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Application of Fractional Calculus on Certain New Subclasses of Analytic Functions

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Abstract: *For function* $f(z)$ *of the form:*

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad z \in U$$

which are analytic and p-valent in the open unit disk $U = \{z : |z| < 1\}$.
The authors study certain new suclasses of starlike, convex and spiralike functions using fractional calculus techniques. Also, they obtained coefficient inequality as well as distortion theorems for the fractional derivative and fractional integration for functions belonging to these subclasses.
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1. Introduction and Preliminary Definitions

Let $S(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad p \in N,\ z \in U \qquad (1)$$

which are analytic and p-valent in the open unit disk $U = \{z : |z| < 1\}$.
Here, we recall the following classes of analytic functions.
The function $f(z)$ of the form (1) is said to be the class of starlike functions with respect to the origin if and only if

$$\Re\left\{\frac{z f'(z)}{f(z)}\right\} > 0 \qquad z \in U \qquad (2)$$

Also, function $f(z)$ of the form (1) is said to be the class of convex functions with respect to the origin if and only if

$$\Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > 0 \qquad z \in U \qquad (3)$$

and for θ real $\left(|\theta| < \frac{\pi}{2}\right)$, $f(z)$ is said to be the class of sprial-like functions with respect to the origin if and only if

$$\Re\left\{e^{i\theta} \frac{z f'(z)}{f(z)}\right\} > 0 \qquad z \in U \qquad (4).$$

The three classes defined above are well known and have been studied by several authors and researchers with their results scattered in many literatures (see[2,3,5] among others) .

Now, in [1], El-Ashwah and Aouf defined certain integral operator $J_p^m(\lambda,l)f(z)$ as follows:

$$J_p^m(\lambda,l)f(z)=\left(\frac{p+l}{\lambda}\right)z^{p-\left(\frac{p+l}{\lambda}\right)}\int_0^z t^{\left(\frac{p+l}{\lambda}\right)-p-1}J_p^{m-1}(\lambda,l)f(t)dt$$

$$=\underbrace{J_p^1(\lambda,l)\left(\frac{z^p}{1-z}\right)*J_p^1(\lambda,l)\left(\frac{z^p}{1-z}\right)*\cdots*J_p^1(\lambda,l)\left(\frac{z^p}{1-z}\right)}_{m-times}*f(z). \qquad (5)$$

$$(f(z)\in S(p);m\in N;z\in U)$$

Also, it was noted that if $f(z)\in S(p)$, then from (1) and (5), we have that

$$J_p^m(\lambda,l)f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m a_k z^k \qquad m\in N_0. \qquad (6)$$

From (6), it is trivial to verify that

$$\lambda z\left(J_p^{m+1}(\lambda,l)f(z)\right)'=(l+p)J_p^m(\lambda,l)f(z)-\left(l+p(1-\lambda)\right)J_p^{m+1}(\lambda,l)f(z)$$

$$\left(\lambda>0:l\geq 0;m\in N_0;p\in N\ and\ z\in U\right).$$

With various choices of parameters $m,l\ and\ \lambda$ several known operators were obtained (see[1]) .
For the sake of our present investigation the following definitions shall be necessary.

Definition 1 [6,7]: The fractional integral of order $\delta\,(0<\delta)$ is defined by

$$D_z^{-\delta}f(z)=\frac{1}{\Gamma(\delta)}\int_0^z\frac{f(t)}{(z-t)^{1-\delta}}dt=\frac{1}{\Gamma(\delta+2)}z^{p+\delta}+\sum_{k=p+1}^{\infty}\frac{\Gamma(k+p)}{\Gamma(k+p-\delta)}a_k z^{k+\delta} \qquad (7)$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 2 [6,7]: The fractional derivative of order $\delta\,(0\leq\delta<1)$ is defined by

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\delta}}dt=\frac{1}{\Gamma(2-\delta)}z^{p-\delta}+\sum_{k=p+1}^{\infty}\frac{\Gamma(k+p)}{\Gamma(k+p-\delta)}a_k z^{k-\delta} \qquad (8)$$

where $f(z)$ is as defined in 1 and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Now, Komatu in [4] defined the integral operator of f denoted by P_λ^μ by the following:

$$P_\lambda^\mu(f(z))=\frac{(\psi+1)^\mu}{\Gamma(\mu)}\int_0^1 t^{\psi-1}\left(\log\frac{1}{t}\right)^{\mu-1}f(t)dt=z^p+\sum_{k=p+1}^{\infty}\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^k \qquad (9)$$

$$\left(\psi>-1;\ \mu>0;\ f\in S(p)\right).$$

The above operator in (9) is usually called Komatu operator.

