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On Strong Diophantine Quadruples With Property D(1)

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Abstract:
This paper concerns with the study of constructing different families of non-zero integer Diophantine Quadruples using the elements of the sequences $\{j_n = 2^n + (-1)^n : n \geq 1\}$, $\{G_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} : n = 1\}$,
$\{H_n = \frac{(1+\sqrt{3})\gamma^{n-2} - (\sqrt{3}-1)\delta^{n-2}}{6} - \frac{(-1)^{n-2}}{3} : n = 1\}$, *and* $\{S_n = \gamma^n + \delta^n : n = 1\}$*; where*
$\gamma = 2 + \sqrt{3}, \delta = 2 - \sqrt{3}$

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1. Introduction

A Set of positive integers ($a_1, a_2, a_3, ...a_m$) is said to have the property D(n), $n \in z - \{0\}$, if $a_i a_j + n$ is a perfect square for all $1 \leq i \lhd j \leq m$ and such a set is called a Diophantine –m-tuple with property D(n). Many mathematicians considered the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n[1] and also for any linear polynomial in n. Further, various authors considered the connections of the problems of Diophantus, Davenport and Fibonacci numbers in[2-26].

In this paper, we construct Strong Diophantine Quadruples [27] with property D(1) starting with the Diophantine-2-tuple with property D(1) by using the elements the Jacobsthal-Lucas number and three other integer sequences $\{G_n : n \geq 1\}, \{H_n : n \geq 1\}, \text{and} \{S_n : n \geq 1\}$
defined by

$G_1 = 1, G_2 = 4$ and $G_{n+2} = 4G_{n+1} - G_n$ for $n \geq 1$;

$H_1 = H_2 = 0$ and $H_{n+2} = 4H_{n+1} - H_n - 2(-1)^n$ for $n \geq 1$;

$S_1 = 4, S_2 = 14$ and $S_{n+2} = 4S_{n+1} - S_n$ for $n \geq 1$;

respectively.

2. Strong Diophantine Quadruple-I

Here we are aiming at constructing the Strong diophantine quadruple from Diophantine-2-tuple $\{\frac{(j_{2n}-1)}{2}, 8j_{2n}\}$ with property D(1)

Let $a=\frac{(j_{2n}-1)}{2}$ and $b=8j_{2n}$ be two integers such that ab+1 is a perfect square

Let 'c' be any non-zero integer such that

$$(8j_{2n})(c)+1=\beta^2 \qquad (1)$$

$$\frac{(j_{2n}-1)}{2}(c)+1=\gamma^2 \qquad (2)$$

Eliminating 'c' from (1) and (2), we obtain

$$\frac{(j_{2n}-1)}{2}(\beta^2)-8j_{2n}(\gamma^2)=(\frac{(j_{2n}-1)}{2}-8j_{2n}) \qquad (3)$$

Using the linear transformations

$$\beta=X+(8j_{2n})T \qquad (4)$$

$$\gamma=X+\frac{(j_{2n}-1)}{2}T \qquad (5)$$

in (3), it leads to the pell equation

$$X^2=(\frac{(j_{2n}-1)}{2})(8j_{2n})T^2+1 \qquad (6)$$

Let $T_0=1$ and $X_0=(2j_{2n}-1)$ be the initial solution of (6). Thus (4) yields $\beta_0=(10j_{2n}-1)$

And using (1), we get $c=\frac{25}{2}(j_{2n})-\frac{5}{2}$

Hence (a,b,c)=($\frac{(j_{2n}-1)}{2}, 8j_{2n}, \frac{25}{2}(j_{2n})-\frac{5}{2}$) is the Strong diophantine triple with property D(1).

By using Euler's solution, the Strong diophantine quadruple (a,b,c,d) is given by

$$\{\frac{(j_{2n}-1)}{2}, 8j_{2n}, \frac{25}{2}(j_{2n})-\frac{5}{2}, 2(2j_{2n}-1)[2(2j_{2n}-1)^2+(2j_{2n}-1)(17j_{2n}-1)+8j_{2n}(j_{2n}-1)]\}$$

Some numerical examples are presented below

Strong diophantine quadruple with property D(1)

n	(a,b,c,d)
1	(2,40,60,19404)
2	(8,136,210,914628)
3	(32,520,810,53916324)
4	(128,2056,3210,3379087908)
5	(512,8200,12810,2.15126059*10^11)

3. Strong Diophantine Quadruple-II

Strong diophantine quadruple extended from Diophantine-2-tuple $\{2G_{n+2}, G_n\}$ with property D(1)

