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Lie Symmetry Solution of Third Order First Degree Nonlinear Wave Equation of Fourth Degree in Second Derivative

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Abstract:
In this paper Lie symmetry analysis is used to find the solution to a third order first degree nonlinear ordinary differential equation (ODE) fourth degree in second derivative that arise in waves of systems like water in shallow oceans. This method gives exact solutions to problems and does not depend on initial boundary values.

Keywords: *Lie groups, infinitesimal generators, Lie algebras, prolongations and invariance under transformations*

1. Introduction

A differential equation is an equation in which at least one term contains any differential coefficients such as

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4} \, and \, \frac{d^5y}{dx^5},$$

whose solution is an equation relating x and y which contains no differential coefficients. They frequently occur in scientific and engineering problems. There are two categories of differential equations namely ordinary differential equations (ODEs) and partial differential equations (PDEs). Any relation between the variables x, y and the derivatives,

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots,$$

is called an ordinary differential equation (ODE). The term ordinary distinguishes it from a partial differential equation which involves the partial derivatives. We have considered a nonlinear ODE in this work. A differential equation with one independent variable present is called an ODE, for example $y''(x) + y(x) = 0$, $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$, $\frac{dy}{dx} = \cos x$ and $y''' + 4y'' + 2y' - 6y = 0$.

The symmetry group of nonlinear ODEs is a group of transformations of independent and dependent variables that leave all solutions invariant. This symmetry group generates new solutions to a given nonlinear ODE which can be used to reduce its highest order to a first order. Norwegian mathematician Sophus Lie introduced the notion of Lie group to study the solutions of ODEs. In this approach, the symmetries of a nonlinear ODE serve the main purpose of identifying its equivalence class or classes, for which a canonical form is known (Riley, *et. al,* 2008). According to Aminer, (2014) Sophus Lie recognized the transformation properties of a nonlinear ODE under certain groups of continuous transformations as being fundamental in analyzing its solution (Mehmet, 2004). Therefore, a symmetry group is a group of transformations that map any solution of the system onto another solution of the same system. The nonlinear ODE:

$$y''' - y'\left(\frac{y''}{y}\right)^4 = 0 \qquad (1)$$

is a third order first degree nonlinear ordinary differential equation of fourth degree in the second derivative. Differential equations may be formed in practice terms from a consideration of the physical problems to which they refer. To solve a differential equation, we have to find the function for which the equation is true. This nonlinear ODE is commonly used in many physical applications especially in engineering field and its very complex to be solved analytically. Differential equations have limited methods for obtaining exact solutions and they exploit their symmetries which lead to exact solutions (Dresner, 1999).

2. Literature Review

Opiyo, (2015) used the method of Lie symmetries to solve a third order first degree nonlinear ordinary differential equation of cubic in the second order derivative and got a solution. The equation was of the form:

$$y''' = y\left(\frac{y''}{y'}\right)^3 \tag{2}$$

whose solution is:

$$V = \frac{1}{A(u')^4}\int u\left(u''\right)^3\left(u'\right)^{-2}\,du \tag{3}$$

Aminer, (2014) applied Lie symmetry analysis in solving a fourth order nonlinear wave equation, a special type of nonlinear ODE. The form of the equation was:

$$\left(yy'\left(y\left(y'\right)^{-1}\right)''\right)' = 0 \tag{4}$$

and its solution is:

$$V = \frac{1}{u^3 + e^{2u'-1}}\int\left(u^3 + e^{2u'-1}\right)\left(4u^{-1}u''^2u'^{-4} - 4u''^3u'^{-6} - u^{-2}u'^{-2}u''\right)du \tag{5}$$

Yulia, (2008) solved the equivalence problem of the third order ordinary differential equations which was quadratic in the second order derivative without a higher degree. For this group of differential equations, the differential invariants of the group of the point equivalence transformations and the invariant differentiation operators were constructed.

3. Mathematical General Solution

Our objective was to solve the special case of the form (1) using the method of Lie symmetry. By expressing (1) as transformation equation gives:

$$y''' - y'\left(y\right)^{-4}\left(y''\right)^4 = 0 \tag{6}$$

after applying the law of indices and removal of the fraction.