At this juncture, we shall recall that given functions $f(z)$ of the form (1) and $g(z)=z^p+\sum_{k=p+1}^{\infty}b_k z^k$ in $S(p)$, the Hadamard product or convolution of $f\ and\ g$ denoted by $(f*g)(z)$ is defined as

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_k b_k z^k.$$

Denoting the convolution of (6) and (9) by $G_{p,\psi}^{m,\mu}(\lambda,l)f(z)$, we have that

$$G_{p,\psi}^{m,\mu}(\lambda,l)f(z) = J_p^m(\lambda,l)f(z) * P_\lambda^\mu(f(z))$$
$$= z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^k \qquad (10)$$
$$(\psi > -1;\ \mu > 0;\ m \in N_0;\ p \in N; l \ge 0;\ \lambda > 0\ \ and\ \ z \in U).$$

Remark:
Supposing $m = 0\ and\ p = 1,$ the operator in (10) immediately yields Komatu operator.

Now, using (10), the author wishes to introduce the new subclass $S_{p,\psi}^{m,\mu}(l,\alpha,\beta,\theta,\lambda)$ which satisfy the following condition:

$$\left|\frac{R-p}{2\alpha\left[R-\left(p+(1-\gamma)e^{-i\theta}\cos\theta\right)\right]-R}\right|<1 \qquad (11)$$

where $R = \dfrac{\beta z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)' + (1-\beta)z^2\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)''}{\beta(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)) + (1-\beta)z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)'}$.

$$\left(|\theta|<\frac{\pi}{2};\ \psi>-1;\ \mu>0;\ m\in N_0;\ 0\le\beta\le1;\ 0<\alpha\le1;\ 0\le\gamma<1\ \ p\in N; l\ge0;\ \lambda>0\ \ and\ \ z\in U\right).$$

2. Main Results

The following theorem is the coefficient inequality for functions in the class $S_{p,\psi}^{m,\mu}(l,\alpha,\beta,\theta,\lambda)$.

Theorem 1: Let $f(z)$ be of the form (1) , then $f(z)$ is in the class $S_{p,\psi}^{m,\mu}(l,\alpha,\beta,\theta,\lambda)$ if the following condition is satisfied.

$$\sum_{k=p+1}^{\infty}\left[\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)|e^{-i\theta}\cos\theta|-(1-\beta)\Phi_1\right]\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k$$
$$\le \alpha p\beta(1-\gamma)|e^{-i\theta}\cos\theta| + (1-\beta)\Phi_2 \qquad (12)$$

where

$\Phi_1 = \left(\alpha k(1-\gamma)|e^{-i\theta}\cos\theta| - k(1-\alpha)(k-p-1)\right) and\ \Phi_2 = \alpha p\left[p-1+(1-\gamma)|e^{-i\theta}\cos\theta|\right]$

and

$$f(z) = z^p + \frac{\alpha p\beta(1-\gamma)|e^{-i\theta}\cos\theta| + (1-\beta)\Phi_2}{\left[\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)|e^{-i\theta}\cos\theta|-(1-\beta)\Phi_1\right]\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu} z^{p+1} \qquad (13)$$

$$\left(|\theta|<\frac{\pi}{2};\ \psi>-1;\ \mu>0;\ m\in N_0;\ 0\le\beta\le1;\ 0<\alpha\le1;\ 0\le\gamma<1\ \ p\in N; l\ge0;\ \lambda>0\ \ and\ \ z\in U\right).$$

