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Theory of Measure and Integration: Daniell's Version of Lebesgue- Integral

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Abstract:
Theory of measure and integration involves the consideration of σ-algebra of subsets of a given space and establish a specific type of set function called a measure defied on the σ-algebra. The integral is defined in terms of a measure of a set. P.J. Daniell gives a direct approach to integration theory as integral is defined as a continuous positive linear functional on a vector lattice. In this paper we here discuss theorems such as Lebesgue Monotone Convergence Theorem, Lebesgue Dominated Convergence Theorem.

Keywords*: Vector lattice, σ-algebra, Linear Functional, Measure, Convergence*

Definition: Let L be a vector space over real numbers of real valued functions defined on a set X. Let for all f and g belonging to L define $f \vee g$ = max{f,g} and $f \wedge g$ =min{f,g}, if L is closed under $f \vee g$ and $f \wedge g$ then L is called a vector lattice of functions on X. We define $f^+ = f \vee 0$ and $f^- = f \wedge 0$ for any f $\in L$ then $f^+ \in L$ and $f^- \in L$. Also |f| = $f^+ + f^-$ thus |f| $\in L$.

Preposition: Let L be a vector space over real numbers of real valued functions defined on a set X. then L is a vector lattice iff $h^+ \in L$ whenever $h \in L$.

Proof: If L be a vector lattice then from the definition of L the said condition is obviously true.

Conversely: Let the condition is satisfied. For f, g $\in L$ then f - g $\in L \Rightarrow (f-g)^+ \in L$.
$\Rightarrow (f-g)^+ + g \in L. \Rightarrow f \vee g \in L$ And $f \wedge g$ =(f+g) - $f \vee g$ shows that $f \wedge g \in L$.
Hence L is a vector Lattice.

Preposition: Let L be a vector space of real valued functions defined on X, then L is a vector lattice iff |h| $\in L$ whenever $h \in L$.

Proof: Suppose L be a vector lattice, let h $\subset L$.Then h , $h \subset L \Rightarrow h^+ \subset L$ and ($h)^+ \subset L$ (from above) $\Rightarrow h^+ + (-h)^+ \in L \Rightarrow h^+ + (h)^- \in L. \Rightarrow$ |h| $\in L \Rightarrow$ the condition is necessary.

Conversely: Assume that the given condition is satisfied, let h$\in L$. Then |h |$\in L \Rightarrow h +$ |h| $\in L$
$\Rightarrow h^+ \quad h^- \mid h^+ \mid h^- \subset L \Rightarrow 2h^+ \subset L \Rightarrow h^+ \subset L$. Thus in view of the above last result L becomes a vector lattice.

Definition: Let L be a vector lattice on X, and I be a +ve linear functional defined on L s.t. when $\phi_n \in L$, $I(\phi_n) \downarrow 0$ whenever $\phi_n \downarrow 0$, Then I is called a Daniell Integral on L.

Note: "$\phi_n \downarrow 0$" means that the sequence $\{\phi_n(x)\}$ is point wise monotone increasing and converging to 0 for each $x \in X$.

Preposition: L be a vector lattice on X, and I be a +ve linear functional defined on L Then I is called Daniell integral iff. Whenever $\phi, \phi_n \in L, \{\phi_n(x)\}$ is any increasing sequence and $\phi \leq Limt\,(\phi_n)$ then $I(\phi) \leq Limt\, I(\phi_n)$.

Proof: Suppose I is a Daniell integral, Let$\phi \in L$, and $\{\phi_n(x)\}$ be any increasing sequence of members of L with $\phi \leq Limt\,(\phi_n)$, $let\ \psi = Limt\,(\phi_n)$ then $\phi - \phi_n \downarrow \phi - \psi$
$\Rightarrow (\phi - \phi_n)^+ \downarrow (\phi - \psi)^+ \Rightarrow (\phi - \phi_n)^- \downarrow 0 \Rightarrow I(\phi - \phi_n)^+ \downarrow o$ ....(1)
Since $\phi - \phi_n \leq (\phi - \phi_n)^-$ we get $I(\phi - \phi_n) \leq I(\phi - \phi_n)^+ \Rightarrow Limt\, I(\phi - \phi_n) \leq Limt\, I(\phi - \phi_n)^+ \Rightarrow Limt\, I(\phi - \phi_n) \leq 0$ as n $\rightarrow \infty$ {from 1}

$\Rightarrow Limt\, [I(\phi) - I(\phi_n)] \leq 0 \quad \Rightarrow I(\phi) \leq Limt(\phi_n)$ Proves that the condition is necessary.