By applying the m^{th} extension of G given as:

$$G^{[m]} = G + \sum_{i=1}^{m}\left(\phi^{(i)} - \sum_{j=1}^{i}\begin{bmatrix} i \\ j \end{bmatrix} y^{(i+1-j)}\omega^{(i)}\right)\frac{\partial}{\partial y^{(i)}}$$

where m is the order, i is the upper limit and j is the lower limit.

Then the third extension of $G^{[3]}$ is:

$$G^{[3]} = \omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \left(\phi' - \omega' y'\right)\frac{\partial}{\partial y'} + \left(\phi'' - 2y''\omega' - y'\omega''\right)\frac{\partial}{\partial y''} + \left(\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega'''\right)\frac{\partial}{\partial y'''} \tag{7}$$

Now, manipulating $G^{[3]}$ on (6) yields:

$$\left(\omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \left(\phi' - \omega' y'\right)\frac{\partial}{\partial y'} + \left(\phi'' - 2\omega' y'' - \omega'' y'\right)\frac{\partial}{\partial y''} + \left(\phi''' - 3\omega' y''' - 3\omega'' y'' - \omega''' y'\right)\frac{\partial}{\partial y'''}\right)$$
$$\left(y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right) = 0 \tag{8}$$

Hence, expansion of (8) gives:

$$\Rightarrow \omega\frac{\partial}{\partial x}\left[y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right] + \phi\frac{\partial}{\partial y}\left[y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right] + \left(\phi' - \omega' y'\right)\frac{\partial}{\partial y'} + \left(\phi'' - 2y''\omega' - y'\omega''\right)\frac{\partial}{\partial y''}$$
$$\left[y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right] + \left(\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega'''\right)\frac{\partial}{\partial y'''}\left[y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right] = 0 \tag{9}$$

From (9) after simplifying and then combining gives:

$$\omega\left[\ y^{(iv)}-(y'')^5(y)^{-4}-4y'(y'')^3(y)^{-4}y'''+4(y')^2(y'')^4(y)^{-5}\ \right]+\phi\left[4y'(y'')^4(y)^{-5}\right]+$$
$$(\phi'-\omega'y')\left[-(y'')^5(y)^{-4}\right]+(\phi''-2y''\omega'-y'\omega'')\left[-4y'(y'')^3(y)^{-4}y'''\right]+(\phi'''-3y'''\omega'-3y''\omega''-y'\omega''')=0 \quad (10)$$

Thus, from (6) we have: $y'''-y'(y'')^4(y)^{-4}=0$

$$\Rightarrow y'''=y'(y'')^4(y)^{-4} \quad (11)$$

and $y^{(iv)}=(y''')'$ hence $y^{(iv)}=\left(y'(y'')^4(y)^{-4}\right)'$ $y^{(iv)}=(y'')^5(y)^{-4}+4y'(y'')^3y'''(y)^{-4}-4(y')^2(y'')^4(y)^{-5}$ (12)

By substituting gives:

$$\omega\left[\ (y'')^5(y)^{-4}+4y'(y'')^3y'''(y)^{-4}-4(y')^2(y'')^4(y)^{-5}-(y'')^5(y)^{-4}-4y'(y'')^3(y)^{-4}y'''+\right.$$
$$\left.4(y')^2(y'')^4(y)^{-5}\ \right]+\left[\ 4y'(y'')^4(y)^{-5}\ \right]\phi-\left[(y'')^5(y)^{-4}\right](\phi'-\omega'y')-$$
$$\left[4y'(y'')^3(y)^{-4}y'''\right](\phi''-2y''\omega'-y'\omega'')+(\phi'''-3y'''\omega'-3y''\omega''-y'\omega''')=0 \quad (13)$$