Let $a=2G_{n+2}$ and $b=G_n$ be two integers such that ab+1 is a perfect square

Let 'c' be any non-zero integer such that

$$(G_n)(c)+1=\beta^2 \qquad (7)$$

$$(2G_{n+2})(c)+1=\gamma^2 \qquad (8)$$

Eliminating 'c' from (7) and (8), we obtain

$$(2G_{n+1})(\beta^2)-(G_n)(\gamma^2)=(2G_{n+1}-G_n) \qquad (9)$$

Using the linear transformations

$$\beta=X+(G_n)T \qquad (10)$$

$\gamma = X + (2G_{n+1})T$ (11)

in (9), it leads to the pell equation

$X^2 = (2G_n G_{n+1})T^2 + 1$ (12)

Let $T_0 = 1$ and $X_0 = (G_{n+1} - G_n)$ be the initial solution of (12). Thus (10) yields $\beta_0 = (3G_{n+1} - G_n)$

And using (7), we get $c = 4(G_{n+1}) - G_n$

Hence (a,b,c)=($2G_{n+1}, G_n, 4(G_{n+1}) - G_n$) is the Strong diophantine triple with property D(1).

By using Euler's solution, the Strong diophantine quadruple (a,b,c,d) is given by

$\{ 2G_{n+1}, G_n, 4(G_{n+1}) - G_n), 4(G_{n+1}) - G_n)[3(G_{n+1}{}^2) - G_n G_{n+1})]\}$

Some numerical examples are presented below

Strong diophantine quadruple with property D(1)

n	(a,b,c,d)
1	(8,1,15,528)
2	(30,4,56,27060)
3	(112,15,209,1405152)
4	(418,56,780,73035468)
5	(1560,209,2911,3796419120)

4. Strong Diophantine Quadruple-III:

Construction of Strong diophantine quadruple from Diophantine-2-tuple $\{2G_{n+2}, \frac{G_n}{2}\}$ with property D(1)

Let $a = 2G_{n+2}$ and $b = \frac{G_n}{2}$ be two integers such that ab+1 is a perfect square

Let 'c' be any non-zero integer such that

$(\frac{G_n}{2})(c) + 1 = \beta^2$ (13)

$(2G_{n+2})(c) + 1 = \gamma^2$ (14)

Eliminating 'c' from (13) and (14), we obtain

$(2G_{n+2})(\beta^2) - (\frac{G_n}{2})(\gamma^2) = (2G_{n+2} - \frac{G_n}{2})$ (15)

Using the linear transformations

$\beta = X + (\frac{G_n}{2})T$ (16)

$\gamma = X + (2G_{n+2})T$ (17)

in (15), it leads to the pell equation

$X^2 = (G_n G_{n+2})T^2 + 1$ (18)

Let $T_0 = 1$ and $X_0 = G_{n+1}$ be the initial solution of (18). Thus (16) yields $\beta_0 = G_{n+1} + \frac{G_n}{2}$

And using (13), we get $c = \frac{G_n}{2} + 2(G_{n+1} + G_{n+2})$

Hence (a,b,c)=($2G_{n+2}, \frac{G_n}{2}, \frac{G_n}{2} + 2(G_{n+1} + G_{n+2})$) is the Strong diophantine triple with property D(1).

By using Euler's solution, the Strong diophantine quadruple (a,b,c,d) is given by

$\{ 2G_{n+2}, \frac{G_n}{2}, \frac{G_n}{2} + 2(G_{n+1} + G_{n+2}), [4(G_{n+1})((G_{n+1}{}^2) + G_{n+1}(2G_{n+2} + \frac{G_n}{2}) + G_n G_{n+2})]\}$

It is to be noted that when n is even, we get Strong diophantine quadruples in integers whereas when n is odd, we get Strong rational diophantine quadruples. A few examples are presented below

n	(a,b,c,d)
1	(30,1/2,77/2,2448)
3	(418,15/2,1075/2,6742176)
5	(5822,209/2,14973/2,1.821914328*10^10)

Table 1: n odd: Strong Rational diophantine quadruple with property D(1)

n	(a,b,c,d)
2	(112,2,143,129540)
4	(1560,28,2006,350495508)
6	(21728,390,27940,9.470453993*10^11)

Table 2: n even: Strong Integer Diophantine quadruple with property D(1)

5. Strong Diophantine Quadruple-IV

Extension of diophantine-2-tuple to Strong diophantine quadruple ($H_n, 2H_{n+1}$) with property D(1)

Let $a = H_n$ and $b = 2H_{n+1}$ be two integers such that ab+1 is a perfect square

