THE INTERNATIONAL JOURNAL OF SCIENCE & TECHNOLEDGE

An Anti Product of Bipolar Anti Fuzzy Sub Groups

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Abstract:
In this paper, we introduce the concept of an anti product of bipolar fuzzy subsets of a set and discussed some of their properties of the anti product of two bipolar anti fuzzy subgroups of a group.We also introduce the concept of an anti image and anti pre-image of a bipolar fuzzy subgroup of a group and discuss in detail series of homomorphic and anti homomorphic properties of bipolar anti fuzzy subgroup.We also introduce the concept of bipolar anti fuzzy normal subgroup of a group and discuss its properties under the operation anti product.

Key words: *Fuzzy set , bipolar-valued fuzzy set , bipolar fuzzy subgroup ,bipolar anti fuzzy subgroup, anti product of bipolar fuzzy subsets, bipolar anti fuzzy normal subgroup, conjugate bipolar anti fuzzy subgroup*

1. Introduction

The concept of fuzzy sets was initiated by Zadeh [14]. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [13] gave the idea of fuzzy subgroup. In fuzzy sets the membership degree of elements range over the interval [0,1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0,1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. The author W.R.Zhang [15] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. In case of Bipolar-valued fuzzy sets membership degree range is enlarged from the interval [0, 1] to [-1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property and the membership degree [-1,0) indicates that elements somewhat satisfy the implicit counter-property. M. Marudai, V.Rajendran [5] introduced the pre-image on bipolar Q fuzzy subgroup. R.Muthuraj et.al.,[9],[10] redefined the concept of pre-image of bipolar fuzzy subgroup and introduced the concept of an image , anti image and anti pre-image of a bipolar fuzzy subgroup and discuss some of its properties with bipolar anti fuzzy subgroups. We define the concept of an anti product of bipolar fuzzy subsets and discuss some properties on bipolar anti fuzzy subgroup of a group, and deals some results under homomorphism and anti homomarphism.

2. Preliminaries

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G,*) is a finite group, e is the identity element of G, and xy we mean x * y.

2.1. Definition [1]
Let X be any non-empty set. A fuzzy subset μ of X is a function $\mu : X \to [0,1]$.

2.2. Definition [5]
Let G be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set μ in G is an object having the form $\mu = \{\langle x, \mu^+(x), \mu^-(x) \rangle : x \in G\}$,where $\mu^+ : G \to [0,1]$ and $\mu^- : G \to [-1,0]$ are mappings. The positive membership degree $\mu^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$ and the negative membership degree $\mu^-(x)$ denotes the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x) \rangle : x \in G\}$. If $\mu^+(x) \neq 0$ and $\mu^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$. If $\mu^+(x) = 0$ and $\mu^-(x) \neq 0$, it is the situation that x does not satisfy the property of $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$, but somewhat satisfies the counter property of $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$. It is

possible for an element x to be such that $\mu^+(x) \neq 0$ and $\mu^-(x) \neq 0$ when the membership function of property overlaps that its counter property over some portion of G. For the sake of simplicity, we shall use the symbol $\mu = (\mu^+, \mu^-)$ for the bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$.

2.3. Definition [11]

A bipolar-valued fuzzy set or bipolar fuzzy set μ of G is a bipolar fuzzy subgroup of G if for all x, y ∈G,

- $\mu^+(xy) \geq \min\{\mu^+(x), \mu^+(y)\}$,
- $\mu^-(xy) \leq \max\{\mu^-(x), \mu^-(y)\}$,
- $\mu^+(x^{-1}) = \mu^+(x)$, $\mu^-(x^{-1}) = \mu^-(x)$.

2.4. Definition [11]

A bipolar-valued fuzzy set or bipolar fuzzy set μ of G is a bipolar anti fuzzy subgroup of G if for all x, y ∈G,

- $\mu^+(xy) \leq \max\{\mu^+(x), \mu^+(y)\}$,
- $\mu^-(xy) \geq \min\{\mu^-(x), \mu^-(y)\}$,
- $\mu^+(x^{-1}) = \mu^+(x)$, $\mu^-(x^{-1}) = \mu^-(x)$.

2.5. Definition

Let $\mu = (\mu^+, \mu^-)$, $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subsets of the sets G_1 and G_2 respectively. Then an anti product $\mu \times \varphi = ((\mu \times \varphi)^+, (\mu \times \varphi)^-)$ where $(\mu \times \varphi)^+ : G_1 \times G_2 \rightarrow [0,1]$ and $(\mu \times \varphi)^- : G_1 \times G_2 \rightarrow [-1,0]$ are mappings defined by

- $(\mu \times \varphi)^+(x,y) = \max\{\mu^+(x), \varphi^+(y)\}$,
- $(\mu \times \varphi)^-(x,y) = \min\{\mu^-(x), \varphi^-(y)\}$, for all $x \in G_1$, $y \in G_2$.