Further simplification gives:

$$\omega(y'')^5(y)^{-4}+4\omega y'(y'')^3(y)^{-4}y'''-4\omega(y')^2(y'')^4(y)^{-5}-\omega(y'')^5(y)^{-4}-$$
$$4\omega y'(y'')^3(y)^{-4}y'''+4\omega(y')^2(y'')^4(y)^{-5}+4\phi y'(y'')^4(y)^{-4}-\phi'(y'')^5(y)^{-4}+$$
$$\omega'y'(y'')^5(y)^{-4}-4\phi''y'(y'')^3(y)^{-4}y'''+8\omega'y'(y'')^4(y)^{-4}y'''+4\omega''(y')^2(y'')^3(y)^{-4}y'''+$$
$$\phi'''-3\omega'y'''-3\omega''y''-y'\omega'''=0 \quad (14)$$

Again when simplified, it gives:

$$4\phi y'(y)^{-4}(y'')^4-\phi'(y)^{-4}(y'')^5-4\phi''y'(y)^{-4}(y'')^3y'''+\phi'''+$$
$$8\omega'y'(y)^{-4}(y'')^4y'''+\omega'y'(y)^{-4}(y'')^5-3\omega'y'''-3\omega''y''+$$
$$4\omega''(y)^{-4}(y')^2(y'')^3y'''-y'\omega'''=0 \quad (15)$$

By expressing first, second and third derivatives of ω and ϕ in terms of partial derivatives given that:

$$\omega=\omega(x,y) \text{ then } d(x)=\left(\frac{\partial\omega}{\partial x}\right)dx+\left(\frac{\partial\omega}{\partial y}\right)dy\ \therefore\ \omega'=\frac{\partial\omega}{\partial x}+y'\frac{\partial\omega}{\partial y} \quad (16)$$

$$\omega''=\frac{\partial^2\omega}{\partial x^2}+2y'\frac{\partial^2\omega}{\partial x\partial y}+y'^2\frac{\partial^2\omega}{\partial y^2}+y''\frac{\partial\omega}{\partial y} \quad (17)$$

$$\omega'''=\frac{\partial^3\omega}{\partial x^3}+3y'\frac{\partial^3\omega}{\partial x^2\partial y}+3y''\frac{\partial^2\omega}{\partial x\partial y}+y'''\frac{\partial\omega}{\partial y}+3y'y''\frac{\partial^2\omega}{\partial y^2}+y'^3\frac{\partial^3\omega}{\partial y^3}+3y'^2\frac{\partial^3\omega}{\partial x\partial y^2} \quad (18)$$

and also $\phi=\phi(x,y)$ then $d(\phi)=\left(\frac{\partial\phi}{\partial x}\right)dx+\left(\frac{\partial\phi}{\partial y}\right)dy\ \therefore\ \phi'=\frac{\partial\phi}{\partial x}+y'\frac{\partial\phi}{\partial y}$ (19)

$$\phi''=\frac{\partial^2\phi}{\partial x^2}+2y'\frac{\partial^2\phi}{\partial x\partial y}+y'^2\frac{\partial^2\phi}{\partial y^2}+y''\frac{\partial\phi}{\partial y} \quad (20)$$

$$\phi'''=\frac{\partial^3\phi}{\partial x^3}+3y'\frac{\partial^3\phi}{\partial x^2\partial y}+3y''\frac{\partial^2\phi}{\partial x\partial y}+y'''\frac{\partial\phi}{\partial y}+3y'^2\frac{\partial^3\phi}{\partial x\partial y^2}+3y'y''\frac{\partial^2\phi}{\partial y^2}+y'^3\frac{\partial^3\phi}{\partial y^3} \quad (21)$$

When we substitute (16), (17), (19), (20), (21) into (15) we get:

$$\Rightarrow 4\phi y'(y)^{-4}(y'')^4-\phi'(y)^{-4}(y'')^5-4\phi''y'(y)^{-4}(y'')^3y'''+\phi'''+8\omega'y'(y)^{-4}(y'')^4y'''+\omega'y'(y)^{-4}(y'')^5-$$
$$3\omega'y'''-3\omega''y''+4\omega''(y)^{-4}(y')^2(y'')^3y'''-y'\omega'''=0 \quad (22)$$