Let 'c' be any non-zero integer such that

$$(2H_{n+1})(c) + 1 = \beta^2 \tag{19}$$

$$(H_n)(c) + 1 = \gamma^2 \tag{20}$$

Eliminating 'c' from (19) and (20), we obtain

$$H_n(\beta^2) - (2H_{n+1})(\gamma^2) = (H_n - 2H_{n+1}) \tag{21}$$

Using the linear transformations

$$\beta = X + (2H_{n+1})T \tag{22}$$

$$\gamma = X + (H_n)T \tag{23}$$

in (21), it leads to the pell equation

$$X^2 = (2H_nH_{n+1})T^2 + 1 \tag{24}$$

Let $T_0 = 1$ and $X_0 = \dfrac{S_{n-1} + (-1)^{n-1}}{3}$ be the initial solution of (24). Thus (22) yields $\beta_0 = \dfrac{S_{n-1} + (-1)^{n-1}}{3} + 2H_{n+1}$ and using (19), we get $c = 2[H_{n+1} + \frac{1}{3}(S_{n-1} + (-1)^{n-1})] + H_n$

Hence (a,b,c)=($H_n, 2H_{n+1}$, $2[H_{n+1} + \frac{1}{3}(S_{n-1} + (-1)^{n-1})] + H_n$) is the Strong diophantine triple with property D(1).

By using Euler's solution, the Strong diophantine quadruple (a,b,c,d) is given by

$$(H_n, 2H_{n+1}, 2[H_{n+1} + \frac{1}{3}(S_{n-1} + (-1)^{n-1})] + H_n,$$

$$\{\frac{4}{27}(S_{n-1} + (-1)^{n-1})[(S_{n-1} + (-1)^{n-1})^2 + 3(S_{n-1} + (-1)^{n-1})(H_n + 2H_{n+1}) + 18H_nH_{n+1}]\})$$

Some numerical examples are presented below

Strong Diophantine quadruple with property D(1)

n	(a,b,c,d)
2	(0,4,6,20)
3	(2,12,24,2380)
4	(6,48,88,101660)
5	(24,176,330,5576740)
6	(88,660,1230,285757556)

6. Strong Diophantine Quadruple-V

We aim at constructing the Strong Diophantine quadruple from Diophantine-2-tuple $(\frac{H_{n+2}}{2}, 2H_n)$ with property D(1)

Let $a=\frac{H_{n+2}}{2}$ and $b=2H_n$ be two integers such that ab+1 is a perfect square

Let 'c' be any non-zero integer such that

$$(2H_n)(c)+1=\beta^2 \quad (25)$$

$$(\frac{H_{n+2}}{2})(c)+1=\gamma^2 \quad (26)$$

Eliminating 'c' from (25) and (26), we obtain

$$\frac{H_{n+2}}{2}(\beta^2)-(2H_n)(\gamma^2)=(\frac{H_{n+2}}{2}-2H_n) \quad (27)$$

Using the linear transformations

$$\beta = X+(2H_n)T \quad (28)$$

$$\gamma = X+(\frac{H_{n+2}}{2})T \quad (29)$$

in (27), it leads to the pell equation

$$X^2=(H_nH_{n+2})T^2+1 \quad (30)$$

Let $T_0=1$ and $X_0=H_{n+1}+(-1)^{n+1}$ be the initial solution of (30). Thus (28) yields $\beta=H_{n+1}+(-1)^{n+1}+2H_n$ and using (25), we get $c=\frac{H_{n+2}}{2}+2(H_{n+1}+(-1)^{n+1})+2H_n$

Hence (a,b,c)=($\frac{H_{n+2}}{2}, 2H_n, \frac{H_{n+2}}{2}+2(H_{n+1}+(-1)^{n+1})+2H_n$) is the Strong diophantine triple with property D(1).

By using Euler's solution, the Strong diophantine quadruple (a,b,c,d) is given by

$$(\frac{H_{n+2}}{2}, 2H_n, \frac{H_{n+2}}{2}+2(H_{n+1}+(-1)^{n+1})+2H_n,$$

$$4(H_{n+1}+(-1)^{n+1})\{(H_{n+1}+(-1)^{n+1})^2+(\frac{H_{n+2}}{2})(H_{n+1}+(-1)^{n+1})+2H_n(H_{n+1}+(-1)^{n+1})]+H_nH_{n+2}\})$$

Some numerical examples are presented below

Strong Diophantine quadruple with property D(1)

n	(a,b,c,d)
1	(1,0,3,8)
2	(3,0,5,16)
3	(12,4,30,5852)
4	(44,12,102,215740)
5	(165,48,391,12388088)

7. Conclusion

In this paper, we have constructed 5 strong diophantine quadruples with property D(1). To conclude, the one may search for strong and almost strong diophantine quadruples for special polygonal numbers with suitable property.