2.6. Definition

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of G. For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, the bipolar set $U[\mu_{<\alpha,\beta>}] = \{x \in G / \mu^+(x) \geq \alpha$ and $\mu^-(x) \leq \beta\}$ is called a upper level subset (or) level subset of the bipolar fuzzy subset μ.

2.7. Definition

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of G. For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, the bipolar set $L[\mu_{<\alpha,\beta>}] = \{x \in G / \mu^+(x) \leq \alpha$ and $\mu^-(x) \geq \beta\}$ is called a lower level subset of the bipolar fuzzy subset μ.

2.8. Theorem

Let μ and φ be bipolar fuzzy subsets of the sets G_1 and G_2 respectively. and let $<\alpha, \beta> \in [0,1]\times[-1,0]$ then $(\mu \times \varphi)_{<\alpha,\beta>} = \mu_{<\alpha,\beta>} \times \varphi_{<\alpha,\beta>}$.

Proof : Let $(x,y) \in (\mu \times \varphi)_{<\alpha,\beta>}$, $x \in G_1$, $y \in G_2$

$\Leftrightarrow (\mu \times \varphi)^+(x,y) \leq \alpha$, $(\mu \times \varphi)^-(x,y) \geq \beta$

$\Leftrightarrow \max\{\mu^+(x), \varphi^+(y)\} \leq \alpha$, $\min\{\mu^-(x), \varphi^-(y)\} \geq \beta$

$\Leftrightarrow \mu^+(x) \leq \alpha$, $\varphi^+(y) \leq \alpha$ and $\mu^-(x) \geq \beta$, $\varphi^-(y) \geq \beta$

$\Leftrightarrow \mu^+(x) \leq \alpha$, $\mu^-(x) \geq \beta$ and $\varphi^+(y) \leq \alpha$, $\varphi^-(y) \geq \beta$

$\Leftrightarrow (x,y) \in \mu_{<\alpha,\beta>}$ and $(x,y) \in \varphi_{<\alpha,\beta>}$

$\Leftrightarrow (x,y) \in \mu_{<\alpha,\beta>} \times \varphi_{<\alpha,\beta>}$

Hence, $(\mu \times \varphi)_{<\alpha,\beta>} = \mu_{<\alpha,\beta>} \times \varphi_{<\alpha,\beta>}$.

2.9. Theorem

Let μ and φ be bipolar anti fuzzy subgroups of the groups G_1 and G_2 respectively. Then $\mu \times \varphi$ is a bipolar anti fuzzy subgroup.

Proof :

Let $x, y \in G_1 \times G_2$ where $x = (x_1,y_1)$, $y = (x_2,y_2)$.

i. $$\begin{aligned}(\mu \times \varphi)^+(xy) &= (\mu \times \varphi)^+((x_1,y_1)(x_2,y_2)) \\ &= (\mu \times \varphi)^+(x_1x_2, y_1y_2) \\ &= \max\{\mu^+(x_1x_2), \varphi^+(y_1y_2)\} \\ &\leq \max\{\max\{\mu^+(x_1), \mu^+(x_2)\}, \max\{\varphi^+(y_1), \varphi^+(y_2)\}\} \\ &= \max\{\max\{\mu^+(x_1), \varphi^+(y_1)\}, \max\{\mu^+(x_2), \varphi^+(y_2)\}\} \\ &= \max\{(\mu \times \varphi)^+(x_1,y_1), (\mu \times \varphi)^+(x_2,y_2)\} \\ &= \max\{(\mu \times \varphi)^+(x), (\mu \times \varphi)^+(y)\}\end{aligned}$$

ii. $(\mu \times \varphi)^-(xy) = (\mu \times \varphi)^-((x_1,y_1)\ (x_2,y_2))$
$= (\mu \times \varphi)^-(x_1x_2\ ,\ y_1y_2)$
$= \min\{ \mu^-(x_1x_2),\ \varphi^-(y_1y_2) \}$
$\geq \min\{ \min\{\mu^-(x_1), \mu^-(x_2)\}, \min\{ \varphi^-(y_1), \varphi^-(y_2) \}\}$
$= \min\{ \min\{\mu^-(x_1), \varphi^-(y_1)\}, \min\{ \mu^-(x_2), \varphi^-(y_2) \}\}$
$= \min\{ (\mu \times \varphi)^-(x_1,y_1), (\mu \times \varphi)^-(x_2,y_2) \}$
$= \min\{ (\mu \times \varphi)^-(x), (\mu \times \varphi)^-(y)\}$

