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Observations on the Non-Homogeneous Equation of the Eighth Degree with Six Unknowns $x^6 - y^6 - 2z^3 = (w^2 - p^2)T^6$.

S. Vidhyalakshmi
Department of Mathematics, Shrimati Indira Gandhi College, Trichy, India
M. A. Gopalan
Department of Mathematics, Shrimati Indira Gandhi College, Trichy, India
K. Lakshmi
Department of Mathematics, Shrimati Indira Gandhi College, Trichy, India

Abstract:
We obtain infinitely many non-zero integer sextuples (x, y, z, w, p, T) satisfying the non-homogeneous equation of eighth degree with six unknowns given by $x^6 - y^6 - 2z^3 = (w^2 - p^2)T^6$. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, Pyramidal numbers, Star numbers, Stella Octangular numbers, Octahedral numbers, Pronic number, Jacobsthal number, Jacobsthal-Lucas number, keynea number, Centered pyramidal numbers are presented

Key words: *Non-homogeneous equation, integral solutions, polygonal numbers, Pyramidal numbers, Centered pyramidal numbers*
MSC 2000 Mathematics subject classification: 11D41

Notations:

$T_{m,n}$ -*Polygonal number of rank* n *with size* m

P_n^m - *Pyramidal number of rank* n *with size* m

SO_n -*Stella octangular number of rank* n

S_n -*Star number of rank* n

PR_n - *Pronic number of rank* n

OH_n - *Octahedral number of rank* n

J_n -*Jacobsthal number of rank of* n

j_n - *Jacobsthal-Lucas number of rank* n

KY_n -*keynea number of rank* n

$CP_{n,3}$ - *Centered Triangular pyramidal number of rank* n

$CP_{n,6}$ - *Centered hexagonal pyramidal number of rank* n

$CP_{n,7}$ - *Centered heptagonal pyramidal number of rank* n

$$GF_n(k,s) = \frac{\alpha^n - \beta^n}{\alpha - \beta}\left(\alpha = \frac{k + \sqrt{k^2 + 4s}}{2}, \beta = \frac{k - \sqrt{k^2 + 4s}}{2}\right)$$ -*Generalised Fibonacci sequence*

$$GL_n(k,s) = \alpha^n + \beta^n\left(\alpha = \frac{k + \sqrt{k^2 + 4s}}{2}, \beta = \frac{k - \sqrt{k^2 + 4s}}{2}\right)$$ -*Generalised Lucas sequence*

1. Introduction

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1-3].Particularly in [4, 5] special equations of sixth degree with four and five unknowns are studied. In [6-9] heptic equations with three and five unknowns are analysed. In [10-11] equations of eighth degree with five and six unknowns are analysed. This paper concerns with the problem of determining non-trivial integral solution of the non- homogeneous equation of eighth degree with six unknowns given by $x^6 - y^6 - 2z^3 = (w^2 - p^2)T^6$. A few relations between the solutions and the special numbers are presented.

2. Method of Analysis:

The Diophantine equation representing the non- homogeneous equation of degree eight is given by

$$x^6 - y^6 - 2z^3 = (w^2 - p^2)T^6 . \qquad (1)$$

Introduction of the transformations

$$x = u + v, y = u - v, z = 2uv, w = uv + 3, p = uv - 3 \qquad (2)$$

in (1) leads to

$$u^2 + v^2 = T^3 \qquad (3)$$

The above equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1)

2.1. Approach 1

Let $T = a^2 + b^2$ (4)

Substituting (4) in (3) and using the method of factorisation, define

$$(u + iv) = (a + ib)^3 \qquad (5)$$

Equating real and imaginary parts in (5) we get

$$\left.\begin{aligned} u &= a^3 - 3ab^2 \\ v &= 3a^2b - b^3 \end{aligned}\right\} \qquad (6)$$

In view of (2), (4) and (6), the corresponding values of x, y, z, w, p, T are represented by

$$\left.\begin{aligned} x &= a^3 - 3ab^2 + 3a^2b - b^3 \\ y &= a^3 - 3a^2b - 3ab^2 + b^3 \\ z &= 2(a^3 - 3ab^2)(3a^2b - b^3) \\ w &= (a^3 - 3ab^2)(3a^2b - b^3) + 3 \\ p &= (a^3 - 3ab^2)(3a^2b - b^3) - 3 \\ T &= a^2 + b^2 \end{aligned}\right\} \qquad (7)$$

The above values of x, y, z, w, p and T satisfies the following relations:

1. $w(a,a) + p(a,a) + z(a,a) - 16[2T_{3,a^3} - 6p_a^3 + 6T_{3,a} + SO_a - 2CP_{a,6}] = 0$
2. $6p_a^3 - x(a,1) - 1 \equiv 0 (\bmod 5)$
3. $3\,[T(2^{2n}, 2^{2n}) - 2KY_{2n} + j_{2n+2} - 1]$ is a nasty number
4. $3[x(a,a) + y(a,a) + w(a,a) - p(a,a) + 2(6p_a^4 - 6T_{3,a} + 3T_{4,a} - T_{8,a})]$ is a cubic integer.
5. $9[64T^2_{3,a^3} - 8CP_{a,6}(2p_a^5 - T_{4,a}) - 32(CP_{a,6})^3 - T^3(a,a) - x(a,a).y(a,a) - w(a,a).p(a,a)]$ is a biquadratic integer

2.2. Approach 2

The solution of (3) can also be obtained as

$$u = a(a^2 + b^2), v = b(a^2 + b^2), T = a^2 + b^2 \quad (8)$$

In view of (8) and (2), the integral solutions of (1) is obtained as

$$\left.\begin{aligned} x &= (a^2 + b^2)(a + b) \\ y &= (a^2 + b^2)(a - b) \\ z &= 2ab(a^2 + b^2)^2 \\ w &= 2ab(a^2 + b^2)^2 + 3 \\ p &= 2ab(a^2 + b^2)^2 - 3 \\ T &= a^2 + b^2 \end{aligned}\right\} \quad (9)$$

Some interesting relations between the values of x, y, z, w, p and T **:**

1. $x(a,a) + y(a,a) - 8CP_{a,3} + 8CP_{a,6} + 4SO_a = 0$
2. $2x(2a,a).y(2a,a) - 75(2P_a^5.SO_a + T_{4,a^2} - SO_a.T_{4,a}) = 0$. 3. $w(a,a) + p(a,a) - 16T_{3,a}.T_{4,a^2} + 8CP_{a,6}.T_{4,a} = 0$
4. $4x(a,b)[x^2(a,b) + y^2(a,b) + 2z(a,b)]$ is a cubic integer.
5. $T(2^{2n}, 2^{2n}) - 2KY_{2n} + 2j_{2n+1} = 0$
6. $z(2^{2n}, 2^{2n}) - 8KY_{6n} + 24j_{6n+1}$ is a biquadratic integer

2.3. Approach 3

Now, rewrite (3) as,

$$u^2 + v^2 = T^3 \times 1 \quad (10)$$

Also 1 can be written as

$$1 = (-i)^n (i)^n \quad (11)$$

Substituting (4) and (9) in (8) and using the method of factorisation, define,

$$(u + iv) = i^n (a + ib)^3 \quad (12)$$

Equating real and imaginary parts in (10) we get

$$\left.\begin{aligned} u &= \cos\frac{n\pi}{2}(a^3 - 3ab^2) - \sin\frac{n\pi}{2}(3a^2b - b^3) \\ v &= \cos\frac{n\pi}{2}(3a^2b - b^3) + \sin\frac{n\pi}{2}(a^3 - 3ab^2) \end{aligned}\right\} \quad (13)$$

In view of (2), (4) and (11), the corresponding values of x, y, z, w, p, T are represented

$$\left.\begin{array}{l} x=\cos\frac{n\pi}{2}(a^3-3ab^2+3a^2b-b^3)+\sin\frac{n\pi}{2}(a^3-3ab^2-3a^2b+b^3) \\ y=\cos\frac{n\pi}{2}(a^3-3ab^2-3a^2b+b^3)-\sin\frac{n\pi}{2}(a^3-3ab^2+3a^2b-b^3) \\ z=2[\cos\frac{n\pi}{2}(a^3-3ab^2)-\sin\frac{n\pi}{2}(3a^2b-b^3)][\cos\frac{n\pi}{2}(3a^2b-b^3)+\sin\frac{n\pi}{2}(a^3-3ab^2)] \\ w=[\cos\frac{n\pi}{2}(a^3-3ab^2)-\sin\frac{n\pi}{2}(3a^2b-b^3)][\cos\frac{n\pi}{2}(3a^2b-b^3)+\sin\frac{n\pi}{2}(a^3-3ab^2)]+3 \\ p=[\cos\frac{n\pi}{2}(a^3-3ab^2)-\sin\frac{n\pi}{2}(3a^2b-b^3)][\cos\frac{n\pi}{2}(3a^2b-b^3)+\sin\frac{n\pi}{2}(a^3-3ab^2)]-3 \\ T=a^2+b^2 \end{array}\right\} \quad (14)$$