Proof: From (11), we have that

$$\left|\frac{(\beta-(1-\beta)p)z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)' - \beta p\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right) + (1-\beta)z^2\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)''}{\begin{array}{l}\left[2\alpha\left(p-(1-\beta)(p-(1-\gamma)e^{-i\theta}\cos\theta)\right)-(\beta-(1-\beta)p)\right]z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)' - \\ \left[2\alpha\beta(p-(1-\gamma)e^{-i\theta}\cos\theta)-\beta p\right]\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right) + \left[(1-\beta)(2\alpha-1)\right]z^2\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)''\end{array}}\right| < 1. \qquad (14)$$

This implies that

$$\left|(1-\beta)z^2\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)'' + (\beta-(1-\beta)p)z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)' - \beta p\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)\right|$$

$$-\left|\begin{array}{l}[(1-\beta)(2\alpha-1)]z^2\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)'' + \left[2\alpha\left(p-(1-\beta)(p-(1-\gamma)e^{-i\theta}\cos\theta)\right)-(\beta-(1-\beta)p)\right]z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)' \\ -\left[2\alpha\beta(p-(1-\gamma)e^{-i\theta}\cos\theta)-\beta p\right]\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)\end{array}\right|$$

$$=\left|\sum_{k=p+1}^{\infty}[\beta(k-p)-(1-\beta)(kp-k(k-1))]\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^k - p(1-\beta)z^p\right|$$

$$-\left|\begin{array}{l}\sum_{k=p+1}^{\infty}\left[\begin{array}{l}2\alpha k[p-(1-\beta)\left(p-(1-\gamma)e^{-i\theta}\cos\theta\right)]-k(\beta-(1-\beta)p)+\beta p-2\alpha\beta\left(p-(1-\gamma)e^{-i\theta}\cos\theta\right)\\ +(1-\beta)(2\alpha-1)k(k-1)\end{array}\right] \\ \left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^k \\ +\left[\begin{array}{l}2\alpha p[p-(1-\beta)\left(p-(1-\gamma)e^{-i\theta}\cos\theta\right)]-p(\beta-(1-\beta)p)+\beta p-2\alpha\beta\left(p-(1-\gamma)e^{-i\theta}\cos\theta\right)\\ +(1-\beta)(2\alpha-1)p(p-1)\end{array}\right]z^p\end{array}\right|$$

$$\leq\sum_{k=p+1}^{\infty}2\left[\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,|+(1-\beta)\left[k(1-\alpha)(k-p-1)-\alpha k(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,|\right]\right]$$

$$\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k-\left[2\alpha p\left[p-\beta\left(p-(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,|\right)\right]+2\alpha p(1-\beta)\left((1-\gamma)\,|\,e^{-i\theta}\cos\theta\,|-1\right)\right]\leq 0.$$

By maximum modulus principle

$$f(z)\in S_{p,\psi}^{m,\mu}(l,\alpha,\beta,\theta,\lambda).$$

Conversely, supposing

$$\left|\frac{R-p}{2\alpha\left[R-\left(p+(1-\gamma)e^{-i\theta}\cos\theta\right)\right]-R}\right|$$

$$=\left|\frac{(\beta-(1-\beta)p)z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)'-\beta p\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)+(1-\beta)z^2\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)''}{\begin{array}{l}\left[2\alpha\left(p-(1-\beta)(p-(1-\gamma)e^{-i\theta}\cos\theta)\right)-(\beta-(1-\beta)p)\right]z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)'-\left[2\alpha\beta(p-(1-\gamma)e^{-i\theta}\cos\theta)-\beta p\right]\\ \left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)+[(1-\beta)(2\alpha-1)]z^2\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)''\end{array}}\right|$$

$$=\left|\frac{\sum_{k=p+1}^{\infty}[\beta(k-p)-(1-\beta)(kp-k(k-1))]\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^{k-p}}{\begin{array}{l}2\alpha p(p-\beta)+2\alpha\beta\left(1-(1-\gamma)e^{-1\theta}\cos\theta\right)-(1-\beta)\left[2\alpha p\left(p-(1-\gamma)e^{-1\theta}\cos\theta\right)+p^2-(2\alpha-1)p(p-1)\right]\\ -\sum_{k=p+1}^{\infty}\left[\beta(k-p)-2\alpha p(k-\beta)-2\alpha\beta(1-\gamma)e^{-1\theta}\cos\theta-(1-\beta)\left(kp-(1-2\alpha)k(k-1)-2\alpha k\left(p-(1-\gamma)e^{-1\theta}\cos\theta\right)\right)\right]\\ \left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^{k-p}\end{array}}\right|<1$$

where $R = \dfrac{\beta z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)' + (1-\beta)z^2\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)''}{\beta(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)) + (1-\beta)z\left(G_{p,\psi}^{m,\mu}(\lambda,l)f(z)\right)'}$.