Conversely: Assume that the given condition is satisfied.
To show that I is Daniell Integral.
Consider any decreasing sequence (ψ_n) of members of L
with $\phi_n \downarrow 0, \Rightarrow 0 \leq \phi_n \Rightarrow I(0) \leq I(\phi_n)\, for\ all\ n \Rightarrow 0 \leq I(\phi_n)\, for\ all\ n \Rightarrow 0 \leq Limt\, I(\phi_n)\, for\ all\ n$ ....(2)
As ϕ_n is decreasing then $(-\phi_n)$ Is increasing sequence of members of L and
$Limt\, I(-\phi_n)=0 \Rightarrow 0 \leq Limt\, I(-\phi_n) \Rightarrow I(0) \leq Limt\, I(-\phi_n) \Rightarrow 0 \leq -Limt\, I(\phi_n)$
$\Rightarrow Limt\, I(\phi_n) \leq 0$ ....(3)
From (2) and (3) we obtain $Limt\, I(\phi_n) = 0$, shows that I is a Daniell Integral.

Preposition: I be a +ve linear functional defined on L_1 Then I is called Daniell integral iff. Whenever $\phi, \in L, u_n \in L, u_n \geq 0$ and $\phi \leq \sum_{n=1}^{\infty} u_n$ then $I(\phi) \leq \sum_{n=1}^{\infty} I(u_n)$.

Proof: Suppose I is Daniell integral let $\phi \in L, u_n \in L\, v_n \geq 0$ and $\phi \leq \sum_{n=1}^{\infty} u_n$
Define $\phi_n = \sum_{i=1}^{n} u_i$ then $\phi_n \in L$ and ϕ_n is increasing and $\phi \leq \sum_{n=1}^{\infty} u_n$= limt$(\sum_{i=1}^{n} u_i)$ = limt (ϕ_n)
$\Rightarrow I(\phi) \leq$ limt $I(\phi_n)$= limt $I(\sum_{i=1}^{n} u_i)$ = limitI$([u_1 + u_2 + u_3 + \cdots .... + u_n])$ = limt$[I(u_1) + I(u_2) + I(u_3) + \cdots .... + I(u_n)]$ = limt$(\sum_{i=1}^{n} I(u_i))$ = $\sum_{n=1}^{\infty} I(u_n)$. Shows that the condition is necessary.

To show that the condition is sufficient
Assume that the condition is satisfied.
Let$\{\varphi_n\}$ be any increasing sequence of members of L and $\varphi \in L$ such that $\varphi \leq limt\, (\varphi_n)$.Define $u_1 = \varphi_1$, $u_2 = \varphi_2 - \varphi_1$, $u_3 = \varphi_3 - \varphi_2$..........
Thus $u_n \geq 0$ and $u_n \in L$ for all n. Further $\varphi \leq$ limt (φ_n)=limt $(\sum_{i=1}^{n} u_i)$= $\sum_{n=1}^{\infty}(u_n)$
Therefore by assumption we get $I(\varphi) \leq \sum_{n=1}^{\infty} I(u_n)$= limt $(\sum_{i=1}^{n} I(u_i))$= limt $I(\sum_{i=1}^{n}(u_i))$ = limit $I(\varphi_n)$.
Shows that I is a Daniell Integral.

