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Bayesian Estimation and Prediction for a Mixture of Weibull and Lomax Distributions

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Abstract:
This paper focuses on the study of a mixture of two components Weibull and Lomax distributions based on a complete sample. Maximum likelihood estimation and Bayes estimation under informative and non-informative priors have been obtained using the symmetric squared error loss function (SELF) and the asymmetric linear exponential (LINEX) and general entropy (GELF) loss functions. Bayesian prediction has been considered for future observation based on the observed sample. Finally, a simulation study as well as a numerical example are carried out to compare the performance of the estimators obtained.

Keywords: *Bayesian estimation, Bayesian prediction, Mixture model, Loss function, Maximum likelihood estimation, Simulation study.*

1. Introduction

The survival analysis of lifetime (failure time) data is an important topic in statistics and has applications in different fields of life and also is a collection of statistical procedures for data analysis for which the outcome variable of interest is the time until an event occurs.

In recent years, mixtures of distributions have gained great interest for the analysis. These models include finite and infinite number of components that can analyze different data sets. The families of mixture models have been widely used in many areas, including medicine, botany, life testing, reliability and etc. Also, there have been studies on mixtures of two identical distributions and mixtures of two different distributions .Abu-Zinadah (2010) considered a mixture of exponentiated Pareto and exponential distributions and obtained the maximum likelihood and Bayes estimators under complete and type- II censored samples. Erisoglu.et.al. (2011) proposed a mixture of two different distribution binaries of Gamma, Weibull and Exponential to model heterogeneous survival data. Turkan and Cahs (2014) used a mixture of two same kind of distributions in addition a mixture of two distributions from pairs of Exponential, Gamma, Lognormal and Weibull combinations of these distributions. Maximum Likelihood estimators of the parameters of the mixture models are obtained by using the EMalgorithm. Daniyal and Rajab (2015) discussed the classical properties of a mixture of Burr-XII and Lomax distributions.

The Weibull distribution has been widely used in modeling of lifetime event data; this is due to the variety of shapes of the probability density function (pdf) based on its parameters and its convenient repetition of the survival function. Many researchers studied Bayes estimation for the Weibull distribution, among them are Berger and San (1993), Kaminskiy and Krivtsov (2005), Zhang and Meeker (2005), Kundu (2008), Ahmed et al. (2010), Zhu et al. (2016) and many others.

The Lomax distribution, sometimes called Pareto of the second kind, has a considerable importance in the field of life testing because of its uses to fit business failure data [Lomax (1945)]. Some authors considered Bayesian estimation of Lomax distribution e.g.Howladerand Hossain (2002), Nasiri and Hosseini (2012), Afaqetal. (2015), AL-Noor and Alwan (2015) and Ferreira et al. (2016).

Many authors have studied Bayesian estimation of two components mixture distribution. Ahmed et. al. (1997) studied approximate method of estimation under type-II censoring for the shape parameters of a mixture of two Weibull distributions. Al-Hussaini et. al. (2001) obtained Bayesian prediction bounds for future observations based on a type -I censored sample from a finite mixture of two Lomax components. Jaheen (2003) studied prediction bounds for s^{th} future observation based type-I censored samples for parameters of a mixture of two component Gompertz distribution. Shawky and Bakoban (2009) considered the problem of estimating reliability, and failure rate functions of a finite mixture of two components from the exponentiated gamma distribution using maximum likelihood and Bayes methods of estimation. Ahmed et al. (2011) considered Bayesian estimation for the parameters of the finite mixture of the Burr type XII distribution with its reciprocal based on the type-I censored data. Mahmoud et al. (2014) developed

prediction bounds for future order statistics under a mixture of two generalized exponential distribution using type-II censored and complete data. Aslam and Feroze (2014) proposed Bayesian estimation for analyzing lifetime data based doubly censored sampling using two component mixture of Weibull distribution. Feroze (2015) discussed the Bayesian estimation for mixture of two component Inverse Weibull distribution under doubly censoring.
The objective of this work is to consider maximum likelihood and Bayesian estimation methods of the two components mixture of Weibull and Lomax distributions as a lifetime model, based on complete data. We derive the maximum likelihood estimators (MLE) and Bayes estimators using Jeffrey's prior and Gamma prior under symmetric and asymmetric loss functions. Also, we discussed the Bayesian two sample prediction bounds for the future observations for the considered model.
The rest of this paper is organized as follows: The two-component mixture of Weibull and Lomax distributions is defined in Section 2. In Section 3, we obtain maximum likelihood estimators and Bayes estimators of the parameters. Bayesian prediction is provided in Section 4. In Section 5, a Monte Carlo simulation study is conducted to compare the performance of different estimation methods of the parameters. In Section 6, a numerical example is provided. Finally, in Section7 we present a summary and a conclusion.