When we expand, it yields: $4\phi y'(y)^{-4}(y'')^4-(y)^{-4}(y'')^5\frac{\partial\phi}{\partial x}-(y)^{-4}(y'')^5y'\frac{\partial\phi}{\partial y}-4y'(y)^{-4}(y'')^3y'''\frac{\partial^2\phi}{\partial x^2}-$

$$8(y')^2(y)^{-4}(y'')^3y'''\frac{\partial^2\phi}{\partial x\partial y}-4(y')^3(y)^{-4}(y'')^3y'''\frac{\partial^2\phi}{\partial y^2}-4y'(y)^{-4}(y'')^4y'''\frac{\partial\phi}{\partial y}+\frac{\partial^3\phi}{\partial x^3}+$$

$$3y'\frac{\partial^3\phi}{\partial x^2\partial y}+3y''\frac{\partial^2\phi}{\partial x\partial y}+y'''\frac{\partial\phi}{\partial y}+3(y')^2\frac{\partial^3\phi}{\partial x\partial y^2}+3y''y'\frac{\partial^2\phi}{\partial y^2}+(y')^3\frac{\partial^3\phi}{\partial y^3}+y'''\frac{\partial\phi}{\partial y}+8y'(y)^{-4}(y'')^4y'''\frac{\partial\omega}{\partial x}+$$
$$8(y')^2(y)^{-4}(y'')^4y'''\frac{\partial\omega}{\partial y}+y'(y)^{-4}(y'')^5\frac{\partial\omega}{\partial x}+(y')^2(y)^{-4}(y'')^5\frac{\partial\omega}{\partial y}-3y'''\frac{\partial\omega}{\partial x}-3y'''y'\frac{\partial\omega}{\partial y}-3y''\frac{\partial^2\omega}{\partial x^2}-$$
$$6y''y'\frac{\partial^2\omega}{\partial x\partial y}-3y''(y')^2\frac{\partial^2\omega}{\partial y^2}-3(y'')^2\frac{\partial\omega}{\partial y}+4(y)^{-4}(y')^2(y'')^3y'''\frac{\partial^2\omega}{\partial x^2}+8(y')^3(y)^{-4}(y'')^3y'''\frac{\partial^2\omega}{\partial x\partial y}+$$
$$4(y')^4(y)^{-4}(y'')^3y'''\frac{\partial^2\omega}{\partial y^2}+4(y'')^4(y)^{-4}(y')^2y'''\frac{\partial\omega}{\partial y}-y'\frac{\partial^3\omega}{\partial x^3}-3(y')^2\frac{\partial^3\omega}{\partial x^2\partial y}-3y'y''\frac{\partial^2\omega}{\partial x\partial y}-3(y')^3\frac{\partial^3\omega}{\partial x\partial y^2}-$$
$$3y''(y')^2\frac{\partial^2\omega}{\partial y^2}-(y')^4\frac{\partial^3\omega}{\partial y^3}-y'y''\frac{\partial\omega}{\partial y}=0 \quad (23)$$

where (23) forms an identity in x, y, y', y'', y'''. Given that ω and ϕ are functions in x and y alone, by equating the combinations of coefficients of the powers of y', y'', y''' to zero and integrating yields:

$$\omega = A_1 y + A_2 \quad (24)$$

where A_1 and A_2 are arbitrary functions of x.

$$\phi = A_1' y^2 + A_3 y + A_4 \quad (25)$$

where A_3 and A_4 are arbitrary functions of x.

$$3A_1'' y + 2A_3' - A_2'' = 0 \quad (26)$$

By equating the coefficients of powers of y^0 and y^1 to zero in (26) and substituting:

$$A_1''' y^{-2} + A_3''(y)^{-3} + A_4''(y)^{-4} = 0 \quad (27)$$

By equating the coefficients of powers of y^{-4}, y^{-3} and y^{-2} to zero and solving :

$$A_1 = B_1 x + B_2 \quad (28)$$

$$A_3 = B_3 x + B_4 \quad (29)$$

$$A_2 = B_3 x^2 + B_5 x + B_6 \quad (30)$$

$$A_4 = B_7 x + B_8 \quad (31)$$

where $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8$ are arbitrary constants. From $\omega = A_1 y + A_2$ and substituting A_1 and A_2 :

$$\omega = B_1 xy + B_2 y + B_3 x^2 + B_5 x + B_6 \quad (32)$$

From $\phi = A_1' y^2 + A_3 y + A_4$ then by substituting A_1, A_3 and A_4:

$$\phi = (B_1 x + B_2)' y^2 + (B_3 x + B_4) y + B_7 x + B_8 \quad (33)$$

Now the infinitesimal generator G is of the form: $G = \omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y}$