Example: Let L be the class of Real valued continuous functions on [a, b], for f $\in L$, Define $I(f) = \int_a^b (f) \Rightarrow$ If $f, g \in L$ and $\alpha, \beta, \in R$ then $\alpha f + \beta g$ is also continuous function, Hence $\alpha f + \beta g \in L$.
Also $f \vee g$, $f \wedge g \in L$
shows that L is a vector lattice .
Further I $(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g)$ =$\alpha \int_a^b f + \beta \int_a^b g$= α I (f) + β I(g) Shows that I is Linear.
And I (f) = $\int_a^b f \geq 0$
Wherever f ≥ 0 Shows that I is a positive linear functional on L.
Let $\{ f_n \}$ be any sequence of members of L s.t. $(f_n) \downarrow 0$, then $f_n \in R[a, b]$ and $|f_n| \to 0$
Therefore by Dominant Convergence Theorem we have $\int_a^b f \to 0$
As f_n is Reimann Integerable over [a, b] therefore $\int_a^b f \to 0 \Rightarrow I(f_n) \to 0$. Proves that I is Daniell Integral.

Example: Let $(\lambda, \mu, \mathcal{O})$be any measurable space. let L be the class of all simple functions which are zero outside sets of finite measure.
Define $I(f) = \int_a^b f d\mu$ for If f, $\in L$, then it is easy to verify that L is a vector lattice and I is a positive linear functional on L as in the above example it can be shown that I is continuous for decreasing null sequence and hence I is Daniell Integral on L.

Definition: Let L be any vector lattice of real valued functions on X By L_u denote the class of real valued functions on X which are limits of increasing sequences of members of L.

Remark (1): Lu is a lattice Let f, g $\in L$, then there exist increasing sequences $\{\phi_n\}$ *and* $\{\psi_n\}$ of members of L s.t. $\phi_n \to f$ *and* $\psi_n \to g$. This shows that $\{\phi_n \vee \psi_n\} \to f \vee g \Rightarrow f \vee g \in Lu$,

Similarly $f \wedge g \in Lu$, $\Rightarrow$ Lu is a lattice.

Remark (2): Let f, g $\in Lu$, and $\alpha, \beta \geq 0$ Then $\alpha f + \beta g \in Lu$
We say that Lu is a cone.

Remark (3): Lu is not a vector space because $\alpha f + \beta g \notin Lu$ for $\alpha, \beta < 0$

Preposition: $Let\ \{\phi_n\}\ and\ \{\psi_n\}$ be increasing sequences of members of L and suppose that $\phi_n \leq Limt\ \psi_n$ then $I(\phi_n) \leq Limt\ I(\psi_n)$.

Proof: $\phi_n \leq Limt I(\phi_n) \leq Limt\ I(\psi_n) \Rightarrow I(\phi_n) \leq Limt\ I(\psi_n)$
$\Rightarrow Limt\ I(\phi_n) \leq Limt\ I(\psi_n)$

Definition: Let f $\in Lu$ and $\{\varphi_n\}$ be increasing sequences and members of L s.t. f= $Limt(\phi_n)$ then we define I (f) = $Limt\ I(\phi_n)$

Preposition: Let f, g $\in Lu$, and $\alpha, \beta \geq 0$, then
(1): f$\leq g \Rightarrow$ I(f) $\leq I(g)$
(2): I $(\alpha f + \beta g) = \alpha$ I$(f) + \beta$ I(g)

Proof (1): $Let\ \{\phi_n\}\ and\ \{\psi_n\}$ be increasing sequences of members of L s.t. $\phi_n \to f\ and\ \psi_n \to g$ as f $\leq g$
$\Rightarrow Limt\ \{\phi_n\} \leq Limt\{\psi_n\} \Rightarrow Limt\ I\{\phi_n\} \leq Limt I\{\psi_n\} \Rightarrow$ I (f) $\leq I(g)$.
(2): As $\phi_n \to f\ and\ \psi_n \to g$ and
$\alpha, \beta \geq 0 \Rightarrow \{\alpha\ \phi_n + \beta\ \psi_n\} \to (\alpha f + \beta g) \Rightarrow I\{\alpha\ \phi_n + \beta\ \psi_n\} \to I(\alpha f + \beta g) \Rightarrow I(\alpha f + \beta g) = Limt\ I\{\alpha\ \phi_n + \beta\ \psi_n\}$
$= Limt\ \{\alpha I(\phi_n) + \beta\ I(\psi_n)\} = \alpha\ Limt\ \{I(\phi_n)\} + \beta\ Limt\ \{I(\psi_n)\}) = \alpha$ I$(f) + \beta$ I(g).