2.3.1. Properties

1. $x(a,a)-y(a,a)-4CP_{a,6}(\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2})=0$
2. $w(a,a)+p(a,a)+4(\cos^2\frac{n\pi}{2}-\sin^2\frac{n\pi}{2})(SO_a.2CP_{a,3}-2T_{3,a^2}+2T_{4,a}]=0$
3. $z(a,a)-(-1)^n(3T_{4,a}.SO_a-13CP_{a,6}+9T_{4,a}-3T_{8,a}]=0$
4. $x(a,a)+y(a,a)+(\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2})(12P_a^4-12T_{3,a}+6T_{4,a}-2T_{8,a}]=0$

2.4. Approach 4

Writing 1 as $1=\frac{(2mn+i(m^2-n^2)(2mn-i(m^2-n^2)}{(m^2+n^2)^2}$

Following the same procedure as above we get the integral solution of (1) as

$$\left.\begin{array}{l} x=(m^2+n^2)^2[f_1(A,B)+g_1(A,B)] \\ y=(m^2+n^2)^2[f_1(A,B)-g_1(A,B)] \\ z=2(m^2+n^2)^4.f_1(A,B).g_1(A,B) \\ w=(m^2+n^2)^4[f_1(A,B).g_1(A,B)]+3 \\ p=(m^2+n^2)^4[f_1(A,B).g_1(A,B)]-3 \\ T=(m^2+n^2)^2(A^2+B^2) \end{array}\right\} \quad (15)$$

Where

$$\left.\begin{array}{l} f_1(A,B)=2mn(A^3-3AB^2)-(m^2-n^2)(3A^2B-B^3) \\ g_1(A,B)=(A^3-3AB^2)(m^2-n^2)+2mn(3A^2B-B^3) \end{array}\right\} \quad (16)$$

2.4.1. Properties

1. $3(m^2-n^2)[6x(a,a)+2(m^2+n^2)^2(m^2-n^2)(S_a+18(OH_a)-6T_{4,a})]$ is a nasty number
2. $2a(m^2+n^2)(2mn+m^2-n^2)[x(a,a)+(a,a)+8(m^2+n^2)^2(2mn+m^2-n^2)P_a^5]$ is a cubic integer

$$3.x(a,a).y(a,a)+T^3(a,a)-8(m^2+n^2)^4(2mn+m^2-n^2)^2 \\ [6F_{4,a,5}.T_{4,a}-3CP_{a,6}.T_{4,a}-2T_{4,a^2}]=0$$

4. $T(a,1)-(6P_a^3-CP_{a,3}-4T_{3,a}+2T_{4,a})\equiv 0 \pmod 3$

5. $z(a,a)+w(a,a)+p(a,a)=32(m^2+n^2)^4(m^4+n^4-6m^2n^2)[2P^8_{a^4}-2T_{3,a^4}+2T_{4,a^4}]$

2.4.2. Note

1 can also be written as

$$1=\frac{((m^2-n^2)+i2mn)((m^2-n^2)-i2mn)}{(m^2+n^2)^2} \quad (17)$$

Following the same procedure as above we get the integral solution of (1) as

$$\left.\begin{aligned} x&=(m^2+n^2)^2[f_2(A,B)+g_2(A,B)] \\ y&=(m^2+n^2)^2[f_2(A,B)-g_2(A,B)] \\ z&=2(m^2+n^2)^4.f_2(A,B).g_2(A,B) \\ w&=(m^2+n^2)^4[f_2(A,B).g_2(A,B)]+3 \\ p&=(m^2+n^2)^4[f_2(A,B).g_2(A,B)]-3 \\ T&=(m^2+n^2)^2(A^2+B^2) \end{aligned}\right\} \quad (18)$$

where

$$\left.\begin{aligned} f_2(A,B)&=(m^2-n^2)(A^3-3AB^2)-2mn(3A^2B-B^3) \\ g_2(A,B)&=(A^3-3AB^2)2mn+(m^2-n^2)(3A^2B-B^3) \end{aligned}\right\} \quad (19)$$