Since $|\Re(z)| < z$ for all z, we have

$$\Re\left\{\frac{\sum_{k=p+1}^{\infty}\left[\beta(k-p)-(1-\beta)(kp-k(k-1))\right]\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^{k-p}}{\begin{array}{l}2\alpha p(p-\beta)+2\alpha\beta-2\alpha\beta(1-\gamma)e^{-i\theta}\cos\theta-(1-\beta)\left(2\alpha p\left(p-(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right)+p\left[p+(1-2\alpha)(p-1)\right]\right)\\ -\sum_{k=p+1}^{\infty}\left[\begin{array}{l}\beta(k-p)-2\alpha p(k-\beta)-2\alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|-\\(1-\beta)\left(kp-(1-2\alpha)k(k-1)-2\alpha kp+2\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right)\end{array}\right]\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^{k-p}\end{array}}\right\} < 1 \quad (15)$$

If we choose the value of z on the real axis so that $G_{p,\psi}^{m,\mu}(\lambda,l)f(z)$ *is real*.

Let $z \to 1^-$, through real values, then we can express (15) as

$$\sum_{k=p+1}^{\infty}\left[\begin{array}{l}\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|-\\(1-\beta)\left(\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|-k(k-p-1)(1-\alpha)\right)\end{array}\right]\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k z^{k-p}$$

$$\leq \alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|-\alpha p(1-\beta)\left[1-p-(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right].$$

Corollary 1: Let $f(z)\in S_{p,\psi}^{m,\mu}(l,\alpha,\beta,\theta,\lambda)$, *then*

$$a_k \leq \frac{\alpha p\beta(1-\gamma)|e^{-i\theta}\cos\theta|-\alpha p(1-\beta)\left(1-p-(1-\gamma)|e^{-i\theta}\cos\theta|\right)\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{-m}\left(\frac{\psi+1}{\psi+k}\right)^{-\mu}}{\left[\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)|e^{-i\theta}\cos\theta|-(1-\beta)\left(\alpha k(1-\gamma)|e^{-i\theta}\cos\theta|-k(k-p-1)(1-\alpha)\right)\right]}.$$

$$k \geq p+1,\ z\in U$$

Corollary 2: Let $f(z)\in S_{p,\psi}^{m,\mu}(l,\alpha,1,\theta,\lambda)$, *then*

$$\sum_{k=p+1}^{\infty}\frac{\left[(k-p)-\alpha p(k-1)+\alpha(1-\gamma)|e^{-i\theta}\cos\theta|\right]}{\alpha p(1-\gamma)|e^{-i\theta}\cos\theta|}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k \leq 1$$

The result is sharp for function $f(z)$ given by

$$f(z)=z^p+\frac{\alpha p(1-\gamma)|e^{-i\theta}\cos\theta|}{\left[(k-p)-\alpha p(k-1)+\alpha(1-\gamma)|e^{-i\theta}\cos\theta|\right]\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu}z^{p+1}\leq 1,\ k\geq p+1.$$

Corollary 3: Let $f(z)\in S_{1,\psi}^{0,\mu}(0,\alpha,1,\theta,1)$, *then*

$$\sum_{k=p+1}^{\infty}\frac{\left[(k-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\theta}\cos\theta|\right]}{\alpha(1-\gamma)|e^{-i\theta}\cos\theta|}\left(\frac{\psi+1}{\psi+k}\right)^\mu a_k \leq 1$$

This result is due to Waggas [7] and it is sharp for function $f(z)$ given by

$$f(z)=z^{p}+\frac{\alpha(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,|}{\left[1+\alpha\left((1-\gamma)\,|\,e^{-i\theta}\cos\theta\,|-1\right)\right]\left(\frac{\psi+1}{\psi+k}\right)^{\mu}}z^{2}\le 1,\ \ k\ge 2.$$