2. The Mixture Model

The probability density function (pdf)$f(x)$of a mixture of two components with mixing weights $(p, 1-p)$is:
$f(x;\alpha_1,\theta_1,\alpha_2,\theta_2) = pf_1(x;\alpha_1,\theta_1) + (1-p)f_2(x;\alpha_2,\theta_2)$
where $f_1(x;\alpha_1,\theta_1)$ is the Weibull distribution with two shape parameters α_1 and θ_1, and is given by
$f_1(x;\alpha_1,\theta_1) = \alpha_1\theta_1 x^{\theta_1-1}e^{-\alpha_1 x^{\theta_1}}$,$x > 0$,$\alpha_1 > 0$,$\theta_1 > 0$.
$f_2(x;\alpha_2,\theta_2)$ is the Lomax distribution with two shape parameters α_2 and θ_2 and is given by
$f_2(x;\alpha_2,\theta_2) = \alpha_2\theta_2(1+\theta_2 x)^{-(\alpha_2+1)}$,$x > 0$,$\alpha_2 > 0$,$\theta_2 > 0$.
It follows that

$$f(x;\alpha_1,\theta_1,\alpha_2,\theta_2) = p\alpha_1\theta_1 x^{\theta_1-1}e^{-\alpha_1 x^{\theta_1}} + (1-p)\alpha_2\theta_2(1+\theta_2 x)^{-(\alpha_2+1)} \quad (1)$$

$x > 0$,$\alpha_1 > 0$,$\theta_1 > 0$,$\alpha_2 > 0$,$\theta_2 > 0$,$p \in [0,1]$.
The corresponding cumulative distribution function (cdf) is given by:

$$F(x;\alpha_1,\theta_1,\alpha_2,\theta_2) = pF_1(x;\alpha_1,\theta_1) + (1-p)F_2(x;\alpha_2,\theta_2)$$

where$F_1(x;\alpha_1,\theta_1) = 1 - e^{-\alpha_1 x^{\theta_1}}$ and $F_2(x;\alpha_2,\theta_2) = 1-(1+\theta_2 x)^{-\alpha_2}$.
It follows that

$$F(x;\alpha_1,\theta_1,\alpha_2,\theta_2) = p\left(1 - e^{-\alpha_1 x^{\theta_1}}\right) + (1-p)[1-(1+\theta_2 x)^{-\alpha_2}] \quad (2)$$

The reliability function of the mixture is then given by:

$$R(x;\alpha_1,\theta_1,\alpha_2,\theta_2) = pR_1(x;\alpha_1,\theta_1) + (1-p)R_2(x;\alpha_2,\theta_2)$$

where $R_1(x;\alpha_1,\theta_1) = e^{-\alpha_1 x^{\theta_1}}$ and$R_2(x;\alpha_2,\theta_2) = (1+\theta_2 x)^{-\alpha_2}$
and $R(x;\alpha_1,\theta_1,\alpha_2,\theta_2) = p\left(e^{-\alpha_1 x^{\theta_1}}\right) + (1-p)(1+\theta_2 x)^{-\alpha_2}$ (3)

3. Estimation of the Parameters

In this section, the maximum likelihood and Bayes methods are applied to estimate the parametersα_1, α_2 given that$\theta_1$and $\theta_2$are known, for different loss functions.