Corollary (1) : I(f+ g) = I (f) + I (g) for all f ,g $\in$ Lu
(2): I (f) ≥ 0 for all f, g $\equiv$ Lu with $f \geq 0$

Preposition: If f $\in$ Lu iff f = $\sum_{n=1}^{\infty} \phi_n$, where $\phi_n \in$ L, $\phi_n \geq 0$ Then I (f) = $\sum_{n=1}^{\infty} I(\phi_n)$

Proof: Suppose f = $\sum_{n=1}^{\infty} \phi_n$, where $\phi_n \in$ Lu, $\phi_n \geq 0$
We first define $\psi_n = \sum_{i=1}^{n} \phi_i$, then ψ_n is an increasing sequence of members of L and
f = $\sum_{n=1}^{\infty} \phi_n$= Limit $(\sum_{i=1}^{n} \psi i)$ = Limt ψ_n as n $\to \infty$
Hence by definition f $\in$ Lu.

Conversely: Suppose f $\in$ Lu then by definition there exist an increasing sequence $\{u_n\}$ of members of L such that $u_n \to f$.
Define $\phi_1 = u_1$, $\phi_2 = u_2 \quad u_1$, ……. , $\phi_n = u_n \quad u_{n-1}$,…………
Then $\phi_n \geq 0$, $\phi_n \in$ L and $\sum_{n=1}^{\infty} \phi_n$= Limt $(\sum_{i=1}^{n} \phi_i)$ = Limt(u_n)= f . Proves that $\sum_{n=1}^{\infty} \phi_n = f$.
Further suppose f = $\sum_{n=1}^{\infty} \phi_n$, where $\phi_n \in$ L, $\phi_n \geq 0$
Define $\sigma_n = \sum_{i=1}^{n} \phi_i$ then $\{\sigma_n\}$ Is an increasing sequence of members of L and $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty}(\sum_{i=1}^{n} \phi_i) = \sum_{n=1}^{\infty} \phi_1$ = f that $\{\sigma_n\} \to f$, $\sigma_n \in L$.
$\Rightarrow$ I(f) = $\lim_{n\to\infty} I(\sigma_n)$ = $\lim_{n\to\infty}(\sum_{i=1}^{n}$ I(ϕ_i)= $\sum_{i=1}^{\infty}$ I$(\phi$i) $\Rightarrow$ I(f) = $\sum_{i=1}^{\infty}$ I$(\phi$i).

Definition: For any function f on X define $\overline{\text{I(f)}}$ =Inf{ I(g)}, where g $\in$Lu, g$\geq f$} and
$\underline{\text{I(f)}}$ = sup{ I(g)}, where g $\in$Lu, g$\geq f$} that is $\underline{\text{I(f)}}$ = $\overline{-\text{I}(-\text{f})}$

Preposition: Let $\{f_n\}$ be any sequence of non–negative functions on X and f = $\sum_{n=1}^{\infty} f_n$, then $\overline{\text{I(f)}} \leq \sum_{n=1}^{\infty} \overline{I(f_n)}$.