ii. $(\mu \times \varphi)^+(x^{-1}) = (\mu \times \varphi)^+((x_1,y_1)^{-1})$
$= (\mu \times \varphi)^+(x_1^{-1}, y_1^{-1})$
$= \max\{ \mu^+(x_1^{-1}),\ \varphi^+(y_1^{-1}) \}$
$= \max\{ \mu^+(x_1),\ \varphi^+(y_1) \}$
$= (\mu \times \varphi)^+(x_1,y_1)$
$= (\mu \times \varphi)^+(x)$, and
$(\mu \times \varphi)^-(x^{-1}) = (\mu \times \varphi)^-((x_1,y_1)^{-1})$
$= (\mu \times \varphi)^-(x_1^{-1}, y_1^{-1})$
$= \max\{ \mu^-(x_1^{-1}),\ \varphi^-(y_1^{-1}) \}$
$= \max\{ \mu^-(x_1),\ \varphi^-(y_1) \}$
$= (\mu \times \varphi)^-(x_1,y_1)$
$= (\mu \times \varphi)^-(x)$.

Hence, an anti product $\mu \times \varphi$ is a bipolar anti fuzzy subgroup .

Remark :

Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subsets of the groups G_1 and G_2 respectively. If $\mu \times \varphi$ is a bipolar anti fuzzy subgroup of $G_1 \times G_2$, then it is not necessary that both μ and φ should be bipolar anti fuzzy subgroups of G_1 and G_2 respectively. Consider the following example,

Let $G_1 = \{ e,u \}$ where $u^2 = e$ and $(G_1, \bullet)$ be a group. Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of the group G_1 and is defined as

$$\mu^+(x) = \begin{cases} 0.6 & \text{for } x = e \\ 0.8 & \text{for } x = u \end{cases} \qquad \mu^-(x) = \begin{cases} -0.5 & \text{for } x = e \\ -0.7 & \text{for } x = u \end{cases}$$

Clearly , $\mu = (\mu^+, \mu^-)$ is a bipolar anti fuzzy subgroup of G_1 .

Let G_2 be the Klein's four group and $G_2 = \{ e, a , b , ab\}$ where $a^2 = e = b^2$, $ab = ba$. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy subsets of the group G_2 is defined as

$$\varphi^+(x) = \begin{cases} 0.4, & \text{for } x = e^1 \\ 0.3, & \text{for } x = a \\ 0.5, & \text{for } x = b \\ 0.6, & \text{for } x = ab \end{cases} \qquad \varphi^-(x) = \begin{cases} -0.5, & \text{for } x = e^1 \\ -0.2, & \text{for } x = a \\ -0.3, & \text{for } x = b \\ -0.6, & \text{for } x = ab \end{cases}$$

Now

$\mu \times \varphi = \{ (e,e^1),(e,a), (e,b), (e,ab) , (u,e^1),(u,a), (u,b), (u,ab)\}$

$$(\mu \times \varphi)^+(x,y) = \begin{cases} 0.6 & \text{for } (x,y) = (e,e^1),(e,a),(e,b),(e,ab) \\ 0.8 & \text{for } (x,y) = (u,e^1),(u,a),(u,b),(u,ab) \end{cases}$$

$$(\mu \times \varphi)^-(x,y) = \begin{cases} -0.7 & \text{for } (x,y) = (e,e^1),(e,a),(e,b),(e,ab) \\ -0.6 & \text{for } (x,y) = (u,e^1),(u,a),(u,b),(u,ab) \end{cases}$$

Clearly, $\mu \times \varphi$ is a bipolar anti fuzzy subgroup of $G_1 \times G_2$, where as μ is a bipolar anti fuzzy subgroup of G_1 but φ is not a bipolar anti fuzzy subgroup of G_2 . Since, $\varphi^+(ab) \leq \max\{ \varphi^+(a) , \varphi^+(b)\}$
$0.6 \leq \max\{ 0.3 , 0.5\}$
$0.6 \leq 0.5$, which is not true.

2.10. Theorem

Let $(\mu \times \varphi)$ be a bipolar fuzzy set of $G_1{\times}G_2$ then $(\mu \times \varphi)$ is a bipolar anti fuzzy subgroup of $G_1{\times}G_2$ if and only if $(\mu \times \varphi)^c$ is a bipolar fuzzy subgroup of $G_1{\times}G_2$.