2.5. Approach 5

The assumption

$$u=UT, v=VT \quad (20)$$

in (3) yields to

$$U^2+V^2=T \quad (21)$$

(i) Taking $T=t^2$ (22)

in (21),we get the solution to(21) as

$$U=2pq, V=p^2-q^2, t=(p^2+q^2),\ p>q>0 \quad (23)$$

$$U=p^2-q^2, V=2pq, t=(p^2+q^2),\ p>q>0 \quad (24)$$

From (22), (20) and (23) we get

$$\left.\begin{aligned} u&=2pq(p^2+q^2)^2 \\ v&=(p^2-q^2)(p^2+q^2)^2 \\ T&=(p^2+q^2)^2 \end{aligned}\right\} \quad (25)$$

In view of (25) and (2), we get the corresponding integral solution of (1).as

$$\left.\begin{array}{l} x=(p^2+q^2)^2(p^2-q^2+2pq) \\ y=(p^2+q^2)^2(2pq-p^2+q^2) \\ z=4pq(p^2+q^2)^4(p^2-q^2) \\ w=2pq(p^2+q^2)^4(p^2-q^2)+3 \\ p=2pq(p^2+q^2)^4(p^2-q^2)-3 \\ T=(p^2+q^2)^2 \end{array}\right\} \quad (26)$$

2.5.1. Properties

1. $x(a,a)-8(6P^3_{a^2}-6T_{3,a^2}+PR_{a^2}-T_{4,a^2})=0$
2. $w(a,a)+p(a,a)-120CP_{a^2,6}(2P^5_{a^2}-T_{4,a^2})=0$
3. $T(a,a)-24F_{4,a,5}+12CP_{a,6}+8T_{4,a}=0$
4. $Z(2a,a)-30[3.SO_a.OH_a+T_{4,a^2}]=0$

Similarly by considering (22), (20), (24) and (2), we get the corresponding integral solution of (1).

(ii) Taking $T=t^n$ (27)
in (21) and considering (20) and performing some algebra, we get

$$\left.\begin{array}{l} u=\frac{1}{2}[(a+ib)^n+(a-ib)^n](a^2+b^2)^n \\ v=\frac{1}{2i}[(a+ib)^n-(a-ib)^n](a^2+b^2)^n \\ T=(a^2+b^2)^n \end{array}\right\} \quad (28)$$

Using (28) and (2), we get the corresponding integral solution to (1).

$$\left.\begin{array}{l} x=(a^2+b^2)^n[f(a,ib)+g(a,ib)] \\ y=(a^2+b^2)^n[f(a,ib)-g(a,ib)] \\ z=2(a^2+b^2)^{2n}.f(a,ib).g(a,ib) \\ w=(a^2+b^2)^{2n}.f(a,ib).g(a,ib)+3 \\ p=(a^2+b^2)^{2n}.f(a,ib).g(a,ib)-3 \\ T=(a^2+b^2)^n \end{array}\right\} \quad (29)$$

where

$$f=\frac{1}{2}[(a+ib)^n+(a-ib)^n]$$

$$g=\frac{1}{2i}[(a+ib)^n-(a-ib)^n]$$

2.5.2. Properties

1. $x(a,a)-2^{n-1}(GL_n(2,-2)+2GF_n(2,-2))(2P^5_{a^n}-2T_{3,a^n}+PR_{a^n}-T_{4,a^n})=0$
2. $y(a,a)-2^{n-1}(GL_n(2,-2)-2GF_n(2,-2))(6P^3_{a^n}-6T_{3,a^n}+2CP_{a^n,6}-SO_{a^n})=0$
3. $w(a,a)+p(a,a)-2^{n-1}GL_n(2,-2).GF_n(2,-2)$
$$(3CP_{a^n,6}.OH_{a^n}-6F_{4,a^n,5}+3CP_{a^n,6}+2T_{4,a^n})=0$$
4. $z(a,a)=2^{2n}GL_n(2,-2).GF_n(2,-2)(4P^5_{a^n}.CP_{a^n,3}-2T_{3,a^n}.2CP_{a^n,6}-CP_{a^n,6})$
5. The following expressions are cubical integers:
   (a) $2^{2n}(GL_n{}^2(2,-2)+4GF_n{}^2(2,-2))[3CP_{a^n,6}.OH_{a^n}-2T_{3,a^n}.T_{4,a^n}+CP_{a^n,6}]$
   (b) $3[2^{4n-2}GL_n{}^2(2,-2).GF_n{}^2(2,-2)(6F_{4,a^{3n},5}-3CP_{a^{3n},6}-2T_{4,a^{3n}})-w(a,a).p(a,a)]$
6. $4x(a,a).y(a,a)-2^{2n}(GL_n{}^2(2,-2)+4GF_n{}^2(2,-2))$
$$(6F_{4,a^n,5}.T_{4,a^n}-18P^3_{a^n}.PR_{a^n}+10T_{4,a^{2n}}+15CP_{a^n,6}+6T_{4,a^n})=0$$