Proof: If $\overline{I(f_n)} = \infty$ for some n, the result is obvious.
Suppose that $\overline{I(f_n)} < \infty$ for all n. Let $\varepsilon > 0$, then $\overline{I(f_n)} < \overline{I(f_n)} + \frac{\varepsilon}{2^n}$ ………………….(*)
$\Rightarrow$ There exist $g_n \in$ Lu s.t. $g_n \geq f_n$ and I (g_n)) $< \overline{I(f_n)} + \frac{\varepsilon}{2^n}$ for all n
Let g = $\sum_{n=1}^{\infty} g_n$ as $g_n \geq f_n \geq 0$, we see that g $\subset$ Lu and I(g) = $\sum_{n=1}^{\infty} I(g_n)$ [By above preposition]
This gives $\sum_{n=1}^{\infty} g_n \geq \sum_{n=1}^{\infty} f_n \Rightarrow$ g $\geq f \Rightarrow$ I(g) $\geq \overline{\text{I(f)}} \Rightarrow \overline{I(f)} \leq$ I(g) $= \sum_{n=1}^{\infty} I(g_n) \leq \sum_{i=1}^{\infty}(\overline{I(f_n)} + \frac{\varepsilon}{2^n})$
…………………..[from *]
$= \sum_{n=1}^{\infty} \overline{(I(f_n)} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \sum_{n=1}^{\infty} \overline{I(f_n)} + \varepsilon \Rightarrow \overline{I(f)} \leq \sum_{n=1}^{\infty} \overline{I(f_n)}$.

Theorem: Prove that

(1) : $\overline{I(f+g)}) \leq \overline{I(f)} + \overline{I(g)}$ and $\underline{I(f+g)} \geq \underline{I(f)} + \underline{I(g)}$

(2): $f \leq g \Rightarrow \overline{I(f)} \leq \overline{I(g)}$ an d $f \leq g \Rightarrow \underline{I(f)} \leq \underline{I(g)}$

(3): $\overline{I(cf)} = c\overline{I(f)}$ and $\underline{I(cf)} = c\underline{I(f)}$ if c > o

(4): $\overline{I(cf)} = c\underline{I(f)}$ and $\underline{I(cf)} = c\overline{I(f)}$ if c < 0

Proof: (1) Let h = f + g and let $\phi \geq f$ an d $\psi \geq g$ then $\phi + \psi \geq f + g \Rightarrow \phi + \psi \geq h$
$\Rightarrow \overline{I(h)} \leq \overline{I(\phi + \psi)} = \overline{I(\phi)} + \overline{I(\psi)} \rightarrow \overline{I(h)} \leq \underset{\phi \geq f}{Inf}\{I(\phi)\} + \underset{\psi \geq g}{Inf}\{I(\psi)\} = \overline{I(f)} + \overline{I(g)}$.

Similarly we can prove that $\underline{I(f+g)} \leq \underline{I(f)} + \underline{I(g)}$

(2): Let $f \leq g$, and $\phi \in$ Lu and $\phi \geq g$, then $f \leq \psi \rightarrow \overline{I(f)} \leq I(\psi) \rightarrow \overline{I(f)} \leq Inf\, I(\psi) - \overline{I(g)} \rightarrow \overline{I(f)} \leq \overline{I(g)}$
Similarly we have $\underline{I(f)} \leq \underline{I(g)}$

(3): If c = 0 then the result is obliviously true

Let c > 0 then $\overline{I(cf)} - Inf\{I(\phi)/\ \phi \in$Lu and $\phi > cf\} = Inf\{I(c\phi)/\ \phi \in$Lu and $\phi > f\}$
$= c\, Inf\{I(\phi)/\ \phi \in$ Lu and $\phi \geq f\} = c\,\overline{I(f)}$

Similarly Let $c < 0$ $\overline{I(f)} = Inf\{I(\phi)/\ \phi \in$Lu and $\phi \geq cf\} = Inf\{I(c\phi)/\ \phi \in$ Lu and $\phi \leq f\}$
$= c\, Sup\{I(\phi)/\ \phi \subset$ Lu and $\phi \leq f\} = c\,\underline{I(f)}$.

Same for the other cases we can prove.

Definition: Let I be the Daniell Integral on a vector lattice L. By L_1 we denote the class of those functions f s.t. $\overline{I(f)} = \underline{I(f)}$ further define that $\overline{I(f)} = \underline{I(f)} = I(f)$ *and it is finite*.