3.1. Maximum Likelihood Estimation

Suppose a sample of n units is put on operation in life testing experiment and that the test is terminated if all n items taken from the population with pdf (1) have failed. After then units have failed each item can be attributed to the appropriate subpopulation. Thus, if the n units have failed during the interval $\left(0, x_{(n)}\right)$: r_1 from the first subpopulation and r_2 from the second subpopulation. Let x_{ij} denote the failure of the j^{th} unit that belongs to the i^{th} subpopulation and $x_{ij} \le x_{(n)}$; $j = 1,2,\dots\dots, r_i$; $i = 1,2$; $n = r_1 + r_2$. where $x_{(n)}$ denotes the failure time of the n^{th} unit. For a two component mixture model, the likelihood function, given$\underline{x} = \left(x_{11},\dots,x_{1r_1},x_{21},\dots,x_{2r_2}\right)$can be written as:

$$L\left(p,\alpha_1,\alpha_2,\theta_1,\theta_2|\underline{x}\right) \propto \prod_{j=1}^{r_1} pf_1\left(x_{1j};\alpha_1,\theta_1\right)\prod_{j=1}^{r_2}(1-p)f_2\left(x_{2j};\alpha_2,\theta_2\right) \quad (4)$$

Assuming that the parameters θ_1 and θ_2 are known, the likelihood function (4) reduces to

$$L\left(p,\alpha_1,\alpha_2,\theta_1,\theta_2|\underline{x}\right) \propto p^{r_1}(1-p)^{r_2}\alpha_1^{r_1}\alpha_2^{r_2}e^{-\alpha_1\sum_{j=1}^{r_1}x_{1j}^{\theta_1}}\prod_{j=1}^{r_2}\left(1+\theta_2 x_{2j}\right)^{-(\alpha_2+1)} \quad (5)$$

Thus, the log-likelihood function of the parameters α_1 , α_2 and p are given by

$$\ln L\left(p,\alpha_1,\alpha_2|\underline{x}\right) \propto r_1\ln p + r_2\ln(1-p) + r_1\ln\alpha_1 + r_2\ln\alpha_2 - \alpha_1\sum_{j=1}^{r_1} x_{1j}^{\theta_1} - (\alpha_2+1)\sum_{j=1}^{r_2} ln\left(1+\theta_2 x_{2j}\right) \quad (6)$$

Partially differentiating equation (6)with respect to α_1 , α_2 and p maximum likelihood estimates (MLEs) can obtained by solving the following equations after equating it to zero

$$\frac{\partial \ln L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial\alpha_1} = \frac{r_1}{\alpha_1} - \sum_{j=1}^{r_1} x_{1j}^{\theta_1}$$

$$\frac{\partial \ln L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial\alpha_2} = \frac{r_2}{\alpha_2} - \sum_{j=1}^{r_2} \ln\left(1+\theta_2 x_{2j}\right)$$

$$\frac{\partial \ln L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial p} = \frac{r_1}{p} - \frac{r_2}{1-p}$$

The second partial derivatives with respect toα_1 , α_2 and p are obtained as

$$\frac{\partial^2 \ln L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial\alpha_1^2} = -\frac{r_1}{\alpha_1^2}$$

$$\frac{\partial^2 \ln L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial\alpha_2\partial\alpha_1} = 0 \qquad \frac{\partial^2\, ln\, L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial\alpha_2^2} = -\frac{r_2}{\alpha_2^2}$$

$$\frac{\partial^2 \ln L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial p\partial\alpha_1} = 0 \quad \frac{\partial^2\, ln\, L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial p\partial\alpha_2} = 0 \quad \frac{\partial^2\, ln\, L\left(p,\alpha_1,\alpha_2|\underline{x}\right)}{\partial p^2} = -\frac{r_1}{p^2} - \frac{r_2}{(1-p)^2}$$

3.2. Bayes Estimation

In this section, we determine Bayes estimator of the parametersα_1 , α_2 and $p$for informative and non-informative prior based on the symmetric (square error) loss function and asymmetric
(LINEX and general Entropy) loss functions.

3.2.1. The Posterior Distribution under the Assumption of Informative Prior

Gamma (or conjugate) prior is the most widely used prior distribution. Let $\alpha_1 \sim Gamma(a_1,b_1)$,$\alpha_2 \sim Gamma(a_2,b_2)$ and $p \sim Beta(c,d)$ for the mixing parameter$p$. Assume the independence of the parameters, the joint prior distribution of α_1 , α_2 and p is

$$g_1(\alpha_1,\alpha_2,p) \propto \alpha_1^{a_1-1}\alpha_2^{a_2-1}e^{-(\alpha_1 b_1+\alpha_2 b_2)}p^{c-1}(1-p)^{d-1} \quad (7)$$

α_1,$\alpha_2 > 0$, $0 < p < 1$, a_1,a_2,b_1,b_2,c,$d > 0$.