8. References

1. A.Baker,H.Davenport,The equations $3x^2-2=y^2$ and $8x^2-7=z^2$, Quart.J. Math.Oxford ser.20(2) (1969),129-137.
2. V.E.Hoggatt,G.E.Bergum, A Problem of Fermat and the Fibonacci sequence, Fibonacci Quart, 15(1977),323-330.
3. A.F.Horadam, Generalization of a result of Morgudo,Portugaliae Math., 44, (1987),131-136.
4. B.W.Jones, Asecond variation on a problem of diophantus and davenport, Fibonacci Quart,16, (1978),155-165.
5. C.Long, G.E.Bergum, On a problem of Diophantus in: Application of Fibonacci numbers, (A.N.Philippou,A.F.Horadam and G.E.Bergum,eds),Kluwer,Dordrecht, Vol.2,(1988)183-191.
6. J.Morgado, Generalization of a result of Hoggatt and bergum on Fibonacci numbers , Portugaliae Math., 42,(1983-1984),441-445.

7. Tao Liqun "On the property P_{-1} ",Electronic Journal of combinatorial number theory 7(2007),#A47.
8. Y.Fujita,"The extensibility of Diophantine pairs (k-1,k+1)",J.number theory 128 (2008), 322-353.
9. H.Gupta and K.Singh, "On k-triad Sequences", Internet.J.Math.Sci., 5(1985), 799-804.
10. A.F.Beardon and M.N.Deshpande, "Diophantine triples", The Mathematical Gazette, 86 (2002),253-260.
11. E.Brown,"Sets in which xy+k is always a square", Math.Comp.45(1985), 613-620.
12. M.N.Deshpande,"Families of Diophantine Triplets",Bulletin of the Marathawada Mathematical Society, 4(2003),19-21.
13. Y.Bugeaud,A.Dujella and Mignotte,"On the family of Diophantine triples $(k-1, k+1, 16k^3-4k)$ ",Glasgow Math.J.49(2007),333-344.
14. Gopalan.M.A,G.Srividhya," Diophantine quadruples for pell no's with property D(1)",Antarctica J.Math.,vol-1,no-3,july 2010,357-362.
15. Gopalan.M.A,S.Mallika, S.Vidhyalakshmi,"On the extendibility of gaussian Diophantine 2-tuple to Gaussian diophantine quadruple withproperty D(1)",IJISM,Vol-2,issue 3,310-311,2014.
16. G.Srividhya,"Diophantine Quadruples for Fibbonacci numbers with property D(1)",Indian Journal of Mathematics and Mathematical Science,Vol.5, No.2,(December 2009),57-59.
17. Gopalan.M.A ,V.Pandichelvi, "The Non Extendibility of the Diophantine Triple $(4(2m-1)^2 n^2, 4(2m-1)n+1, 4(2m-1)^4 n^4 - 8(2m-1)^3 n^3)$, Impact.J.sci.Tech, 5(1),25-28, 2011.
18. Yasutsugu Fujita,Alain Togbe, "Uniqueness of the extension of the $D(4k^2)$ -triple $(k^2-4, k^2, 4k^2-4)$" NNTDM 17 (2011),4,42-49.
19. Gopalan.M.A , G.Srividhya,"Some non extendable P_{-5} sets ", Diophantus J.Math.,1(1) , (2012), 19-22.
20. Gopalan.M.A, G.Srividhya," Two Special Diophantine Triples ", Diophantus J.Math.,1(1),(2012), 23-27.
21. Gopalan.M.A , G.Srividhya," Diophantine Quadruple for Fionacci and Lucas Numbers with property D(4) ", Diophantus J.Math.,1(1),(2012),1`5-18.
22. Andrej Dujella,Zagreb,Croatia,"The problem of Diophantus and Davenport for Gaussian Integers", Glas. Mat.Ser.III, 32 (1997), 1-10.
23. S.Vidhyalakshmi,M.A.Gopalan and K.Lakshmi,"Gaussian Diophantine quadruples with property D(1)", IOSR Journal of Mathematics, Vol 10, issue 3, Ver.II,(may-jun 2014),12-14.
24. M.A.Gopalan, S.Vidhyalakshmi E.Premalatha and R.Presenna ," On the extendibility of 2-tuple to 4-tuple withproperty D(4)", Bulletin of Mathematical Sciences and Applications,Vol 3, issue 2, (2014),100-104.
25. M.A.Gopalan ,G.Sumathi , S.Vidhyalakshmi ,"Special Dio-quadruple involviong Jacobsthal and Jacobsthal lucas Numbers with property $D(k^2+1)$ ",IJMSEA, Vol 8, no 3,2014,221-225.
26. K.Meena, S.Vidhyalakshmi, M.A.Gopalan,G.Akila and R.Presenna,"Formation of special diophantine quadruples with property $D(6kpq)^2$ "The IJST,Vol 2,issue 2,2014,11-14.
27. Andrej Dujella, Vinko petricevic,"Strong Diophantine triples",Experimental Math,Vol 17,No 1 (2008), 83-89