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On the Cubic Equation with Five Unknowns $z^3 + w^3 - x^3 - y^3 = 12T^2(x+y)$

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Abstract:
The cubic equation $z^3 + w^3 - x^3 - y^3 = 12T^2(x+y)$ is analysed for its patterns of non-zero integer solutions. Three patterns of solutions are illustrated. A few properties among the solutions are presented.

Key words: *cubic equation with five unknowns, Integral solutions*
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1. Introduction

Diophantine equations are rich in variety and they have been an interesting topic to many mathematicians since antiquity. In particular, searching for integral solutions for the homogeneous or non-homogeneous Diophantine cubic equation is an interesting concept as it can be seen from [1, 2, 3]. In [4-10] a few special cases of cubic Diophantine equation with four unknowns are studied. In [11-14] cubic equation with five unknowns are studied for their integral solutions. This communication concerns with yet another cubic Diophantine equation with five unknowns $z^3 + w^3 - x^3 - y^3 = 12T^2(x+y)$

2. Notations

- $T_{m,n}$: Polygonal number of rank n with m sides
- Pr_n : Pronic number of rank n
- P_n^m : Pyramidal number of rank n with m sides

3. Method of Analysis

The cubic Diophantine equation with five unknowns to be solved is given by

$$z^3 + w^3 - x^3 - y^3 = 12T^2(x+y) \tag{1}$$

The substitution of the linear transformations

$$x = u+v\,,\ y = u-v\,,\ z = u+p\,,\ w = u-p\,,\ T = 2u \tag{2}$$

in (1) leads to

$$p^2 = v^2 + (4u)^2 \tag{3}$$

(3) is solved through different approaches and the different patterns of solutions of (1) obtained are presented below:

3.1. Pattern: 1

Note that (3) is the well known Pythagorean equation. Using the standard solution of the Pythagorean equation, the values of u, v and p satisfy (3) are given by,

$$p = 4m^2 + n^2$$
$$v = 4m^2 - n^2$$
$$u = mn$$

In view of (2), the integer solution of (1) are given by,

$$x(m,n) = mn + 4m^2 - n^2$$
$$y(m,n) = mn - 4m^2 + n^2$$
$$z(m,n) = mn + 4m^2 + n^2$$
$$w(m,n) = mn - 4m^2 - n^2$$
$$T(m,n) = 2mn$$

3.1.1. Properties

- $y(m,1) + w(m,1) + T_{18,m} \equiv 0 \pmod 5$
- $x(m,1) - T_{10,m} \equiv -1 \pmod 4$
- $z(1,n) + T(1,n) - \Pr_n \equiv 0 \pmod 2$
- $3\{z(1,n) - x(1,n)\}$ is a nasty number

3.2. Pattern: 2

Write (3) as

$$p^2 - v^2 = 16u^2$$

This equation is written in the form of ratio as

$$\frac{p+v}{16u} = \frac{u}{p-v} = \frac{a}{b}, \; b \neq 0 \tag{4}$$

which is equivalent to the system of double equations

$$bp + bv - 16ua = 0 \tag{5}$$

and $$-ap + av + ub = 0 \tag{6}$$

Solving (5) and (6) by method of cross multiplication we've,

$$\left.\begin{aligned} p &= 16a^2 + b^2 \\ v &= 16a^2 - b^2 \\ u &= 2ab \end{aligned}\right\} \tag{7}$$

Substituting (7) in (2), the integer solutions of (1) are given by,

$$x(a,b) = 16a^2 + 2ab - b^2$$
$$y(a,b) = -16a^2 + 2ab + b^2$$
$$z(a,b) = 16a^2 + 2ab + b^2$$
$$w(a,b) = -16a^2 + 2ab - b^2$$
$$T(a,b) = 4ab$$

3.2.1. Properties

- $y(1,b) + z(1,b) - 2\Pr_b \equiv 0 \pmod 2$
- $x(a,1) + z(a,1) - T_{66,a} \equiv 0 \pmod{35}$
- $w(a,1) + T(a,1) + T_{34,a} \equiv -1 \pmod 9$
- $3\{y(1,b) - w(1,b)\}$ is a nasty number
- $x(a,a(a+1)) + y(a,a(a+1)) + z(a,a(a+1)) + w(a,a(a+1)) + T(a,a(a+1)) = 24P_a^5$

3.3. Note
Write (4) as

$$\frac{p+v}{8u}=\frac{2u}{p-v}=\frac{a}{b}\,,\ b\neq 0$$

Following the procedure as in pattern: 2, the corresponding integer solutions to (1) are obtained as,

$$x(a,b)=8a^2+2ab-2b^2$$
$$y(a,b)=-8a^2+2ab+2b^2$$
$$z(a,b)=8a^2+2ab+2b^2$$
$$w(a,b)=-8a^2+2ab-2b^2$$
$$T(a,b)=4ab$$

3.3.1. Properties

- $y(a,1)+T(a,1)+16T_{3,a}\equiv 0(\bmod 2)$
- $x(a,a(a+1))+y(a,a(a+1))=P_a^5$
- $z(a,1)+w(a,1)\equiv 0(\bmod 4)$
- $x(a,1)-y(a,1)-16\,\mathrm{Pr}_a\equiv 0(\bmod 2)$
- $6\{z(1,b)+x(1,b)\}$ is a nasty number

3.4. Pattern: 3
Consider (3) as

$$v^2+(4u)^2=p^2*1 \tag{8}$$

Assume

$$p=a^2+(4b)^2 \tag{9}$$

Write 1 as

$$1=\frac{(4m^2-n^2+i4mn)(4m^2-n^2-i4mn)}{(4m^2+n^2)^2} \tag{10}$$

Substituting (9) and (10) in (8) and employing the method of factorization, define

$$v+i4u=(a+i4b)^2\left[\frac{4m^2-n^2+i4mn}{4m^2+n^2}\right]$$

Equating the real and imaginary parts in the above equation, we get

$$u=\frac{1}{4m^2+n^2}\left[mn(a^2-16b^2)+2ab(4m^2-n^2)\right]$$
$$v=\frac{1}{4m^2+n^2}\left[(4m^2-n^2)(a^2-16b^2)-32abmn\right]$$

Replacing 'a' by $(4m^2+n^2)A$ and 'b' by $(4m^2+n^2)B$ in the above equation, we have

$$u=(4m^2+n^2)\left[mn(A^2-16B^2)+2AB(4m^2-n^2)\right]$$
$$v=(4m^2+n^2)\left[(4m^2-n^2)(A^2-16B^2)-32ABmn\right]$$

Substituting the above values of u and v in (2), one obtains the corresponding integer solutions to (1)

4. Conclusion

In this paper, we analysed different patterns of non-zero distinct integer solutions to the cubic equation with five unknowns. It is worth mentioning here that the Diophantine equations are rich in variety due to their definition. To conclude, one may search for patterns of non-zero distinct integer solutions to other forms of cubic Diophantine equations with multi variables and their corresponding properties.

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