By combining the likelihood function given in (5) and the joint prior (7)the joint posterior distribution given data is:

$$p_1(\alpha_1,\alpha_2,p|\underline{x}) = K_1^{-1}p^{r_1+c-1}(1-p)^{r_2+d-1}\alpha_1^{r_1+a_1-1}\alpha_2^{r_2+a_2-1}e^{-\alpha_1\varphi_1}e^{-\alpha_2\varphi_2} \quad (8)$$

whereK_1 is the normalizing constant given by

$$K_1 = \beta(r_1+c,r_2+d)\frac{\Gamma(r_1+a_1)}{(\varphi_1)^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{(\varphi_2)^{r_2+a_2}}$$

with

$$\varphi_1 = b_1 + \sum_{j=1}^{r_1} x_{1j}^{\theta_1}\, and \varphi_2 = b_2 + \sum_{j=1}^{r_2} \ln\left(1+\theta_2 x_{2j}\right)$$

3.2.1.1. Bayes Estimator under Squared Error Loss Function (SELF)

The estimators of α_1,α_2 and p using the squared error loss function is the mean of the corresponding marginal posterior density functions and are given by:

$$\hat{\alpha}_{1,SELF} = K_1^{-1}\beta(r_1+c,r_2+d)\frac{\Gamma(r_1+a_1+1)}{(\varphi_1)^{r_1+a_1+1}}\frac{\Gamma(r_2+a_2)}{(\varphi_2)^{r_2+a_2}}$$

$$\hat{\alpha}_{2,SELF} = K_1^{-1}\beta(r_1+c,r_2+d)\frac{\Gamma(r_1+a_1)}{(\varphi_1)^{r_1+a_1}}\frac{\Gamma(r_2+a_2+1)}{(\varphi_2)^{r_2+a_2+1}}$$

$$\hat{p}_{SELF} = K_1^{-1}\beta(r_1+c+1,r_2+d)\frac{\Gamma(r_1+a_1)}{(\varphi_1)^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{(\varphi_2)^{r_2+a_2}}$$

3.2.1.2. Bayes Estimator under Linear-Exponential Loss Function (LINEX)

Considering the LINEX loss function, the Bayes estimator of any parameter A is obtained from:

$$\hat{A}_{LINEX} = -\frac{1}{q}\ln\left[E\left(e^{-qA}|\underline{x}\right)\right]$$

provided that the expected value with respect to the posterior function of A,$E\left(e^{-qA}|\underline{x}\right)$,exists and is finite.

The Bayes estimator of α_1,α_2 and p can be obtained as

$$\hat{\alpha}_{1,LINEX} = -\frac{1}{q}\ln\left[K_1^{-1}\beta(r_1+c,r_2+d)\frac{\Gamma(r_1+a_1)}{(\varphi_1+q)^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{(\varphi_2)^{r_2+a_2}}\right]$$

$$\hat{\alpha}_{2,LINEX} = -\frac{1}{q}\ln\left[K_1^{-1}\beta(r_1+c,r_2+d)\frac{\Gamma(r_1+a_1)}{(\varphi_1)^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{(\varphi_2+q)^{r_2+a_2}}\right]$$

$$\hat{p}_{LINEX} = -\frac{1}{q}\ln\left[K_1^{-1}\frac{\Gamma(r_1+a_1)}{(\varphi_1)^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{(\varphi_2)^{r_2+a_2}}\sum_{k=0}^{\infty}\frac{(-q)^k}{k!}\beta(r_1+c+k,r_2+d)\right]$$

3.2.1.3. Bayes Estimator Under General Entropy Loss Function (GELF):

The Bayes estimator of any parameter Acan be obtained as:

$$\hat{A}_{GELF} = \left[E\left(A^{-h}|\underline{x}\right)\right]^{-\frac{1}{h}}$$

provided that the expected value with respect to the posterior function of A,$E\left(A^{-h}|\underline{x}\right)$,exists and is finite.

