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Lattice Points on the Homogeneous Cone

$$7(x^2+y^2)-13xy=28z^2$$

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Abstract:
We obtain infinitely many non-zero distinct integral points on the homogeneous cone given by $7(x^2+y^2)-13xy=28z^2$. A few interesting relations between the solutions and special number patterns are presented.

Keywords: *Ternary Quadratic, Lattice Points, Homogeneous cone.*
MSC 2000 subject classification:11D09.

Notations:

- $t_{m,n}$: Polygonal number of rank n with size m
- P_n : Pronic number of rank n
- G_n : Gnomonic number of rank n
- S_n : Star number of rank n
- M_n : Mersenne number of rank n
- HG_n : Hexagonal number of rank n
- Ky_n : Kynea number of rank n
- W_n : Woodall number of rank n
- TK_n : Thabit ibn Kurrah number of rank n
- P_n^m : Pyramidal number of rank n with size m
- SO_n : Stella octagonal number of rank n

1. Introduction

The ternary quadratic diophantine equations (homogeneous and non-homogeneous) offer an unlimited field for research of variety in [1-2].For an extensive review of various problems one may refer[3-19].This communication concerns with yet another interesting ternary quadratic equation $7(x^2+y^2)-13xy=28z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations between the solutions and special number patterns are presented.

2. Method of Analysis

The ternary quadratic equation representing homogeneous equation is

$$7(x^2+y^2)-13xy=28z^2 \quad (1)$$

2.1. Pattern I

Introducing the linear transformations,

$$x=u+v, y=u-v \quad (2)$$

in (1) leads to

$$u^2+27v^2=28z^2 \quad (3)$$

Let

$z = a^2 + 27b^2$ (4)

write 28 as,

$28 = (1 + i3\sqrt{3})(1 - i3\sqrt{3})$ (5)

Using (4) and (5) in (3) and applying the method of factorization, define

$u + i3\sqrt{3}v = (1 + i3\sqrt{3})(a + i3\sqrt{3}b)^2$

Equating the real and imaginary parts, we get

$u = a^2 - 27b^2 - 54ab$

$v = a^2 - 27b^2 + 2ab$

The corresponding integral values of x, y, z satisfying (1) are obtained as,

$x(a,b) = 2a^2 - 54b^2 - 52ab$

$y(a,b) = -56ab$

$z(a,b) = a^2 + 27b^2$

2.1.1. Properties

1. Each of the following expression is a nasty number.

$6\left[x(a,b) - y(a,b) - z(a,b) + 85b^2\right]$

$z(a,a) - 4a^2$

$6\left[z(a,2^a) - 9(TK_{2a} + 1)\right]$

2. $y(a,a-1) + 56P_{a-1} = 0$

3. $x(a,a(a-1)) + 54P_{(a-1)^2} + 104t_{3,a} = 0$

2.2. Pattern II

(3) can be written as,

$27v^2 = 28z^2 - u^2$ (6)

write 27 as,

$27 = (\sqrt{28} + 1)(\sqrt{28} - 1)$ (7)

Let

$v = 28a^2 - b^2$ (8)

Substituting (7) and (8) in (6) and employing the method of factorization, define

$(\sqrt{28} + 1)(\sqrt{28} - 1)(\sqrt{28}a + b)^2(\sqrt{28}a - b)^2 = (\sqrt{28}z + u)(\sqrt{28}z - u)$

Equating the rational and irrational parts, we get

$z = 28a^2 + b^2 + 2ab$

$u = 28a^2 + b^2 + 56ab$

The corresponding integral values of x, y, z satisfying (1) are obtained as,

$x(a,b) = 56a^2 + 56ab$

$y(a,b) = 2a^2 + 56ab$

$z(a,b) = 28a^2 + b^2 + 2ab$

2.2.1. Properties

1. $x(a,b) \equiv y(a,b) \pmod{54}$

2. $t_{56,a} + (a+b)^2 - z(a,b) \equiv 0 \pmod{26}$

3. $x(a,a) + y(a,a) - z(a,a) \equiv 0 \pmod{139}$

4. $z(a,a-1) - S_a - 2a$ is a perfect square.

5. $x(a,1) - 56P_a = 0$

2.3. Pattern III

(3) can be written as,

$$u^2 + 27v^2 = 27z^2 + z^2$$
$$u^2 - z^2 = 27z^2 - 27v^2$$
$$\frac{u+z}{z-v} = 27\frac{z+v}{u-z} = \frac{A}{B}, B \neq 0$$

