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A Study on Balance Equation with the System of Shortest Queue Using Poisson Processes

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Abstract:
A Poisson stream of customers arrives at a service center which consists of two single-server queues in parallel. The service times of the customer are exponentially distributed, and both server at the same rate .Arriving customers join the shorter of the two queues, with ties broken in any plausible manner. No jockeying between the queues is allowed. Employing linear programming techniques, we calculate bounds for the probability distributor of the number of customers in the system, and its expected value in equilibrium. The bounds are asymptotically tight in heavy traffic.

JEL: Single –server Queue.
Key words: *Asymptotically bounds, exponentially distributed, jockeying, linear programming techniques, single-server queue*

1. Introduction

A Poisson stream of customer with rate λ arrives at a service system which consists of two single-server queues in parallel[3].The service time of the customers are independent and exponentially distributed with rate µ.Both servers serve at an equal rate of 1.Each server has an associated queuing space of unlimited capacity. An arriving customer joins the shorter of the two queues, if their sizes are unequal, otherwise he joins any queue. Jockeying between the queues is not allowed.
This system, which is known as the 'shortest queue', has received a lot of attention in the literature, because it (with its generalizations to many servers and to general service times) models many real-life situations such as vehicles going through toll booths, jobs scheduled on a multiprocessor system, etc, .On the other hand, no simple method for analyzing this system is known. The problem was originally introduced by Haight (1958).Kingman (1961) proved that an equilibrium distribution exists whenever $\lambda/\mu < 2$.Both he and Flatto and McKean(1977)treated the problem by applying techniques of complex-function theory to obtain representation for the general functions of the state probability .The above –mentioned papers derive some asymptotic approximations for the state probabilities for large number of customers in the system, and for heavy traffic.Conolly(1984) discussed the finite –waiting –room version of the problem. In the current paper we derive upper and lower bounds for the state probabilities, tail probabilities, and mean number of customers in the system, in equilibrium. The bounds are derived by considering a subset of the balance equation, which gives rise to linear programs which in turn produce the bounds .We also derive lower bounds for the tail distribution by comparison with an M/M/2 system. These rather elementary techniques produce bound which are within 10% of the true values for $1\leq \lambda/\mu <2$, and which are asymptotically tight in heavy traffic. Some notation is introduced in section 2, and preliminary results which are needed for the probability are obtained in section3.In section 4 we derive formulae for the probability distribution of the total number of customers in the system, and introduce our main vehicle - the linear program which generates the bounds. We calculate explicit solutions for the upper bounds [6] and treat the lower bounds. Bounds for the tail of the distribution of the total number of customers in the system are discussed, and the quality of the bounds is examined deals with bounds for the mean number in the system.

2. Notation

The state space consists of pairs (i,j) where i,j=0,1,2,3………,and i≥j.we say that the system is in state (i,j) if number of customers in the longer queue is i and the number in the shorter queue is j. Note that under this description, servers are not associated with a particular Component of the state vector, and, there is no need to specify what happens when a customer finds that both queues are of equal length. Clearly the system behaves as a Markov chains on the state space, with transition, intensities as described in following Figure.Let a= λ/μ .We assume that a <2, which implies that the system in stable. Let p_{ij} be the equilibrium probability of the state (i, j).

Let $\pi_n = \sum_{i+j=n} p_{ij}$, n=0, 1, 2……..Thus π_n is the probability that there are exactly n customers in the system in equilibrium.

Denote by q_k the sum of the probabilities in the k^th diagonal.

$$q_k = \sum_{i=0}^{\infty} p_{i+ki},\ \text{, k=0, 1, 2…}$$

And let q*= $\sum_{k=1}^{\infty} q_k$ be the probability that the queues are unequal.

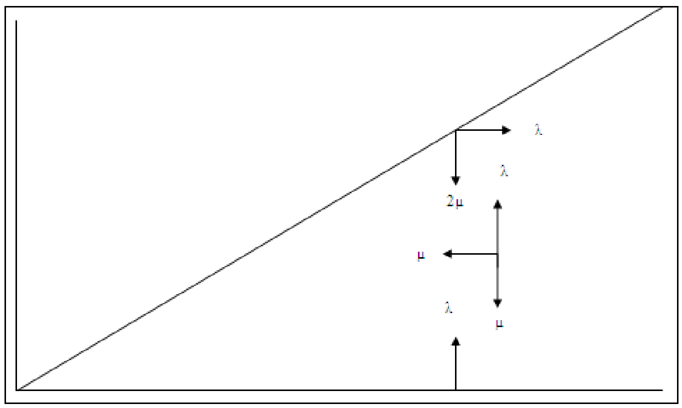

Figure 1

3. Preliminary Results

It is well known that for a Markov chain in equilibrium, if the state space is divided into two disjoint subsets, then the 'intensity flows' from one subset into the other are equal for both subsets.