The Bayes estimator of α_1,α_2 and $p$are obtained as:

$$\hat{\alpha}_{1,GELF} = \left[K_1^{-1}\beta(r_1+c,r_2+d)\frac{\Gamma(r_1+a_1-h)}{(\varphi_1)^{r_1+a_1-h}}\frac{\Gamma(r_2+a_2)}{(\varphi_2)^{r_2+a_2}}\right]^{-\frac{1}{h}}$$

$$\hat{\alpha}_{2,GELF} = \left[K_1^{-1}\beta(r_1+c,r_2+d)\frac{\Gamma(r_1+a_1)}{(\varphi_1)^{r_1+a_1}}\frac{\Gamma(r_2+a_2-h)}{(\varphi_2)^{r_2+a_2-h}}\right]^{-\frac{1}{h}}$$

$$\hat{p}_{GELF} = \left[K_1^{-1}\beta(r_1+c-h,r_2+d)\frac{\Gamma(r_1+a_1)}{(\varphi_1)^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{(\varphi_2)^{r_2+a_2}}\right]^{-\frac{1}{h}}$$

3.2.2. The Posterior Distribution under the Assumption of non-Informative prior

Jeffery (1946) proposed a formal rule for obtaining a non-informative prior. The uninformative priors are recommended when no formal prior information about the parameters exist. This is defined as the density of the parameters proportional to the square root of the determinant of the Fisher information matrix. The prior distribution for the mixing parameter$p$ is uniform, i.e., $p\sim Uniform(0,1)$. Assume independence, the joint prior is given by:

$$g_2(\alpha_1,\alpha_2,p) \propto \frac{1}{\alpha_1\alpha_2}\alpha_1,\alpha_2 > 0,\quad 0<p<1 \tag{9}$$

By combining the likelihood function given in (5) and the joint prior(9), the posterior distribution given data is:

$$p_2\left(\alpha_1,\alpha_2,p\middle|\underline{x}\right) = K_2^{-1}p^{r_1}(1-p)^{r_2}\,\alpha_1^{r_1-1}\alpha_2^{r_2+-1}e^{-\alpha_1\varphi_1^*}e^{-\alpha_2\varphi_2^*} \tag{10}$$

whereK_2 is the normalizing constant given by

$$K_2 = \beta(r_1+1,r_2+1)\frac{\Gamma(r_1+a_1)}{(\varphi_1^*)^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{(\varphi_2^*)^{r_2+a_2}}$$

$$with\varphi_1^* = \sum_{j=1}^{r_1}x_{1j}^{\theta_1}\,and\varphi_2^* = \sum_{j=1}^{r_2}\ln\left(1+\theta_2x_{2j}\right)$$

3.2.2.1. Bayes Estimator under Squared Error Loss Function (SELF)

Under the squared error loss function the estimator of any parameter is the mean of the marginal posterior density function of that parameter. It follows that:

$$\hat{\alpha}_{1,SELF} = K_2^{-1}\beta(r_1+1,r_2+1)\frac{\Gamma(r_1+1)}{(\varphi_1^*)^{r_1+1}}\frac{\Gamma(r_2)}{(\varphi_2^*)^{r_2}}$$

$$\hat{\alpha}_{2,SELF} = K_2^{-1}\beta(r_1+1,r_2+1)\frac{\Gamma(r_1)}{(\varphi_1^*)^{r_1}}\frac{\Gamma(r_2+1)}{(\varphi_2^*)^{r_2+1}}$$

$$\hat{P}_{SELF} = K_2^{-1}\beta(r_1+2,r_2+1)\frac{\Gamma(r_1)}{(\varphi_1^*)^{r_1}}\frac{\Gamma(r_2)}{(\varphi_2^*)^{r_2}}$$

