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On Special Families of Hyperbola $x^2=(4k^2\pm k)y^2+\alpha^{2t}$, $\alpha>1$

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Abstract

The binary quadratic equation $x^2=(4k^2\pm k)y^2+\alpha^{2t},\alpha>1$ *is considered for finding its integer solutions and a few interesting properties among the solutions are presented. Also,we present infinitely many positive integer solutions interms of Generalized Fibonacci sequences of numbers, Generalized Lucas sequences of numbers.Six other families of hyberbolas along with their integer solutions and corresponding recurrence relations among the integer solutions are exhibited.*

Mathematics Subject Classification*:11D09*
Key words: *Binary quadratic,integral solutions.*

Notations

$GF_n(k,s)$: *Generalized Fibonacci Sequences of rank* n

$GL_n(k,s)$: *Generalized Lucas Sequences of rank* n

1. Introduction

The binary quadratic equation of the form $y^2=Dx^2+1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values $[1,2,3,4]$.In [5] infinitely many Pythogorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2=3x^2+1$.In [6],a special Pythogorean triangle is obtained by employing the integral solutions of $y^2=10x^2+1$. In [7], different patterns of infinitely many Pythogorean triangles are obtained by employing the non-integral solutions of $y^2=12x^2+1$.In this context one may also refer [8-14]. In [15],These results have motivated us to search for the integral solutions of yet another binary quadratic equation $x^2=19y^2-3^t$ representing a hyberbola. A few interesting properties among the solutions are presented.

2. Method of Analysis

The binary quadratic diophantine equation represents a hyperbola to be solved for its non-zero integral solutions is

$$x^2=(4k^2\pm k)y^2+\alpha^{2t},\quad \alpha>1 \qquad (1)$$

whose smallest positive integer solution is $(x_0,y_0)=\left((8k\pm1)\alpha^t,4\alpha^t\right)$ (2)

Consider the pellian equation

$$x^2=(4k^2\pm k)y^2+1 \qquad (3)$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is represented by

$$\left.\begin{aligned} \tilde{x}_n &= \frac{f_n}{2} \\ \tilde{y}_n &= \frac{g_n}{2\sqrt{4k^2 \pm k}} \end{aligned}\right\} \qquad (4)$$

Where

$$f_n = \left[(8k \pm 1) + 4\sqrt{4k^2 \pm k}\right]^{n+1} + \left[(8k \pm 1) + 4\sqrt{4k^2 \pm k}\right]^{n+1}$$

$$g_n = \left[(8k \pm 1) + 4\sqrt{4k^2 \pm k}\right]^{n+1} - \left[(8k \pm 1) + 4\sqrt{4k^2 \pm k}\right]^{n+1}$$

Then the general solution of (1) is found to be

$$\left.\begin{aligned} x_n &= \alpha^t \frac{f_n}{2} \\ y_n &= \alpha^t \frac{g_n}{2\sqrt{4k^2 \pm k}} \end{aligned}\right\} \qquad (5)$$

where $n = 0,1,2,$

The recurrence relations satisfied by the values of $(x_{n,} y_n)$ are correspondingly exhibited below:

$$x_{n+2} - (16k \pm 2)x_{n+1} + x_n = 0$$
$$y_{n+2} - (16k \pm 2)y_{n+1} + y_n = 0$$

3. A Few Interesting Properties Are Given Below

1.Each of the following is a nasty number

(a) $6\left(\frac{1}{\alpha^t}\left[(16k \pm 2)x_{2n+2} - x_{2n+3} + x_{2n+1}\right] + 2\right)$

(b) $\left(\frac{4}{\alpha^t}\left[y_{2n+2} - (8k \pm 1)y_{2n+1} - x_{2n+1}\right] + 12\right)$

2. Each of the following is a cubical integer

(a) $\left(\frac{4}{\alpha^t}\left[y_{3n+3} - (8k \pm 1)y_{3n+2} + 12x_{2n}\right]\right)$

(b) $\left(\frac{1}{\alpha^t}\left[(16k \pm 2)x_{3n+3} - x_{3n+4} + x_{3n+2} + 6x_n\right]\right)$

3. Each of the following is a biquadratic integer

(a) $8\left(\frac{1}{\alpha^t}\left[y_{4n+4}-(8k\pm1)y_{4n+3}\right]+32\frac{x_n^2}{\alpha^{2t}}-4\right)$

(b) $\frac{1}{\alpha^t}\left[(16k\pm2)x_{4n+4}-x_{4n+5}+x_{4n+3}\right]+16\frac{x_n^2}{\alpha^{2t}}-2$

4. The solutions of (1) interms of special integers namely, Generalized Lucas GL_n and Fibonacci GF_n Sequences are exhibited below:

$$x_n=\frac{\alpha^t}{2}GL_{n+1}(16k\pm2,-1)$$
$$y_n=4\alpha^t GF_{n+1}(16k\pm2,-1)$$

For simplicity, we present in the following, six different families of hyperbola along with their corresponding solutions in terms of Generalized Fibonacci Sequences and Generalized Lucas Sequences and recurrence relations.

Equation	Solution interms of Generalized Fibonacci Sequences and Generalized Lucas Sequences	Recurrence relation
$x^2=(4k^2+7k+3)y^2+\alpha^{2t},\alpha>1$	$x_s=\frac{\alpha^t}{2}GL_{n+1}(16k+14,-1)$ $y_s=4\alpha^t GF_{n+1}(16k+14,-1)$	$x_{n+2}-(16k+14)x_{n+1}+x_n=0$ $y_{n+2}-(16k+14)y_{n+1}+y_n=0$
$x^2=(9k^2\pm2k)y^2+\alpha^{2t},\alpha>1$	$x_s=\frac{\alpha^t}{2}GL_{n+1}(18k\pm2,-1)$ $y_s=4\alpha^t GF_{n+1}(18k\pm2,-1)$	$x_{n+2}-(18k\pm2)x_{n+1}+x_n=0$ $y_{n+2}-(18k\pm2)y_{n+1}+y_n=0$
$x^2=(9k^2\pm k)y^2+\alpha^{2t},\alpha>1$	$x_s=\frac{\alpha^t}{2}GL_{n+1}(36k\pm2,-1)$ $y_s=4\alpha^t GF_{n+1}(36k\pm2,-1)$	$x_{n+2}-(36k\pm2)x_{n+1}+x_n=0$ $y_{n+2}-(36k\pm2)y_{n+1}+y_n=0$
$x^2=(9k^2+17k+8)y^2+\alpha^{2t},\alpha>1$	$x_s=\frac{\alpha^t}{2}GL_{n+1}(36k+34,-1)$ $y_s=4\alpha^t GF_{n+1}(36k+34,-1)$	$x_{n+2}-(36k+34)x_{n+1}+x_n=0$ $y_{n+2}-(36k+34)y_{n+1}+y_n=0$

Table 1

4. Conculsion

To conclude, one may search other formats of hyperbolas for finding their integer solutions and corresponding properties.

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