First, for a fixed i, we divide the states into those with first component less than i, and those with first component greater than or equal to i. We get

$$\lambda p_{i-1i-1} = \mu \sum_{j=0}^{i-1} p_{ij}, i = 1, 2, ...\ \ \ ………………. \ .\ (3.1)$$

Summing (3.1) over i we get

$$\lambda q_0 = \mu q^* \qquad ………………………… \qquad (3.2)$$

since $q_0 + q^* = 1$,we can solve them:

$$q_{0=} \frac{1}{1+a}, q^* = \frac{a}{1+a}.\ .\ ………………………………………..(3.3)$$

Next, define the 'diagonals'

$$D_k = \bigcup_{i=0}^{\infty} (i+k, i)\text{, k=0,1,……………………………………..…(3.4)}$$

And let us separate the diagonals $D_0, D_1, ...D_k$ from D_{k+1}, D_{k+2}.........

The resulting cut yields

$$(1+a)q_1 = (2+a)q_0 - 2p_{00},\ \text{ for k=0,………………….(3.5)}$$

$$(1+a)q_{k+1} = q_k - p_{k0},\ \text{ for k=1,2,3,……………………..….(3.6)}$$

Summing (3.5) and (3.6) over k we get

$$(1+a)q^* = (2+a)q_0 + q^* - (2p_{00} + \sum_{k=1}^{\infty} p_{k0}),$$

and subtracting the values of q0 and q* ,we get the following result .

3.1. Lemma

$$2p_{00}+\sum_{k=1}^{\infty}p_{k0}=2-a,\ for 0\le a\prec 2\ .\ \ldots\ldots\ldots\ldots\ldots\ldots(3.7)$$

It is introduce to note that $q_0, q_1, \ldots\ldots\ldots\ldots$ exist even when a≥2,if we interpret q_k as the limit of the probability of being in D_k at time t ,when t→∞, independent of the initial conditional. To see that, when a≥2 the probability that any queue is empty at time t converges to 0 for t→∞. Thus the processes behaves asymptotically as birth-and-death process on the diagonals Dk,with the birth rate being μ and the death rate being $\lambda+\mu$ for k>0,and for k=0 the birth rate being $\lambda+2\mu$ and the death rate 0.

Thus a limiting distribution exists, independent of the initial conditions, and it can be calculated explicitly as follows.

Theorem1.

$$q_0=\begin{cases}\frac{1}{1+a},\ for 0\le a\prec 2\\ \frac{a}{2(1+a)}\ for 2<a\end{cases}$$

$$q_1=\frac{a(a+2)}{2(1+a)^2}\ for 2<a,$$

and $q_k=q_1\dfrac{1}{(1+a)^{k-1}}\ for 2<a.$

4. The Total Number in the System

Let us divide the state space into those states with n or more customers, and those with n-1 or less. The intensity flow equation, over these cuts yields:

$$\pi_n={a\!/\!2}\pi_{n-1}+{1\!/\!2}p_{n0},\quad n=1,\ 2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4.1)$$

Iterating (4.1) by substituting $\pi_{n-1}, \pi_{n-2}, \ldots\ldots\ldots\ldots$ and noticing that

$$\pi_0=p_{00}\ \pi_n=({a\!/\!2})^n p_{00}+{1\!/\!2}(({a\!/\!2})^{n-1}p_{10}+({a\!/\!2})^{n-2}p_{20}+\ldots..+p_{n0}),\ n=0,1,2,\ldots\ldots(4.2)$$

We are now going to use (4.2) to derive bounds for the $\pi_n S$, by employing the relation found in section 3.From(3.3),(3.5)and (3.6)we get

$$p_{00}=\frac{2+a}{2(1+a)}-\frac{1+a}{2}q_1$$

$$p_{i0}=q_i-(1+a)q_{i+1},\quad i=1,2,\ldots\ldots n,$$

Substituting in(4.3) and rearranging we get

$$\pi_n=({a\!/\!2})^n\frac{2+a}{2(1+a)}+\frac{1}{4}(2-a-a^2)\sum_{i=1}^{n}({a\!/\!2})^{n-i}q_i-\frac{1+a}{2}q_{n+1}\ ,\ n=0,1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

Consider the linear program with n+ 1 variable $X_1, X_2, \ldots\ldots. X_{n+1},$ with the objective function

$$f(X)=\frac{1}{4}(2-a-a^2)\sum_{i=1}^{n}({a\!/\!2})^{n-i}X_i-\frac{1+a}{2}X_{n+1},\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.4)$$

and with the constraints:

$$\begin{cases} X_k \geq 0, k = 1, 2, ...n+1 \\ C_0 : X_1 \leq \dfrac{2+a}{(1+a)^2} \\ \overline{C} : X_1 + ...X_n + X_{n+1} \leq \dfrac{a}{1+a} \\ \underline{C} : X_1 + ...X_n + \dfrac{1+a}{a} X_{n+1} \geq \dfrac{a}{1+a}. \end{cases}$$

5. Conculsion

We have developed a method to obtain bounds for the shortest queue problem by using a subset of the balance equation and deriving the corresponding linear program. The system discussed about the finite number of customer finds queue when both queues are equal length, the state of Markov chain is equilibrium.

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