Theorem 2: Let $f(z)$ be of the form (1). If $f(z)\in S_{p,\psi}^{m,\mu}(l,\alpha,\beta,\theta,\lambda,\delta)$, then

$$\left|D_{z}^{-\delta}f(z)\right| \le \frac{1}{\Gamma(\delta+2)}|z|^{p+\delta}\left\{1+\frac{\Gamma(p+2)\Gamma(\delta+2)\begin{Bmatrix}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|-\alpha p(1-\beta)\\ \left[1-p-(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{Bmatrix}}{\Gamma(p+\delta+2)\begin{Bmatrix}\beta(k-p)-\alpha p(k-\beta)+\\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -(1-\beta)\begin{bmatrix}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -k(k-p-1)(1-\alpha)\end{bmatrix}\end{Bmatrix}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}}\right\}|z| \tag{16}$$

and

$$\left|D_{z}^{-\delta}f(z)\right| \ge \frac{1}{\Gamma(\delta+2)}|z|^{p+\delta}\left\{1-\frac{\Gamma(p+2)\Gamma(\delta+2)\begin{Bmatrix}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|-\alpha p(1-\beta)\\ \left[1-p-(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{Bmatrix}}{\Gamma(p+\delta+2)\begin{Bmatrix}\beta(k-p)-\alpha p(k-\beta)+\\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -(1-\beta)\begin{bmatrix}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -k(k-p-1)(1-\alpha)\end{bmatrix}\end{Bmatrix}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}}\right\}|z|. \tag{17}$$

The inequalities (16) and (17) are attained for the function *f(z)* given by (13).

Proof: Using theorem 1, we have that

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{\alpha p\beta(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,| - \alpha p(1-\beta)\left(1-p-(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,|\right)\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{-m}\left(\frac{\psi+1}{\psi+k}\right)^{-\mu}}{\left[\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,| - (1-\beta)\left(\alpha k(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,| - k(k-p-1)(1-\alpha)\right)\right]}.$$

$$k \geq p+1,\ z\in U \qquad (18)$$

By definition (7), we have that

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{\delta+p} + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+\delta)} a_k z^{k+\delta}$$

and

$$\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+1+\delta)} a_k z_k = z^p + \sum_{k=p+1}^{\infty} \Phi(k) a_k z_k \qquad (19)$$

where $\Phi(k) = \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+1+\delta)}$.

We can observe that $\Phi(k)$ is an increasing function of k and $0 \leq \Phi(k)\Phi(p+1) = \sum_{k=p+1}^{\infty} \frac{\Gamma(p+2)\Gamma(2+\delta)}{\Gamma(p+\delta+2)}$.

Now, using (18) and (19), we obtain

$$\left|\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)\right| \leq |\,z\,|^p + \Phi(p+1)\,|\,z\,|^{p+1} \sum_{k=p+1}^{\infty} a_k \leq |z|^p$$

$$+ \frac{\Gamma(p+2)\Gamma(\delta+2)\left\{\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| - \alpha p(1-\beta)\left[1-p-(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\right\}}{\Gamma(p+\delta+2)\left\{\begin{matrix}\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -(1-\beta)\left[\begin{matrix}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -k(k-p-1)(1-\alpha)\end{matrix}\right]\end{matrix}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}} |z|^{p+1}$$

which gives (16). Also, we can have that

$$\left|\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)\right| \geq |\,z\,|^p + \Phi(p+1)\,|\,z\,|^{p+1} \sum_{k=p+1}^{\infty} a_k \geq |z|^p$$

$$- \frac{\Gamma(p+2)\Gamma(\delta+2)\left\{\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| - \alpha p(1-\beta)\left[1-p-(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\right\}}{\Gamma(p+\delta+2)\left\{\begin{matrix}\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -(1-\beta)\left[\begin{matrix}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -k(k-p-1)(1-\alpha)\end{matrix}\right]\end{matrix}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}} |z|^{p+1}.$$

which gives (17).