Theorem: I be the Daniell Integral on a vector lattice L. Let L_1 be denote the class of those functions f s.t. $\overline{I(f)} = \underline{I(f)}$, which is finite .For $f \in L_1$ define that $\overline{I(f)} = \underline{I(f)} = I(f)$. Then L_1 be a vector lattice and I on L_1 is an extension of I on L.

Proof: (1) Let $f, g \in L_1$ then $\overline{I(f+g)} \leq \overline{I(f)} + \overline{I(g)} = I(f) + I(g) = \underline{I(f)} + \underline{I(g)} \leq \underline{I(f+g)}$
$\Rightarrow \overline{I(f+g)} = \underline{I(f+g)} = I(f) + I(g)$ And it is finite shows that $f+g \in L_1$
And $I(f+g) = I(f) + I(g)$.

(2): Let $f \in L_1$

And $\alpha \in R$ let $\alpha \geq 0$ then $\overline{I(\alpha f)} = \alpha\overline{I(f)} = \alpha\underline{I(f)} = \underline{I(\alpha f)} \Rightarrow \overline{I(\alpha f)} = \underline{I(\alpha f)} = I(\alpha f)$

If $\alpha < 0$ then $\overline{I(\alpha f)} = \alpha\,\underline{I(f)} = I(\alpha f) - \alpha\overline{I(f)} = \underline{I(\alpha f)}$

Shows that $\overline{I(\alpha f)} = \underline{I(\alpha f)} = I(\alpha f) \Rightarrow \alpha f \in L_1$ and $I(\alpha f) = \alpha\, I(f)$

Proves that L_1 is a vector space and I is a linear transformation.

(3): Let $f \in L_1$ and $f \geq 0$ an d $g \in L_u$ and $g \geq 0$ then $I(g) \geq 0$
$\Rightarrow$ Inf $\{I(g)\} \geq 0 \Rightarrow \overline{I(f)} \geq 0 \Rightarrow$ I is positive linear functional.

(4): let $\phi \in L$ then $\phi \in Lu \Rightarrow \overline{I(\phi)} = \underline{I(\phi)} = I(\phi)$ shows that I on L_1 is an extension of Ion L.

(5): Let $f \in L_1$ take any $\varepsilon > 0$ then $I(f) < \underline{I(f)} + \epsilon \Rightarrow \overline{I(f)} < \overline{I(f)} + \epsilon \Rightarrow$

There exist $g \in L_u$ s.t. $g \geq f$ and $\overline{I(f)} \leq \overline{I(g)} \leq \overline{I(f)} + \epsilon$ ....(*)

Similarly there exist $h \in L_u$ s.t. $h \geq -f$ and $\overline{I(-f)} \leq \overline{I(h)} \leq \overline{I(-f)} + \epsilon$.

$\Rightarrow$- $h \leq f$ and -$\underline{I(f)} \leq I(h) \leq$ -$\underline{I(f)}$+ϵ ....(**)

Define $g_1 = g \vee 0$, $h_1 = h \wedge 0$, then $g_1 = g^+$, $h_1 = -h^-$, from _h $\leq f \leq g \Rightarrow (-h)^+ \leq f^+ \leq g^+ \Rightarrow h^- \leq f^+ \leq g_1 \Rightarrow -h_1 \leq f^+ \leq g_1$
.....................................(***)

Consider any $x \in X$, If $g(x) \geq 0$ then $g_1(x) = g(x) \Rightarrow h_1(x) \leq h(x) \Rightarrow g_1(x) + h_1(x) \leq g(x) + h(x)$
If $g(x) < 0$ then $g_1(x) = 0$, from $-h \leq f \leq g$ we get $-h(x) \leq g(x) < 0 \Rightarrow h(x) > 0 \Rightarrow h_1(x) = 0$, $g_1(x) = 0$ also Hence $\Rightarrow$ $g_1(x) + h_1(x) \leq g(x) + h(x)$ thus $g_1(x) + h_1(x) \leq g(x) + h(x)$ for all x
$\Rightarrow g_1 + h_1 \leq g + h \Rightarrow I(g_1 + h_1) \leq I(g+h) \Rightarrow I(g_1) + I(h_1) \leq I(g) + I(h) \Rightarrow I(g_1) + I(h_1) \leq 2\epsilon$ ....(****)
(From (*) and (**))

