f\, d\mu$
Suppose now $\mu(E_0) = 0$ since $v \ll \mu$ we get $v(E_0) = 0$
Let R and N be any positive integers such that N be fixed for some time then define
$E_k = \{x \in E \;/\; \frac{k-1}{N} \leq f(x) \leq \frac{k}{N}\}$
From (5) *and* (6) we get $\frac{k-1}{N}\,\mu(E_k) \leq v(E_k) \leq \frac{k}{N}\,\mu(E_k)$ ……………….(7)
It is easy to verify that
E = $\bigcup_{k=0}^{\infty} E_k$ and the sets E_k are disjoint and measurable for k= 0, 1, 2, …………
Therefore v (E) = $\sum_{k=0}^{\infty} v(E_k)$ and $\mu(E) = \sum_{k=0}^{\infty}(\mu(E_k))$
v (E) = $\sum_{k=1}^{\infty} v(E_k)$ and $\mu(E) = \sum_{k=1}^{\infty}(\mu(E_k))$ ………………….(8)
because $v(E_0) = 0 = (E_0)$, Further $\int_E f\, d\mu = \sum_{k=0}^{\infty} \int_{E_k} f\, d\mu \Rightarrow \int_E f\, d\mu = \sum_{k=1}^{\infty} \int_{E_k} f\, d\mu$
[As $\mu(E_0) = 0$] …………….. (9)
On the set E_k we have $\frac{k-1}{N} \leq f(x) \leq \frac{k}{N}$, it gives
$\int_{E_k} \frac{k-1}{N}\, d\mu \leq \int_{E_k} f\, d\mu \leq \int_{E_k} \frac{k}{N}\, d\mu \Rightarrow \frac{k-1}{N}(\mu(E_k)) \leq \int_{E_k} f\, d\mu \leq \frac{k}{N}(\mu(E_k)$……….(10)
From (7) *we have* $-\frac{k}{N}(\mu(E_k)) \leq -v(E_k) \leq -\frac{k-1}{N}(\mu(E_k)$ …………………(11)
From (10) and (11) we get $-\frac{1}{N}(\mu(E_k)) \leq \int_{E_k} f\, d\mu - v(E_k) \leq \frac{1}{N}(\mu(E_k)$
$-\frac{1}{N}(\sum_{k=1}^{\infty}\mu(E_k)) \leq \sum_{k=1}^{\infty}\int_{E_k} f\, d\mu - \sum_{k=1}^{\infty} v(E_k) \leq \frac{1}{N}\sum_{k=1}^{\infty}\mu(E_k)$
$\Rightarrow -\frac{1}{N}\,\mu(E) \leq \int_E f\, d\mu - v(E) \leq \frac{1}{N}\,\mu(E)$ from (8) and (9) taking limits as n → ∞ we get
$\int_E f\, d\mu = v(E)$

Further now suppose that μ is σ finite then there exist a disjoint sequence of measurable sets {Xn} such that $\mu(Xn) < \infty$ ∀ n and X = $\bigcup_1^{\infty} X_n$.
Let v_n be the restriction of v to Xn and μ_n is a finite measure on X, v_n is a measure on Xn s.t. $v_n \ll \mu_n$ by what has been proved above there exist a non negative measurable function f_n on Xn such that $v_n(E) = \int_E f_n\, d\mu_n$ ∀ measurable sub sets E of Xn.
Define f on X by $f = f_n$ on Xn. Since Xn are disjoint it is clear that f is well defined on X. For any real number α we have $\{f \leq \alpha\}$ = $\{f \leq \alpha\} \cap X = \{f \leq \alpha\} \cap \{\bigcup_1^{\infty} X_n\}$
= $\bigcup_{n=1}^{\infty}\{f \leq \alpha\} \cap Xn\} = \bigcup_{n=1}^{\infty}\{f_n \leq \alpha\}$ [Because $f = f_n$ on Xn]
$\Rightarrow \{f \leq \alpha\}$ is measurable $\Rightarrow f$ is measurable.
Let E be any measurable sub set of X then E = E ∩ X = E ∩ $\{\bigcup_1^{\infty} X_n\} = \bigcup_{n=1}^{\infty}(E \cap Xn)$
$\Rightarrow v(E) = \sum_{n=1}^{\infty} v(E \cap Xn) = \sum_{n=1}^{\infty} v_n(E \cap Xn) = \sum_{n=1}^{\infty} \int_{E\cap X_n} f_n\, d^{\mu}n = \sum_{n=1}^{\infty} \int_{E\cap X_n} f\, d\mu = \int_{\cup(E\cap X_n)} f\, d\mu = \int_E f\, d\mu$. Hence Proved.