Theorem 3: Let $f(z)$ be of the form (1). If $f(z) \in S_{p,\psi}^{m,\mu}(l,\alpha,\beta,\theta,\lambda,\delta)$, then

$$\left|D_z^{\delta} f(z)\right|$$

$$\leq \frac{1}{\Gamma(2-\delta)}|z|^{p-\delta}\left\{1+\frac{\Gamma(p+2)\Gamma(2-\delta)\left\{\begin{array}{l}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|+\alpha p(1-\beta)\\ \left[p-1+(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{array}\right\}}{\Gamma(p-\delta+2)\left\{\begin{array}{l}\beta(k-p)-\alpha p(k-\beta)+\\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -(1-\beta)\left[\begin{array}{l}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -k(k-p-1)(1-\alpha)\end{array}\right]\end{array}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}}\right\}|z| \text{ and}$$

(20)

$$\left|D_z^{\delta} f(z)\right|$$

$$\geq \frac{1}{\Gamma(2-\delta)}|z|^{p-\delta}\left\{1-\frac{\Gamma(p+2)\Gamma(2-\delta)\left\{\begin{array}{l}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|+\alpha p(1-\beta)\\ \left[p-1+(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{array}\right\}}{\Gamma(p-\delta+2)\left\{\begin{array}{l}\beta(k-p)-\alpha p(k-\beta)+\\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -(1-\beta)\left[\begin{array}{l}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -k(k-p-1)(1-\alpha)\end{array}\right]\end{array}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}}\right\}|z|$$

(21)

Proof: Using definition 2, we have that

$$D_z^{\delta} f(z)=\frac{1}{\Gamma(2-\delta)} z^{p-\delta}+\sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+\delta)} a_k z^{k-\delta} \quad .$$

This implies that,

$$\Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z)=z^p+\sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)} a_k z_k=z^p+\sum_{k=p+1}^{\infty} \Phi(k) a_k z_k ,$$

where $\Phi(k)$ is a decreasing function of k and for $k \geq p+1$

$$\Phi(k)\Phi(p+1)=\sum_{k=p+1}^{\infty} \frac{\Gamma(p+2)\Gamma(2-\delta)}{\Gamma(p+2-\delta)} .$$

Since, $\sum_{k=p+1}^{\infty} a_k \le \dfrac{\alpha p\beta(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,| - \alpha p(1-\beta)\left(1-p-(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,|\right)\left(\dfrac{p+l}{p+\lambda(k-1)+l}\right)^{-m}\left(\dfrac{\psi+1}{\psi+k}\right)^{-\mu}}{\left[\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,| - (1-\beta)\left(\alpha k(1-\gamma)\,|\,e^{-i\theta}\cos\theta\,| - k(k-p-1)(1-\alpha)\right)\right]}.$

$$k \ge p+1,\ z \in U$$

Then,

$$\left|\Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z)\right| \le |\,z\,|^{p} + \Phi(p+1)\,|\,z\,|^{p+1} \sum_{k=p+1}^{\infty} a_k \le |z|^{p}$$

$$+ \frac{\Gamma(p+2)\Gamma(2-\delta)\left\{\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| + \alpha p(1-\beta)\left[p-1+(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\right\}}{\Gamma(p-\delta+2)\left\{\begin{array}{l}\beta(k-p)-\alpha p(k-\beta)+\alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -(1-\beta)\left[\begin{array}{l}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -k(k-p-1)(1-\alpha)\end{array}\right]\end{array}\right\}\left(\dfrac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\dfrac{\psi+1}{\psi+k}\right)^{\mu}} |z|^{p+1}$$