This is equivalent to the following two equations

$$Bu + z(B-A) + Av = 0$$
$$-Au + z(27B + A) + 27Bv = 0$$

On employing the method of cross multiplication, we get

$$u = 27b^2 - 54ab - 2a^2$$
$$v = 27b^2 + 2ab - a^2$$
$$z = -27b^2 - a^2$$

The corresponding integral values of x, y, z satisfying (1) are obtained as,

$$x(a,b) = 56b^2 - 52ab - 2a^2$$
$$y(a,b) = -56ab$$
$$z = -27b^2 - a^2$$

2.3.1. Properties

1. $x(a,b) - y(a,b) + z(a,b) \equiv 1 \pmod 3$
2. $z(a,b) + a^2 \equiv 0 \pmod{27}$
3. $y(a, 2^a) + 56(W_a + 1) = 0$

2.4. Pattern IV

Introducing the linear transformation,

$$u = X + 27T, v = X - T \quad (9)$$

Substituting (9) in (3) we get,

$$X^2 + 27T^2 = z^2 \quad (10)$$

Solving (10)

$$T = 2ab, z = 27a^2 + b^2, X = 27a^2 - b^2$$

Then the corresponding integral values of x, y, z satisfying (1) are obtained as,

$$x(a,b) = 54a^2 - 2b^2 + 52ab$$
$$y(a,b) = 56ab$$
$$z(a,b) = 27a^2 + b^2$$

2.4.1. Properties

1. $x(a,b) - y(a,b) - 2z(a,b) \equiv 0 \pmod 4$
2. $y(a, a-1) - z(a, a-1) + 23G_a$ can be written as difference between two perfect square.
3. $z(2^a, 1) - 27M_a \equiv 0 \pmod{28}$

2.5. Pattern V

(3) can be written as,

$$1 = \sqrt{28} - \sqrt{27} \quad (11)$$

Let

$$u = 28a^2 - 27b^2 \quad (12)$$

write 1 as

$$1 = (\sqrt{28} - \sqrt{27})(\sqrt{28} + \sqrt{27}) \quad (13)$$

Substituting (12) and (13) in (11) and applying the method of factorization we define,

$$(\sqrt{28} + \sqrt{27})(\sqrt{28}a + \sqrt{27}b)^2 = (\sqrt{28}z + \sqrt{27}v)$$

Equating the rational and irrational parts, we get

$$z = 28a^2 + 27b^2 + 54ab$$
$$u = 28a^2 - 27b^2$$
$$v = 28a^2 + 27b^2 + 56ab$$

Then the corresponding integral values of x, y, z satisfying (1) are obtained as,

$$x(a,b) = 56a^2 + 56ab$$
$$y(a,b) = -54b^2 - 56ab$$
$$z(a,b) = 28a^2 + 27b^2 + 54ab$$

2.5.1. Properties

1. $12\left[x(a,a) + y(a,a)\right]$ is a nasty number.
2. $x(a,a-1) - z(a,a-1) - HG_a \equiv 25 \pmod{53}$
3. $56\left[Ky_a + 1\right] - x(2^a, 2) = 0$

3. Remarkable Observations

1. If (x_0, y_0, z_0) is any known solution of (1) then the triple (X, Y, Z) given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 21^{2n-2} \begin{pmatrix} 21 & 0 & 0 \\ -13 & 35 & -56 \\ -13 & 14 & -35 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

also satisfies (1).

2. $7\left[\left(\frac{6P_{x-1}^4}{t_{3,2(x-1)}}\right)^2 + \left(\frac{P_y^5}{t_{3,y}}\right)^2\right] - 13\left(\frac{6P_{x-1}^4}{t_{3,2(x-1)}}\right)\left(\frac{P_y^5}{t_{3,y}}\right) \equiv 0 \pmod{28}$

3. $7\left[\left(\frac{3P_{x-2}^3}{t_{3,x-2}}\right)^2 + \left(\frac{3P_y^3}{t_{3,y+1}}\right)^2\right] - 13\left(\frac{3P_{x-2}^3}{t_{3,x-2}}\right)\left(\frac{3P_y^3}{t_{3,y+1}}\right) \equiv 0 \pmod{28}$

4. $7\left[\left(\frac{6P_x^4}{HG_{,x+1}}\right)^2 + \left(\frac{2P_{y-1}^8}{t_{3,2y-3}}\right)^2\right] - 13\left(\frac{6P_x^4}{HG_{,x+1}}\right)\left(\frac{2P_{y-1}^8}{t_{3,2y-3}}\right) \equiv 0 \pmod{28}$

5. Each of the following expression is a nasty number

   i. $42\left[7\left[\left(\frac{6P_x^4}{t_{3,2x+1}}\right)^2 + \left(\frac{HG_y}{G_y}\right)^2\right] - 13\left(\frac{6P_x^4}{t_{3,2x+1}}\right)\left(\frac{HG_y}{G_y}\right)\right]$

   ii. $42\left[7\left[\left(\frac{3P_x^3}{t_{3,x+1}}\right)^2 + \left(\frac{HG_y}{SO_y}\right)^2\right] - 13\left(\frac{3P_x^3}{t_{3,x+1}}\right)\left(\frac{HG_y}{SO_y}\right)\right]$

4. Conclusion

To conclude one may search for other patterns of solutions and their properties:

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