Proof :

Let x , y $\in G_1{\times}G_2$ where $x = (x_1,y_1)$, $y = (x_2,y_2)$.

i. $(\mu \times \varphi)^+(xy) \leq \max\{(\mu \times \varphi)^+(x), (\mu \times \varphi)^+(y)\}$

$\Leftrightarrow 1-((\mu \times \varphi)^+)^c(xy) \leq \max\{1-((\mu \times \varphi)^+)^c(x), 1-((\mu \times \varphi)^+)^c(y)\}$

$\Leftrightarrow ((\mu \times \varphi)^+)^c(xy) \geq 1-\max\{1-((\mu \times \varphi)^+)^c(x), 1-((\mu \times \varphi)^+)^c(y)\}$

$\Leftrightarrow ((\mu \times \varphi)^+)^c(xy) \geq \min\{((\mu \times \varphi)^+)^c(x), ((\mu \times \varphi)^+)^c(y)\}$

ii. $(\mu \times \varphi)^-(xy) \geq \min\{(\mu \times \varphi)^-(x), (\mu \times \varphi)^-(y)\}$

$\Leftrightarrow -1-((\mu \times \varphi)^-)^c(xy) \geq \min\{-1-((\mu \times \varphi)^-)^c(x), -1-((\mu \times \varphi)^-)^c(y)\}$

$\Leftrightarrow ((\mu \times \varphi)^-)^c(xy) \leq -1-\min\{-1-((\mu \times \varphi)^-)^c(x), -1-((\mu \times \varphi)^-)^c(y)\}$

$\Leftrightarrow ((\mu \times \varphi)^-)^c(xy) \leq \max\{((\mu \times \varphi)^-)^c(x), ((\mu \times \varphi)^-)^c(y)\}$

iii. $(\mu \times \varphi)^+(x^{-1}) = (\mu \times \varphi)^+((x)$

$\Leftrightarrow 1-((\mu \times \varphi)^+)^c(x^{-1}) = 1-((\mu \times \varphi)^+)^c(x)$

$\Leftrightarrow ((\mu \times \varphi)^+)^c(x^{-1}) = ((\mu \times \varphi)^+)^c(x)$, and

iv. $(\mu \times \varphi)^-(x^{-1}) = (\mu \times \varphi)^-((x)$

$\Leftrightarrow 1-((\mu \times \varphi)^-)^c(x^{-1}) = 1-((\mu \times \varphi)^-)^c(x)$

$\Leftrightarrow ((\mu \times \varphi)^-)^c(x^{-1}) = ((\mu \times \varphi)^-)^c(x)$.

Hence, $(\mu \times \varphi)^c$ is a bipolar fuzzy subgroup of $G_1{\times}G_2$.

2.11. Theorem

Let μ and φ be bipolar fuzzy subsets of the groups G_1 and G_2 respectively. Suppose that e and e^1 are the identity elements of G_1 and G_2 respectively. If an anti product $\mu \times \varphi$ is a bipolar anti fuzzy subgroup of $G_1{\times}G_2$ then atleast one of the following two statements must hold

- $\varphi^+(e^1) \leq \mu^+(x)$, $\varphi^-(e^1) \geq \mu^-(x)$ for all $x \in G_1$,
- $\mu^+(e) \leq \varphi^+(y)$, $\mu^-(e) \geq \varphi^-(y)$ for all $y \in G_2$.

Proof :

Let $\mu \times \varphi = ((\mu \times \varphi)^+, (\mu \times \varphi)^-)$ be a bipolar anti fuzzy subgroup of $G_1{\times}G_2$. By contraposition , suppose that none of the statements (1) and (2) holds. Then we can find 'a' in G_1 and 'b' in G_2 such that

$\mu^+(a) < \varphi^+(e^1)$, $\mu^-(a) > \varphi^-(e^1)$ and $\varphi^+(b) < \mu^+(e)$, $\varphi^-(b) > \mu^-(e)$. we have

i. $(\mu \times \varphi)^+(a,b) = \max\{\mu^+(a), \varphi^+(b)\}$

$< \max\{\varphi^+(e^1), \mu^+(e)\}$

$= \max\{\mu^+(e), \varphi^+(e^1)\}$

$= (\mu \times \varphi)^+(e, e^1)$

$(\mu \times \varphi)^+(a,b) < (\mu \times \varphi)^+(e, e^1)$ and

ii. $(\mu \times \varphi)^-(a,b) = \min\{\mu^-(a), \varphi^-(b)\}$

$> \min\{\varphi^-(e^1), \mu^-(e)\}$

$= \min\{\mu^-(e), \varphi^-(e^1)\}$

$= (\mu \times \varphi)^-(e, e^1)$

$(\mu \times \varphi)^-(a,b) > (\mu \times \varphi)^-(e, e^1)$

Thus the anti product $\mu \times \varphi$ is not a bipolar anti fuzzy subgroup of $G_1{\times}G_2$.

Hence, either $\varphi^+(e^1) \leq \mu^+(x)$, $\varphi^-(e^1) \geq \mu^-(x)$ for all $x \in G_1$.

or $\mu^+(e) \leq \varphi^+(y)$, $\mu^-(e) \geq \varphi^-(y)$ for all $y \in G_2$.