By substituting ω and ϕ, this form is then given as :

$$G = (B_1 xy + B_2 y + B_3 x^2 + B_5 x + B_6)\frac{\partial}{\partial x} + (B_1 y^2 + B_3 xy + B_4 y + B_7 x + B_8)\frac{\partial}{\partial y} \therefore G = B_1\left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right) +$$
$$B_2\left(y\frac{\partial}{\partial x}\right) + B_3\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right) + B_4\left(y\frac{\partial}{\partial y}\right) + B_5\left(x\frac{\partial}{\partial x}\right) + B_6\left(\frac{\partial}{\partial x}\right) + B_7\left(x\frac{\partial}{\partial y}\right) + B_8\left(\frac{\partial}{\partial y}\right) \quad (34)$$

which is $eight$ parameter symmetry. We can generate an $eight - one$ parameter symmetry given by:

$$G_1 = \frac{\partial}{\partial x},\ G_2 = \frac{\partial}{\partial y},\ G_3 = x\frac{\partial}{\partial x},\ G_4 = y\frac{\partial}{\partial x},\ G_5 = y\frac{\partial}{\partial y},\ G_6 = x\frac{\partial}{\partial y},\ G_7 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},$$
$$G_8 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \quad (35)$$

Now, the non - zero Lie brackets are given as follows:

$[G_1,G_3]=G_1$, $[G_1,G_6]=G_2$, $[G_2,G_4]=G_1$, $[G_2,G_5]=G_2$, $[G_2,G_8]=G_6$, $[G_3,G_4]=-G_4$,
$[G_3,G_6]=G_6$, $[G_4,G_5]=-G_4$, $[G_4,G_8]=G_7$, $[G_5,G_7]=G_7$

The process of finding the symmetries of ordinary differential equations is highly systematic.

Thus let $S_1=\frac{\partial}{\partial x}$, $S_3=x\frac{\partial}{\partial x}$ which are the Lie solvable algebra of the admitted $eight-one$ parameter symmetry (35) .

By solving using prolongation: $G^{[3]}=G^{[2]}+\left(\phi'''-3y'''\omega'-3y''\omega''-y'\omega'''\right)\frac{\partial}{\partial y'''}$ (36)

Consider the third order prolongation for the differential invariant operator:

$$G=S=\omega\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial y}.$$

$S_1^{[0]}=\frac{\partial}{\partial x}$, $S_1^{[1]}=\frac{\partial}{\partial x}$, $S_1^{[2]}=\frac{\partial}{\partial x}$,

$$S_1^{[3]}=1\bullet\frac{\partial}{\partial x}+0\bullet\frac{\partial}{\partial y} \quad (37)$$

By solving for the characteristic: $\frac{dx}{1}=\frac{dy}{0}$

$dy=0$ and integrating yields the differential invariant: $y=U$ (39)

where U is a constant , a function of x .

Again, consider the third order prolongation of the differential invariant operator:

$G=S=\omega\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial y}$, if $S_3=x\frac{\partial}{\partial x}$ it follows that : $S_3^{[0]}=x\frac{\partial}{\partial x}$, $S_3^{[1]}=x\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y'}$,

$S_3^{[2]}=x\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y'}-2y''\frac{\partial}{\partial y''}$,

$$S_3^{[3]}=x\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y'}-2y''\frac{\partial}{\partial y''}-3y'''\frac{\partial}{\partial y'''} \quad (40)$$

By solving for the characteristics : $\frac{dy}{1}=\frac{dy'}{-y'}=\frac{dy''}{-2y''}=\frac{dy'''}{-3y'''}$ (41)

Then by integrating (41) the differential invariants are as follows:

(i) $\frac{dy}{1}=\frac{dy'}{-y'}$

$$\therefore y=\ln\left|\frac{C_1}{y'}\right| \quad (42)$$

where C_1 is a constant.