3.2.2.2. Bayes Estimator under Linear Exponential Loss Function (LINEX)

Bayes estimators of α_1,α_2 and p based on LINEX loss function are given by

$$\hat{\alpha}_{1,LINEX} = -\frac{1}{q}\ln\left[K_2^{-1}\beta(r_1+1,r_2+1)\frac{\Gamma(r_1)}{(\varphi_1^*+q)^{r_1}}\frac{\Gamma(r_2)}{(\varphi_2^*)^{r_2}}\right]$$

$$\hat{\alpha}_{2,LINEX} = -\frac{1}{q}\ln\left[K_2^{-1}\beta(r_1+1,r_2+1)\frac{\Gamma(r_1)}{(\varphi_1^*)^{r_1}}\frac{\Gamma(r_2)}{(\varphi_2^*+q)^{r_2}}\right]$$

$$\hat{p}_{LINEX} = -\frac{1}{q}\ln\left[K_2^{-1}\frac{\Gamma(r_1)}{(\varphi_1^*)^{r_1}}\frac{\Gamma(r_2)}{(\varphi_2^*)^{r_2}}\sum_{k=0}^{\infty}\frac{(-q)^k}{k!}\beta(r_1+k+1,r_2+1)\right]$$

3.2.2.3. Bayes Estimator under General Entropy Loss Function (GELF)

Bayes estimators of α_1,α_2 and p based on general entropy loss function are given by

$$\hat{\alpha}_{1,GELF} = \left[K_2^{-1}\beta(r_1+1,r_2+1)\frac{\Gamma(r_1-h)}{(\varphi_1^*)^{r_1-h}}\frac{\Gamma(r_2)}{(\varphi_2^*)^{r_2}}\right]^{-\frac{1}{h}}$$

$$\hat{\alpha}_{2,GELF} = \left[K_2^{-1}\beta(r_1+1,r_2+1)\frac{\Gamma(r_1)}{(\varphi_1^*)^{r_1}}\frac{\Gamma(r_2-h)}{(\varphi_2^*)^{r_2-h}}\right]^{-\frac{1}{h}}$$

$$\hat{p}_{GELF} = \left[K_2^{-1}\beta(r_1-h+1,r_2+1)\frac{\Gamma(r_1)}{(\varphi_1^*)^{r_1}}\frac{\Gamma(r_2)}{(\varphi_2^*)^{r_2}}\right]^{-\frac{1}{h}}$$

4. Bayesian Prediction

In this section, the Bayesian two sample prediction of a future order statistics is considered. Based on a random sample of size n drawn from the population with pdf (1), a future unobservable independent random sample of size m from the same population is under consideration. Lety_s represents the s^{th} ordered statistic in the future sample, $1 \le s \le m$. The s^{th}order statistic in a sample of size m represents the life length of a $(m-s+1)$ out of m system. The distribution function of $y_s$the ordered future sample is given [see Arnold et al. (1992) and Jaheen (2003)] by

$$F_{Y_s}(y_s|p,\alpha_1,\alpha_2) = \sum_{l=s}^{m}\binom{m}{l}[F_X(y_s|p,\alpha_1,\alpha_2)]^l[1-F_X(y_s|p,\alpha_1,\alpha_2)]^{m-l}$$
$$= \sum_{l=s}^{m}\sum_{j_1=0}^{l}\binom{m}{l}\binom{l}{j_1}(-1)^{j_1}[R(y_s)]^{m-l+j_1} \quad (11)$$

where $F_X(y_s|p,\alpha_1,\alpha_2)$ is the distribution function of the mixture model and $R(y_s)$ is the reliability function of the mixture model after replacing x by y_s

Using the binomial expansion for $[R(y_s)]^{m-l+j_1}$ it follows that

$$[R(y_s)]^{m-l+j_1} = \left[pe^{-\alpha_1 y_s^{\theta_1}} + (1-p)(1+\theta_2 y_s)^{-\alpha_2}\right]^{m-l+j_1}$$
$$= \sum_{j_2=0}^{m-l+j_1}\binom{m-l+j_1}{j_2}p^{\delta_1}(1-p)^{j_2}\left(e^{-\alpha_1 y_s^{\theta_1}}\right)^{\delta_1}(1+\theta_2 y_s)^{-\alpha_2 j_2}$$

Therefore,

$$F_{Y_s}(y_s|p,\alpha_1,\alpha_2) = \sum_{l=s}^{m}\sum_{j_1=0}^{l}\sum_{j_2=0}^{m-l+j_1}\binom{m}{l}\binom{l}{j_1}\binom{m-l+j_1}{j_2}(-1)^{j_1}p^{\delta_1}(1-p)^{j_2}\left(e^{-\alpha_1 y_s^{\theta_1}}\right)^{\delta_1}$$
$$\times (1+\theta_2 y_s)^{-\alpha_2 j_2} \quad (12)$$

The Bayes predictive pdf of $y_s$given $\underline{x}$, is defined by:

$$f^*(y_s|\underline{x}) = \int_0^1\int_0^\infty\int_0^\infty f(y_s|p,\alpha_1,\alpha_2)P(p,\alpha_1,\alpha_2|\underline{x})d\alpha_1 d\alpha_2 dp \quad (13)$$

where $P(p,\alpha_1,\alpha_2|\underline{x})$is the joint posterior density for parametersα_1,$\alpha_2$and $p$and $f(y_s|p,\alpha_1,\alpha_2)$ is the pdf ofs^{th} component in a future sample.