From (***) we get $\underline{I(-h)} + \leq \underline{I(f^+)}$ and

$\overline{I(f^+)} \leq \overline{I(g_1)} \Rightarrow -\overline{I(h_1)} \leq \underline{I(f^+)}$ and $\overline{I(f^+)} \leq \overline{I(g_1)} \Rightarrow \overline{I(f^+)} - \underline{I(f^+)} \leq I(g_1) + I(h_1) \leq 2\epsilon \Rightarrow \overline{I(f^+)} = \underline{I(f^+)}$ and it is finite $\Rightarrow f^+ \in L_1$.

Theorem: (Lebesgue's Monotone convergence Theorem)
Let $\{f_n\}$ be any increasing sequence of functions in L_1. And f= limt $\{f_k\}$, then f $\in L_1$ iff limt I(f_n) $< \infty$, also I(f) = limt I(f_n).
Proof: Suppose limt I(f_n)$= \infty$,
As $(f_n) \uparrow f$, we have $f \geq f_n$ for all n $\Rightarrow \overline{I(f)} \geq \overline{I(f_n)}$ for all n $\Rightarrow \overline{I(f)} \geq \overline{I(f_n)} = I(f_n)$
$\Rightarrow I(f) \geq$ limt I(f_n) $\Rightarrow I(f) = \infty \Rightarrow$ f $\notin L_1$ shows if f $\in L_1$ then limt I(f_n) $< \infty$.
Suppose now limt I(f_n) $< \infty$,
Define g = f-f_1 and $g_n = f_{n+1} - f_n$ then $g \geq 0$ *and* $g_n \geq 0$ and $g_n \to g$
$\sum_{k=1}^{n}(f_{k+1} - f_k) = g_n \Rightarrow$ limt [$\sum_{k=1}^{n}(f_{k+1} - f_k)$] = limt (g_n) = g$\Rightarrow$ g = $\sum_{k=1}^{\infty}(f_{k+1} - f_k) \Rightarrow$
$\overline{I(g)} \leq \sum_{n=1}^{\infty} \overline{I}\,(f_{n+1} - f_n) = \sum_{n=1}^{\infty}[\,I(f_{n+1})$-I($f_n$)] = limt[$\sum_{k=1}^{n}\{I(f_{k+1}) - I(f_k)\}$] = limt[$I(f_{n+1}) - I(f_1)$) = limt [$I(f_n) - I(f_1)$]
$\Rightarrow I(f_1) + \overline{I(g)} \leq$ limt $I(f_n)$ ....(*)
As $f_n \leq f \Rightarrow \underline{I(f_n)} \leq \underline{I(f)}$ for all n $\Rightarrow I(f_n) \leq \underline{I(f)} \Rightarrow limt\, I(f_n) \leq \underline{I(f)}$ ....(**)
From g = f-f_1 , we get f = g+f_1 $\Rightarrow \overline{I(f)} \leq \overline{I(g)} + \overline{I(f_1)} = \overline{I(g)} + I(f_1) \leq limt I(f_n)$
{From *}
$\overline{I(f)} \leq limt\, I(f_n)$ ... ... (***)
From (**) and (***) we have $\overline{I(f)} \leq limt\, I(f_n) \leq \underline{I(f)} \Rightarrow \overline{I(f)} = \underline{I(f)} = limt\, I(f_n) \Rightarrow$ f $\in L_1$
And $I(f) = limt\, I(f_n)$, *Proved*.

Fatous Lemma: Let $\{f_n\}$ be any sequence of non–negative functions in L_1, then Inf $\{f_n\} \in L_1$. Further $\underline{limt}\{f_n\} \in L_1$ if $\underline{limt}\{f_n\} < \infty$ and I ($\underline{limt} f_n) \leq \underline{limt} I(f_n)$.