Uniqueness: Suppose that f & g be two non negative measurable functions on X such that $v(E) = \int_E f\, d\mu$ and $v(E) = \int_E g\, d\mu$ for every measurable set E. Which gives that $\int_E f\, d\mu = \int_E g\, d\mu$ on E $\Rightarrow \int_E f\, d\mu - \int_E g\, d\mu = 0 \Rightarrow \int_E (f-g)\, d\mu = 0 \Rightarrow f - g = 0$
$\Rightarrow f = g$ a. e. [μ].

Radon Nikodym Theorem for signed measures: Let (X, $\mathcal{A}, \mu$) be a σ –finite measure space. Let v be any finite signed measure on $\mathcal{A}$ such that $v \ll \mu$ then there exist an integrable function h on X such that $v(E) = \int_E h\, d\mu$ for every measurable set E of X.
Moreover h is unique a.e.(almost everywhere) [μ].

Proof: *Consider* v^+ and v^- these both are finite measures and $v^+ \ll \mu$ and $v^- \ll \mu$. Hence there exist integrable function f and g on X such that v^+ (E) = $\int_E f\, d\mu$ and v^- (E) = $\int_E f\, d\mu$ for every measurable set E of X. Take h = $f - g$ then h is integerable and for measurable set E
$v(E) = (v^+ - v^-)(E) = v^+(E) - v^-(E) = \int_E f\, d\mu - \int_E g\, d\mu = \int_E (f-g)\, d\mu = \int_E h\, d\mu$ [As f & g are integerable]

Uniqueness: Suppose h *and* h^* be two integerable functions on X such that
$v(E) = \int_E h\, d\mu$ and $v(E) = \int_E h^*\, d\mu$ for every measurable set E of X.

Then we have $\int_E h\, \mathrm{d}\mu = \int_E h^*\, \mathrm{d}\mu$ on E
$\Rightarrow \int_E (h - h^*)\, \mathrm{d}\mu = 0$ on E $\Rightarrow h - h^* = 0$ a. e. $[\mu] \Rightarrow h = h^*$ a. e. $[\mu]$. Proved.

Remark: If the condition of σ- finiteness is dropped Radon Nikodym Theorem may not hold. Consider the following example.
Take X = [0, 1] and $\mathcal{A}$ be the σ-algebra of Lebesuge measurable sets contained in X. Let μ be the counting measure and ϑ be the restriction of Lebesgue measure on X.
If $\mu(E) = 0$ then E = $\phi \Rightarrow \vartheta(E) = 0$. Which shows that $\ll \mu$.
Assume that Radon Nikodym Theorem holds.
Let f be non negative measurable function s. t. $v(E) = \int_E f\, \mathrm{d}\mu$ $\forall$ E measurable sub set of X.
Consider any x ∈ X and let E = {x} then $\vartheta(E) = 0 \Rightarrow \int_E f\, \mathrm{d}\mu = 0 \Rightarrow \int_{\{x\}} f\, \mathrm{d}\mu = 0 \Rightarrow f(x)\mu(x) = 0 \Rightarrow f(x) = 0$ $\forall$ x ∈ X
We also have $v(X) = \int_X f\, \mathrm{d}\mu \Rightarrow 1 = 0$ which is absurd, thus our assumption is wrong. Thus Radon Nikodym Theorem does not hold in this case.

References

i. Halmos. P.R., Measure Theory (Springer, New York, 1974).
ii. Qiang. G, Further discussion on the Hann decomposition theorem for signed fuzzy measure, Fuzzy Sets and Systems 70 (1995) 89-95.
iii. Kumar. A. ,& Rao. J.V. (2010) Jordan Decomposition and its Uniqueness of Signed Lattice Measure. Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 9, 431 – 438.