2.12. Theorem

Let μ and φ be bipolar fuzzy subsets of the groups G_1 and G_2 respectively. Such that $\mu^+(x) \geq \varphi^+(e^1)$, $\mu^-(x) \leq \varphi^-(e^1)$ for all $x \in G_1$, e^1 be the identity element of G_2 . If an anti product $\mu \times \varphi$ is a bipolar anti fuzzy subgroup of $G_1{\times}G_2$ then μ is a bipolar anti fuzzy subgroup of G_1.

Proof :

Let $\mu \times \varphi = ((\mu \times \varphi)^+, (\mu \times \varphi)^-)$ be a bipolar anti fuzzy subgroup of $G_1{\times}G_2$ and x , y $\in G_1$, then $(x, e^1), (y, e^1) \in G_1{\times}G_2$.
Given (i) $\mu^+(x) \geq \varphi^+(e^1)$, (ii) $\mu^-(x) \leq \varphi^-(e^1)$ for all $x \in G_1$ we get,

i. $\mu^+(xy) = \max\{\mu^+(xy), \varphi^+(e^1e^1)\}$,by (i)
$= (\mu \times \varphi)^+((xy), (e^1e^1))$
$= (\mu \times \varphi)^+((x, e^1)\ (y,e^1))$
$\leq \max\{(\mu \times \varphi)^+(x, e^1), (\mu \times \varphi)^+(y, e^1)\}$
$= \max\{\max\{\mu^+(x), \varphi^+(e^1)\}, \max\{\mu^+(y), \varphi^+(e^1)\}\}$
$= \max\{\mu^+(x), \mu^+(y)\}$
$\mu^+(xy) \leq \max\{\mu^+(x), \mu^+(y)\}$

ii. $\mu^-(xy) = \min\{\mu^-(xy), \varphi^-(e^1e^1)\}$,by (ii)
$= (\mu \times \varphi)^-((xy), (e^1e^1))$
$= (\mu \times \varphi)^-((x, e^1)\ (y,e^1))$
$\geq \min\{(\mu \times \varphi)^-(x, e^1), (\mu \times \varphi)^-(y, e^1)\}$
$= \min\{\min\{\mu^-(x), \varphi^-(e^1)\}, \min\{\mu^-(y), \varphi^-(e^1)\}\}$
$= \min\{\mu^-(x), \mu^-(y)\}$
$\mu^-(xy) \geq \min\{\mu^-(x), \mu^-(y)\}$

iii. $\mu^+(x^{-1}) = \max\{\mu^+(x^{-1}), \varphi^+((e^1)^{-1})\}$,by (i)
$= (\mu \times \varphi)^+(x^{-1}, (e^1)^{-1})$
$= (\mu \times \varphi)^+((x, e^1)^{-1})$
$= (\mu \times \varphi)^+((x, e^1))$
$= \max\{\mu^+(x), \varphi^+(e^1)\}$
$\mu^+(x^{-1}) = \mu^+(x)$, and

$\mu^-(x^{-1}) = \min\{\mu^-(x^{-1}), \varphi^-((e^1)^{-1})\}$,by (ii)
$= (\mu \times \varphi)^-(x^{-1}, (e^1)^{-1})$
$= (\mu \times \varphi)^-((x, e^1)^{-1})$
$= (\mu \times \varphi)^-((x, e^1))$
$= \min\{\mu^-(x), \varphi^-(e^1)\}$
$\mu^-(x^{-1}) = \mu^-(x)$.

Hence, $\mu = (\mu^+, \mu^-)$ is a bipolar anti fuzzy subgroup of G_1.

2.13. Theorem

Let μ and φ be bipolar fuzzy subsets of the groups G_1 and G_2 respectively. Such that $\varphi^+(x) \geq \mu^+(e)$, $\varphi^-(x) \leq \mu^-(e)$ for all $x \in G_2$, e be the identity element of G_1 . If an anti product $\mu \times \varphi$ is a bipolar anti fuzzy subgroup of $G_1 \times G_2$ then φ is a bipolar anti fuzzy subgroup of G_2.

Proof :

Let $\mu \times \varphi = ((\mu \times \varphi)^+, (\mu \times \varphi)^-)$ be a bipolar anti fuzzy subgroup of $G_1 \times G_2$ and x , y $\in G_2$, then $(e,x), (e,y) \in G_1 \times G_2$. Given (i) $\varphi^+(x) \geq \mu^+(e)$, (ii) $\varphi^-(x) \leq \mu^-(e)$ for all $x \in G_2$, we get,