Thus,

$$\left|D_z^{\delta} f(z)\right|$$

$$\le \frac{1}{\Gamma(2-\delta)}|z|^{p-\delta}\left\{1+\frac{\Gamma(p+2)\Gamma(2-\delta)\left\{\begin{array}{l}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| + \alpha p(1-\beta) \\ \left[p-1+(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{array}\right\}}{\Gamma(p-\delta+2)\left\{\begin{array}{l}\beta(k-p)-\alpha p(k-\beta)+ \\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -(1-\beta)\left[\begin{array}{l}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -k(k-p-1)(1-\alpha)\end{array}\right]\end{array}\right\}\left(\dfrac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\dfrac{\psi+1}{\psi+k}\right)^{\mu}}\right\}|z| \text{ and}$$

$$\left|D_z^{\delta} f(z)\right|$$

$$\geq \frac{1}{\Gamma(2-\delta)}|z|^{p-\delta}\left\{1-\frac{\Gamma(p+2)\Gamma(2-\delta)\left\{\begin{array}{l}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|+\alpha p(1-\beta)\\ \left[p-1+(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{array}\right\}}{\Gamma(p-\delta+2)\left\{\begin{array}{l}\beta(k-p)-\alpha p(k-\beta)+\\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -(1-\beta)\left[\begin{array}{l}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -k(k-p-1)(1-\alpha)\end{array}\right]\end{array}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}}\right\}|z| .$$

Corollary 4: Let $f(z)\in S^{m,\mu}_{p,\psi}(l,\alpha,1,\theta,\lambda,\delta)$, *then*

$$\frac{|z|^{p+1}}{2}\left\{1-\frac{2\left\{\begin{array}{l}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|+\alpha p(1-\beta)\\ \left[p-1+(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{array}\right\}}{\Gamma(p+2)\left\{\begin{array}{l}\beta(k-p)-\alpha p(k-\beta)+\\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -(1-\beta)\left[\begin{array}{l}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -k(k-p-1)(1-\alpha)\end{array}\right]\end{array}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}}\right\}|z| .$$

$$\leq \left|\int_0^z f(t)\,dt\right| \leq \frac{|z|^{p+1}}{2}\left\{1+\frac{2\left\{\begin{array}{l}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|+\alpha p(1-\beta)\\ \left[p-1+(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{array}\right\}}{\Gamma(p+2)\left\{\begin{array}{l}\beta(k-p)-\alpha p(k-\beta)+\\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -(1-\beta)\left[\begin{array}{l}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\\ -k(k-p-1)(1-\alpha)\end{array}\right]\end{array}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^{m}\left(\frac{\psi+1}{\psi+k}\right)^{\mu}}\right\}|z| .$$

Proof:

(a.) Supposing we follow definition (1) and theorem 2 with $\delta=1$, we have that

$D_z^{-1}f(z)=\int_0^z f(t)\,dt$. Thus, the following corollary follows immediately.

(b.) By definition 2 and theorem 3 with $\delta = 1$, we have that $D_z^0 f(z) = \frac{d}{dz}\int_0^z f(t)dt = f(z)$

Corollary 5: $D_z^{-\delta} f(z)$ and $D_z^{\delta} f(z)$ are included in the unit disk with centre at the origin and radii

$$\leq \frac{1}{\Gamma(\delta+2)}\left\{1+\frac{\Gamma(p+2)\Gamma(\delta+2)\left\{\begin{matrix}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| - \alpha p(1-\beta) \\ \left[1-p-(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{matrix}\right\}}{\Gamma(p+\delta+2)\left\{\begin{matrix}\beta(k-p)-\alpha p(k-\beta)+ \\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -(1-\beta)\begin{bmatrix}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -k(k-p-1)(1-\alpha)\end{bmatrix}\end{matrix}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu}\right\}|z|$$

and

$$\frac{1}{\Gamma(2-\delta)}\left\{1+\frac{\Gamma(p+2)\Gamma(2-\delta)\left\{\begin{matrix}\alpha p\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| + \alpha p(1-\beta) \\ \left[p-1+(1-\gamma)\left|e^{-i\theta}\cos\theta\right|\right]\end{matrix}\right\}}{\Gamma(p-\delta+2)\left\{\begin{matrix}\beta(k-p)-\alpha p(k-\beta)+ \\ \alpha\beta(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -(1-\beta)\begin{bmatrix}\alpha k(1-\gamma)\left|e^{-i\theta}\cos\theta\right| \\ -k(k-p-1)(1-\alpha)\end{bmatrix}\end{matrix}\right\}\left(\frac{p+l}{p+\lambda(k-1)+l}\right)^m\left(\frac{\psi+1}{\psi+k}\right)^\mu}\right\}|z|.$$

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