2.6. Approach 6

Using (22) in (21) and arranging we get

$$(t-U)(t+U)=1\times V^2 \qquad (30)$$

Writing (29) as a set of double equations in two different ways as shown below:

Set1: $t+U=V^2, t-U=1,$

Set2: $t+U=1, t-U=V^2$

Solving **set1,** the corresponding values of U,V *and* t are given by

$$U=2k^2+2k, V=2k+1, t=2k^2+2k+1 \qquad (31)$$

In view of (30), (22), (20), (4) and (2), the corresponding solutions to (1) obtained from set1 are represented as shown below:

$$x=(2k^2+4k+1)(2k^2+2k+1)^2$$
$$y=(2k^2-1)(2k^2+2k+1)^2$$
$$z=4(k^2+k)(2k+1)(2k^2+2k+1)^4$$
$$w=2(k^2+k)(2k+1)(2k^2+2k+1)^4+3$$
$$P=2(k^2+k)(2k+1)(2k^2+2k+1)^4-3$$
$$T=(2k^2+2k+1)^2$$

2.6.1. Properties

1. $x(a)-(3T_{4,a}-T_{8,a})(4T_{3,a}+1)^2$ is a cubic integer
2. $T(a)-(8T_{3,a^2}+48P_a^3-40T_{3,a}+10T_{4,a}-2T_{12,a})=1$
3. $4[T(2^{2n})-KY_{4n+1}-KY_{2n+1}-j_{6n+3}]$ is a biquadratic integer
4. $w(a)+p(a)+z(a)-48P_a^4.T^2(a)=0$
5. $\dfrac{3(x(a)-y(a))}{4CP_{a,3}-2CP_{a,6}+1}$ is a nasty number

Similarly, the solutions corresponding to set2 can also be obtained.

3. Remark

Using (22) in (21) and arranging we get

$$(t-V)(t+V)=1.U^2 \quad (32)$$

Using the same procedure as in approach6, two more sets of solutions to (1) can be obtained.

- All the above approaches satisfy the following interesting relations

1. The following are nasty numbers:

(a). $\frac{3(x-y)z}{x+y}$

(b). $12((w^2+p^2)-6z^2$

(c). $6(x^4+x^3y+x^2y^2-xy^3-y^4-4T^3(w+p)-2xyz)$

(d). $6(T^3+w+p)$

2. The following are cubic integers:

(a). $4(x^2+y^2)(x^2y^2+z^2)$

(b). $6(z^2-4wp)$

(c). $2x^2-2z-y^2-w-p$

(d). $\frac{(x+y)}{2z}[(x^3-y^3-2(w-3)(x+y)]$

3. The following are sextic integers

(a). $32(2x^2y^2+(w+p)^2+z^2)$

(b). $(x^2y^2+z^2)$

4. $x^2-y^2-2w-2p=0$

5. $w^2-p^2-2T^3\equiv 0(\mathrm{mod}\,6)$

6. If u *and* v are the generators are of the Pythagorean triangle ABC, then

(i). the area of the triangle ABC= $\frac{xyz}{2}$

(ii). the perimeter of the triangle ABC= $xy+z+T^3$

7. $x^2+y^2+z^2+w^2+p^2-2T^3-6(p+3)^2\equiv 0(\mathrm{mod}\,18)$

8. $4(x+y)(x^3+y^3)-(x+y)^4+24z^2=0$

9. $x^4-y^4-4zx^2+8z(p+3)=0$

10. $x^4+y^4-4(w+p)^2-2[2T^3-(x-y)^2]^2=0$

11. $\frac{z(x+y)}{x-y}+(w-2)^2-xy-pw-T^3-z\equiv 0(\mathrm{mod}\,10)$

12. $4(x-y)(x^5+y^5)=2z(4xyT^3+10(w-3)^2+6(x-y)^4)$

13. $(w-3)[80T^3+40z^2+(x-y)^2\{(x-y)^2-20\}]-4(x+y)(x^5-y^5)=0$

4. Conclusion

In conclusion, one may search for different patterns of solutions to (1) and their **c**orresponding properties.

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