i. $\varphi^+(xy) = \max\{\mu^+(ee), \varphi^+(xy)\}$, by (i)
$= (\mu \times \varphi)^+((ee), (xy))$
$= (\mu \times \varphi)^+((e,x)\ (e,y))$
$\leq \max\{(\mu \times \varphi)^+(e,x), (\mu \times \varphi)^+(e,y)\}$
$= \max\{\max\{\mu^+(e), \varphi^+(x)\}, \max\{\mu^+(e), \varphi^+(y)\}\}$
$= \max\{\varphi^+(x), \varphi^+(y)\}$
$\varphi^+(xy) \leq \max\{\varphi^+(x), \varphi^+(y)\}$

ii. $\varphi^-(xy) = \min\{\mu^-(ee), \varphi^-(xy)\}$,by (ii)
$= (\mu \times \varphi)^-((ee), (xy))$
$= (\mu \times \varphi)^-((e,x)\ (e,y))$
$\geq \min\{(\mu \times \varphi)^-(e,x), (\mu \times \varphi)^-(e,y)\}$
$= \min\{\min\{\mu^-(e), \varphi^-(x)\}, \min\{\mu^-(e), \varphi^-(y)\}\}$
$= \min\{\varphi^-(x), \varphi^-(y)\}$
$\varphi^-(xy) \geq \min\{\varphi^-(x), \varphi^-(y)\}$

iii. $\varphi^+(x^{-1}) = \max\{\mu^+(e^{-1}), \varphi^+(x^{-1})\}$,by (i)
$= (\mu \times \varphi)^+(e^{-1}, x^{-1})$
$= (\mu \times \varphi)^+((e, x)^{-1})$
$= (\mu \times \varphi)^+((e,x))$
$= \max\{\mu^+(e), \varphi^+(x)\}$
$\varphi^+(x^{-1}) = \varphi^+(x)$, and
$\varphi^-(x^{-1}) = \min\{\mu^-(e^{-1}), \varphi^-(x^{-1})\}$,by (ii)
$= (\mu \times \varphi)^-(e^{-1}, x^{-1})$
$= (\mu \times \varphi)^-((e, x)^{-1})$
$= (\mu \times \varphi)^-(e,x)$
$= \min\{\mu^-(e), \varphi^-(x)\}$
$\varphi^-(x^{-1}) = \varphi^-(x)$.
Hence, $\varphi = (\varphi^+, \varphi^-)$ is a bipolar anti fuzzy subgroup of G_2.

Corollary :
Let μ and φ be bipolar fuzzy subsets of the groups G_1 and G_2 respectively. If $\mu \times \varphi$ is a bipolar anti fuzzy subgroup of $G_1 \times G_2$, then either μ is a bipolar anti fuzzy subgroup of G_1 or φ is a bipolar anti fuzzy subgroup of G_2.

2.14. Definition
A bipolar anti fuzzy subgroup $\mu = (\mu^+, \mu^-)$ of group G is said to be bipolar anti fuzzy normal subgroup of G if ,
- $\mu^+(xyx^{-1}) \le \mu^+(y)$
- $\mu^-(xyx^{-1}) \ge \mu^-(y)$, for all $x,y \in G$.

2.15. Theorem [12]
Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subgroup of a group G then $\mu = (\mu^+, \mu^-)$ is a bipolar anti fuzzy normal subgroup of G if and only if any one of the following conditions is satisfied
- $\mu^+(xyx^{-1}) = \mu^+(y)$, $\mu^-(xyx^{-1}) = \mu^-(y)$ for all $x,y \in G$
- $\mu^+(xy) = \mu^+(yx)$, $\mu^-(xy) = \mu^-(yx)$ for all $x,y \in G$.

2.16. Theorem
Let $\mu = (\mu^+, \mu^-)$, $\varphi = (\varphi^+, \varphi^-)$ be bipolar anti fuzzy subgroups of the groups G_1 , G_2 respectively . If μ , φ are bipolar anti fuzzy normal then $\mu \times \varphi$ is bipolar anti fuzzy normal .

Proof :
By Theorem [2.9]. Let μ and φ be bipolar anti fuzzy subgroups of the groups G_1 and G_2 respectively. Then $\mu \times \varphi$ is a bipolar anti fuzzy subgroup.
Now, let us Show the normality condition,.
For (x_1, x_2) , $(y_1, y_2) \in G_1 \times G_2$,
$(\mu \times \varphi)^+((x_1, x_2)(y_1, y_2)) = (\mu \times \varphi)^+(x_1y_1, x_2y_2)$
$= \max\{\mu^+(x_1y_1), \varphi^+(x_2y_2)\}$
$= \max\{\mu^+(y_1x_1), \varphi^+(y_2x_2)\}$
$= (\mu \times \varphi)^+((y_1, y_2)(x_1, x_2))$ and

$(\mu \times \varphi)^-((x_1, x_2)(y_1, y_2)) = (\mu \times \varphi)^-(x_1y_1, x_2y_2)$
$= \min\{\mu^-(x_1y_1), \varphi^-(x_2y_2)\}$
$= \min\{\mu^-(y_1x_1), \varphi^-(y_2x_2)\}$
$= (\mu \times \varphi)^-((y_1, y_2)(x_1, x_2))$.
Thus, $\mu \times \varphi$ is bipolar anti fuzzy normal.