(ii) $\frac{dy'}{-y'}=\frac{dy''}{-2y''}$, $t_1=\frac{1}{C_2}$

$$\therefore t_1=\frac{y''}{(y')^2} \quad (43)$$

where C_2 and t_1 are constants.

(iii) $\frac{dy'}{-y'}=\frac{dy'''}{-3y'''}$, $t_2=\frac{1}{C_3}$

$$\therefore t_2=\frac{y'''}{(y')^3} \quad (44)$$

where C_3 and t_2 are constants.

(iv) $\frac{dy''}{-2y''} = \frac{dy'''}{-3y'''}$, $t_3 = \frac{1}{C_4}$ $\therefore t_3 = \frac{(y''')^2}{(y'')^3}$ (45)

where C_4 and t_3 are constants.

By taking (39) and (43) such that: $y = U$ and $t_1 = \frac{y''}{(y')^2}$. Let $t_1 = V$ then (43) becomes :

$$\therefore V = \frac{y''}{(y')^2} \quad (46)$$

Now reducing (1) to first order ODE (Dresner, 1999) yields: $\frac{dV}{dU} = \frac{D_x(V)}{D_x(U)}\ \frac{dV}{dU} = (y'')^4 (y')^{-2} (y)^{-4} - \frac{2(y'')(y'')}{(y')^2 (y')^2}$

From (39) and (46) through substitution leads to :

$$\therefore \frac{dV}{dU} + 2V^2 = (U'')^4 (U')^{-2} (U)^{-4} \quad (47)$$

Then (47) is of the form :

$$\frac{dV}{dU} + P(U)V = Q(U) \quad (48)$$

implying that we have managed to reduce a third order equation (1) to a simple first order linear equation (47) that is easily solvable by other known simpler methods.

This can be easily integrated using integrating factors given by : $I(U)$

Thus $I(U) = e^{\int P(U)dU}$ (49)

$I(U) = e^{\int 2VdU} = Me^{\ln |U'|^4}$ (if $e^C = M$) = $e^{\ln |U'|^4}$ (since C = 0, then M = 1)

then $\therefore I(U) = e^{\ln |U'|^4} = (U')^4$ where C and M are constants. From the form :

$$V = \frac{1}{I(U)} \int (U')^4 Q(U) dU \quad (50)$$

whose simplification leads to : $V = \frac{1}{(U')^4} \int (U'')^4 (U')^2 (U)^{-4} dU$ (51)

or $V = \frac{1}{(y')^4} \int (y'')^4 (y')^2 (y)^{-4} dy$

4. Conclusion

To achieve this, we developed symmetry generators, infinitesimal transformations, extended transformations (prolongations), differential invariants, invariant transformations, integrating factor and order reduction of the third order nonlinear ODE. This then enabled us to reduce the wave equation to a simple first order linear ordinary differential equation.

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6. References

i. Aminer, T. (2014). *Lie Symmetry Solution of Fourth Order Nonlinear Ordinary Differential Equation, PhD Thesis.* Bondo, Kenya: Jaramogi Oginga Odinga University of Science and Technology.
ii. Dresner, L. (1999). *Applications of Lie's Theory of Ordinary and Partial Differential Equations.* London: London Institute of Physics.
iii. Mehmet, C. (2004). *Lie Symmetries of Differential Equations by Computer Algebra.* Istanbul Turkiye: Instanbul Technical University, Mathematics Department Maslak 80626.
iv. Opiyo, N. (2015). *Symmetry Analysis of Third Order Nonlinear Ordinary Differential Equation which is Cubic in the Second Order Derivative, Masters Thesis.* Bondo, Kenya: Jaramogi Oginga Odinga University of Science and Technology.
v. Riley, K. F; Hobson, M.P and Bence, S.J. (2008). *Mathematical Methods for Physics and Engineering Third Edition.* Cambridge, United Kingdom: Cambridge University Press.
vi. Yulia, Y. B. (2008). *Equivalence of Third Order Ordinary Differential Equations to Chazy Equations IXIII.* Massachusetts: Massachusetts Institute of Technology.