Proof: Let Inf $\{f_n\} = g$, Define g_n= min $\{f_1, f_2, f_3, \dots\dots, f_n\}$ then g_n is a monotone decreasing sequence of non-negative functions in L_1 and $\{g_n\} \downarrow g$ then $\{-g_n\} \uparrow -g$. Then I $(g_n) \geq 0$
$\Rightarrow$ -I $(g_n) \leq 0 \Rightarrow$ I$(-g_n) \leq 0 \Rightarrow$ limt I$(-g_n) \leq 0 \Rightarrow$ limt I$(-g_n)$ } $\in L_1$. $\Rightarrow -g \in L_1 \Rightarrow g \in L1 \Rightarrow$ Inf $\{f_n\} \in L_1$.
To prove the second part assume that $\underline{limt} I(f_n) < \infty$ Define h = $\underline{limt}\,(f_n)$,
h_n = inf $\{f_n, f_{n+1}, f_{n+2}, \dots\dots\}$ then h = limt(h_n) and (h_n) is an increasing sequence of members of L_1 then $h_n \leq f_v$ where $v \geq n \Rightarrow I(h_n) \leq I(f_v)$ for all v where $v \geq n \to I(h_n) \leq \underline{limt} I(f_v)$
$\Rightarrow$ limt $I(h_n) \leq \underline{limt} I(f_v) < \infty \Rightarrow$ limt $I(h_n) < \infty \Rightarrow h \in L_1$ and I(h) = limt $I(h_n)$) $\Rightarrow$ I($\underline{limt}(f_n)$) $\leq \underline{limt}\, I(f_n)$ which completes the proof.

Theorem :(Lebesgue's Dominated Convergence Theorem): Let $\{f_n\}$ be any sequence of non–negative functions in L_1 and $g \in L_1$, s.t. $|f_n| \leq g$ for all n, let f = limt(f_n) then $f \in L_1$ and $I(f)$ = limt$I(f_n)$.

Proof: Consider the functions f_n and g s.t. $|f_n| \leq g$ for all n thus -g $\leq f_n \leq g$, we can see that $\{f_n + g\}$ is a sequence of non-negative functions in L_1 thus I$(f_n + g)$ = I$(f_n) + I(g) \leq I(g) + I(g) \to \underline{limtI}(f_n + g) \leq 2\,I(g) < \infty$
[Because $g \in L_1$ thus $I(g) < \infty$]
$\Rightarrow \underline{limt}(f_n + g) \in L_1 \Rightarrow (f + g) \in L_1 \Rightarrow f \in L_1$ [Because $g \in L_1$]
$\Rightarrow I(f + g)$ = I {$\underline{limt}(f_n + g)$} $\leq \underline{limtI}(f_n + g)$ ... ... (**)
$\Rightarrow I(f) \leq \underline{limtI}(f_n)$.
Now consider g-f_n. Since $f_n \leq g$ for all n
Note that {g-f_n} is a sequence of non-negative members in L_1 $\Rightarrow$ I(g-f_n) = $I(g)$- $I(f_n) \leq 2\,I(g)$
[Because -$f_n \leq g$]
Shows that $\underline{limt}\, I$(g-f_n) $\leq 2\,I(g) < \infty \Rightarrow \underline{limt}\, I$(g-$f_n$) $\in L_1$ $\Rightarrow$ g-$f \in L_1$ $\Rightarrow f \in L_1$
and $I(g - f)$ = I {$\underline{limt}$(g-f_n)} $\leq \underline{limtI}$(g-
f_n) $\Rightarrow I(g) - I(f) < I(g) - \overline{limt}$ I(f_n) $\Rightarrow \overline{limt}$ I(
f_n) $\leq I(f)$ ... ... (***)
From (**) and (***) $\overline{limt}$ I(f_n) $\leq I(f) \leq \underline{limtI}(f_n) \Rightarrow \underline{limtI}(f_n)$= $\overline{limt}$I(f_n) $= I(f)$

$\Rightarrow I(f) = limt\ I(f_n)$, proves the theorem.

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