2.17. Definition
A bipolar anti fuzzy subgroup $\mu = (\mu^+, \mu^-)$ of group G is said to be conjugate to bipolar anti fuzzy subgroup $\varphi = (\varphi^+, \varphi^-)$ of G if there exist $x \in G$ such that for all $g \in G$,
- $\mu^+(g) = \varphi^+(x^{-1}gx)$
- $\mu^-(g) = \varphi^-(x^{-1}gx)$.

2.18. Theorem
Let a bipolar anti fuzzy subgroup μ of a group G_1 be conjugate to a bipolar anti fuzzy subgroup σ of G_1 and a bipolar anti fuzzy subgroup φ of a group G_2 be conjugate to a bipolar anti fuzzy subgroup η of G_2 . Then the bipolar anti fuzzy subgroup $\mu \times \varphi$ of the group $G_1 \times G_2$ is conjugate to the bipolar anti fuzzy subgroup $\sigma \times \eta$ of the group $G_1 \times G_2$.

Proof : Let a bipolar anti fuzzy subgroup μ of a group G_1 be conjugate to a bipolar anti fuzzy subgroup σ of G_1 the if there exist $x_1 \in G_1$ such that for all $g_1 \in G_1$,

- $\mu^+(g_1) = \sigma^+(x_1^{-1} g_1 x_1)$
- $\mu^-(g_1) = \sigma^-(x_1^{-1} g_1 x_1)$.

and a bipolar anti fuzzy subgroup μ of a group G_2 be conjugate to a bipolar anti fuzzy subgroup σ of G_2 the if there exist $x \in G_2$ such that for all $g \in G_2$,

- $\varphi^+(g_2) = \eta^+(x_2^{-1} g_2 x_2)$
- $\varphi^-(g_2) = \eta^-(x_2^{-1} g_2 x_2)$.

if there exist $(x_1, x_2) \in G_1\times G_2$ such that for all $(g_1, g_2) \in G_1\times G_2$

now i. $(\mu\times\varphi)^+(g_1, g_2) = \max\{\mu^+(g_1), \varphi^+(g_2)\}$

$$= \max\{\sigma^+(x_1^{-1} g_1 x_1), \eta^+(x_2^{-1} g_2 x_2)\}$$
$$= (\sigma\times\eta)^+(x_1^{-1} g_1 x_1, x_2^{-1} g_2 x_2)$$
$$= (\sigma\times\eta)^+((x_1^{-1}, x_2^{-1})(g_1, g_2)(x_1, x_2))$$
$$= (\sigma\times\eta)^+((x_1, x_2)^{-1}(g_1, g_2)(x_1, x_2))$$

and

ii. $(\mu\times\varphi)^-(g_1, g_2) = \max\{\mu^-(g_1), \varphi^-(g_2)\}$

$$= \max\{\sigma^-(x_1^{-1} g_1 x_1), \eta^-(x_2^{-1} g_2 x_2)\}$$
$$= (\sigma\times\eta)^-(x_1^{-1} g_1 x_1, x_2^{-1} g_2 x_2)$$
$$= (\sigma\times\eta)^-((x_1^{-1}, x_2^{-1})(g_1, g_2)(x_1, x_2))$$
$$= (\sigma\times\eta)^-((x_1, x_2)^{-1}(g_1, g_2)(x_1, x_2))$$

Hence, the bipolar anti fuzzy subgroup $\mu\times\varphi$ of the group $G_1\times G_2$ is conjugate to the bipolar anti fuzzy subgroup $\sigma\times\eta$ of the group $G_1\times G_2$.

3. Properties of Bipolar Anti Fuzzy Subgroup of a Group Under Homomorphism and Anti Homomorphism

We discuss the properties of a bipolar anti fuzzy subgroup of a group under homomorphism and anti homomorphism. Throughout this section the finite groups G_1, G_2, H_1 and H_2 are not necessarily commutative.

3.1. Definition [9]

A mapping f from a group G_1 to a group G_2 is said to be a homomorphism if $f(xy) = f(x) f(y)$ for all $x,y \in G_1$.

3.2. Definition [9]

A mapping f from a group G_1 to a group G_2 (G_1 and G_2 are not necessarily commutative) is said to be an anti homomorphism if $f(xy) = f(y) f(x)$ for all $x,y \in G_1$.

3.3. Definition [10]

Let G_1 and G_2 be any two groups. Let $\mu = (\mu^+,\mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subsets in G_1 and G_2 respectively, Let $f : G_1\to G_2$ be a mapping then the anti image $f_a(\mu)$ of μ is a bipolar fuzzy subset $f_a(\mu) = ((f_a(\mu))^+, (f_a(\mu))^-)$ of G_2 defined by for each $u\in G_2$,

$$(f_a(\mu))^+(u) = \begin{cases} \min\{\mu^+(x) : x\in f^{-1}(u)\}, & \text{if } f^{-1}(u) \neq \phi \\ 1, & \text{otherwise} \end{cases}$$

and

$$(f_a(\mu))^-(u)) = \begin{cases} \min\{\mu^-(x) : x\in f^{-1}(u)\}, & \text{if } f^{-1}(u) \neq \phi \\ -1, & \text{otherwise} \end{cases}$$

also the anti pre-image $f_a^{-1}(\varphi)$ of φ under f is a bipolar fuzzy subset of G_1 defined by for $x \in G_1$, $((f_a^{-1}(\varphi)^+)(x) = \varphi^+(f_a(x))$, $((f_a^{-1}(\varphi)^-)(x) = \varphi^-(f_a(x))$.

3.4. Theorem

Let f be homomorphism from $G_1\times G_2$ to $H_1\times H_2$. If $(\mu\times\varphi)$ is a bipolar anti fuzzy subgroup of $G_1\times G_2$, then an anti image $f_a(\mu\times\varphi)$ of $(\mu\times\varphi)$ under f is a bipolar anti fuzzy subgroup of $H_1\times H_2$.

Proof : It is clear , by Theorem 3.3 [10].

3.5. Theorem

Let f be homomorphism from $G_1\times G_2$ to $H_1\times H_2$. If $(\sigma\times\eta)$ is a bipolar anti fuzzy subgroup of $H_1\times H_2$ then an anti pre-image $f^{-1}(\sigma\times\eta)$ is a bipolar anti fuzzy subgroup of $G_1\times G_2$.

Proof : It is clear ,by Theorem 3.5 [10].

3.6. Theorem

Let f be an anti homomorphism from $G_1 \times G_2$ to $H_1 \times H_2$. If $(\mu \times \varphi)$ is a bipolar anti fuzzy subgroup of $G_1 \times G_2$, then an anti image $f_a(\mu \times \varphi)$ of $(\mu \times \varphi)$ under f is a bipolar anti fuzzy subgroup of $H_1 \times H_2$.

Proof : It is clear , by Theorem 3.4 [10].

3.7. Theorem

Let f be an anti homomorphism from $G_1 \times G_2$ to $H_1 \times H_2$. If $(\sigma \times \eta)$ is a bipolar anti fuzzy subgroup of $H_1 \times H_2$ then an anti pre-image $f^{-1}(\sigma \times \eta)$ is a bipolar anti fuzzy subgroup of $G_1 \times G_2$.

Proof : It is clear , by Theorem 3.6 [10].

3.8. Theorem

Let f be homomorphism from $G_1 \times G_2$ to $H_1 \times H_2$. If $\mu \times \varphi$ is a bipolar fuzzy subset of $G_1 \times G_2$ then

i. $f((\mu \times \varphi)^c) = [f_a(\mu \times \varphi)]^c$.

ii. $f_a((\mu \times \varphi)^c) = (f(\mu \times \varphi))^c$.

Proof : It is clear ,by Theorem 3.2 [10].

3.9. Theorem

Let f be a homomorphism from a $G_1 \times G_2$ to $H_1 \times H_2$. If $\sigma \times \eta$ is a bipolar fuzzy subset of $H_1 \times H_2$ then $f^{-1}((\sigma \times \eta)^c) = [f^{-1}(\sigma \times \eta)]^c$.

Proof : It is clear , by theorem 3.1 [10].

4. Conclusion

We have given the notion of an anti product of bipolar fuzzy subsets of a set in a bipolar anti fuzzy groups and studied some of their properties.We also proved that an anti product of bipolar anti fuzzy subgroups of groups G_1 and G_2 is a bipolar anti fuzzy subgroup of $G_1 \times G_2$. We proved that an anti product of bipolar anti fuzzy normal subgroups of groups G_1 and G_2 is a bipolar anti fuzzy normal subgroup of $G_1 \times G_2$. We have proved that the homomorphic and anti homomorphic anti image of an anti product of bipolar anti fuzzy subgroups of $G_1 \times G_2$ is a bipolar anti fuzzy subgroup of $H_1 \times H_2$ and the homomorphic and anti homomorphic anti pre-image of an anti product of bipolar anti fuzzy subgroups of $H_1 \times H_2$ is a bipolar anti fuzzy subgroup of $G_1 \times G_2$.We hope that our results can also be extended to other algebraic system.

5. References

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