Therefore, using the Bayesian predictive density of y_s , for a given value ν, we obtain

$$Pr[y_s \ge \nu|\underline{x}] = \int_\nu^\infty f^*(y_s|\underline{x})dy_s$$

$$= 1 - \int_0^1 \int_0^\infty \int_0^\infty F_{y_s}(\nu|p,\alpha_1,\alpha_2) P\big(p,\alpha_1,\alpha_2|\underline{x}\big) d\alpha_1 d\alpha_2 dp \qquad (14)$$

where $F_{y_s}(\nu|p,\alpha_1,\alpha_2)$ is the cumulative distribution of the s^{th} component in a future sample as given by (12)
Substitution of (8) and (12) in (14), we get

$$Pr\big[y_s \geq \nu|\underline{x}\big] = 1 - K_1^{-1} \sum B\, \beta(\delta_1+\delta_2,\delta_3) \frac{\Gamma(r_1+a_1)}{(\varphi_1^{**})^{r_1+a_1}} \frac{\Gamma(r_2+a_2)}{(\varphi_2^{**})^{r_2+a_2}}$$

If the prior distribution is non-informative, Bayes predictive distribution bounds with value ν for y_s, can be obtained by substituting (10)and (12) in (14),and obtain

$$Pr\big[y_s \geq \nu|\underline{x}\big] = 1 - K_2^{-1} \sum B\, \beta(\delta_1^*+1,\delta_2^*+1) \frac{\Gamma(r_1)}{(\varphi_1^{***})^{r_1}} \frac{\Gamma(r_2)}{(\varphi_2^{***})^{r_2}}$$

where

$$\sum = \sum_{l=s}^{m} \sum_{j_1=0}^{l} \sum_{j_2=0}^{m-l+j_1}, \qquad B = \binom{m}{l}\binom{l}{j_1}\binom{m-l+j_1}{j_2}(-1)^{j_1}$$

$\delta_1 = m - l + j_1 - j_2,\quad \delta_2 = r_1 + c,\quad \delta_3 = r_2 + d + j_2$
$\delta_1^* = \delta_1 + r_1,\quad \delta_2^* = j_2 + r_2,$
$\varphi_1^{**} = \nu^{\theta_1}\delta_1 + \varphi_1,\quad \varphi_2^{**} = j_2 \ln\big(1+\theta_2 x_{2j}\big) + \varphi_2$
$\varphi_1^{***} = \nu^{\theta_1}\delta_1 + \varphi_1^*, \varphi_2^{***} = j_2 \ln\big(1+\theta_2 x_{2j}\big) + \varphi_2^*$
A 100 γ%predictioninterval for y_s is given by

$$P\big[L\big(\underline{x}\big) < y_s < U\big(\underline{x}\big)\big] = \gamma$$

where $L\big(\underline{x}\big)$ and $U\big(\underline{x}\big)$ are obtained respectively by solving the following two equations:

$$P\big[y_s > L\big(\underline{x}\big)|\underline{x}\big] = \frac{1+\gamma}{2} \quad and \quad P\big[y_s > U\big(\underline{x}\big)|\underline{x}\big] = \frac{1-\gamma}{2} \qquad (15)$$

5. Simulation Study

In this section, we present simulation study to compare the performance of the various estimates based on samples from a mixture of Weibull and Lomax distribution. The following steps were followed:

- In this study the following parameters values were used $(\alpha_1,\alpha_2,p) = (2,3,0.45)$ and $(1,2,0.45)$ along with$(\theta_1,\theta_2) = (1.2,1.6)$. The values chosen for the constants$q$ and h are $(0.5,-0.5,0.9,-0.9,1,-1)$ for the LINEX and GELF loss functions. The following values are used for the hyper parameters$(a_1 = 0.2, a_2 = 0.12, b_1 = 0.35, b_2 = 0.15, c = 1.5\ and\ d = 3.5)$ for informative prior. In case of non-informative prior, we take $\{(a_1 = a_2 = b_1 = b_2 = 0),(c = d = 1)\}$.In all these cases samples of sizes$n$=25,50,75,100 and 200 are generated.
- Generate a uniform $(0,1)$ random number u.
- If $u \leq p$ the observation is randomly generated from first subpopulation. If $u > p$ then the observation is generated from the second subpopulation.
- Calculate the maximum likelihood estimates and Bayes estimates of the parameters α_1,α_2 and p when θ_1 and θ_2 are known according to sect3.
- Bayesian prediction bounds for the s^{th} future observation$Y_s$are obtained by solving (15)numerically, with$\gamma = 0.95$.
- The above steps are repeated 1000 times, and the calculated average of the estimates are presented in the Tables (1-13). The average of the lower and the average of the upper bounds of the interval of $Y_s$when $s = \frac{m}{2}$ is even or $\frac{m+1}{2}$ is odd for different sample size $n$and different future sample size m. The simulated coverage probability and average interval lengths are presented in Tables (14) and (15).

The computations are done using Mathematica 10.

n	MLEs	Loss Function								
		Squared Error	LINEX				General Entropy			
			$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$	$h=-1$
25	2.20319 (0.66989)	2.07547 (0.44259)	1.97417 (0.34620)	2.19594 (0.62460)	1.88665 (0.29639)	2.34701 (1.01339)	1.93302 (0.38105)	2.02837 (0.41694)	1.88469 (0.37107)	2.07547 (0.44259)
50	2.10357 (0.22871)	2.04773 (0.18682)	1.99975 (0.16680)	2.09921 (0.21554)	1.95484 (0.15379)	2.15468 (0.25521)	1.97833 (0.17231)	2.02469 (0.18085)	1.9550 (0.16977)	2.04773 (0.18682)
75	2.06443 (0.15206)	2.03086 (0.13702)	1.99939 (0.12747)	2.06376 (0.14972)	1.95821 (0.10818)	2.09822 (0.16607)	1.97312 (0.11618)	2.01559 (0.13423)	1.95809 (0.11536)	2.03086 (0.13702)
100	2.05664 (0.10113)	2.03210 (0.09321)	2.00871 (0.08794)	2.05625 (0.10001)	1.98606 (0.08406)	2.08122 (0.10850)	1.99785 (0.08897)	2.02070 (0.09153)	1.98639 (0.08809)	2.03210 (0.09321)
200	2.03262 (0.04894)	2.02095 (0.04695)	2.00957 (0.04554)	2.03251 (0.04866)	1.99836 (0.04443)	2.04425 (0.05070)	2.00413 (0.04574)	2.01535 (0.04648)	1.99851 (0.04547)	2.02095 (0.04695)

Table 1: Average values of the different estimators and corresponding MSE based informative prior when $\alpha_1 = 2$

n	MLEs	Loss Function								
		Squared Error	LINEX				General Entropy			
			$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$	$h=-1$
25	3.24560 (1.03010)	3.14827 (0.84715)	2.9640 (0.63592)	3.37307 (1.28135)	2.80818 (0.54497)	3.66333 (2.23496)	2.97220 (0.73166)	3.08996 (0.80110)	2.91270 (0.70858)	3.14827 (0.84715)
50	3.11830 (0.39990)	3.07557 (0.34682)	2.98887 (0.30354)	3.16977 (0.41606)	2.90866 (0.28049)	3.27272 (0.51912)	2.99072 (0.32284)	3.04737 (0.33717)	2.96225 (0.31820)	3.07557 (0.34682)
75	3.09708 (0.25626)	3.05065 (0.22968)	2.97841 (0.20641)	3.11102 (0.25853)	2.92474 (0.19659)	3.17484 (0.29856)	2.97908 (0.21481)	3.03191 (0.22534)	2.96040 (0.21312)	3.05065 (0.22968)
100	3.04604 (0.17243)	3.02697 (0.16578)	2.98523 (0.15617)	3.07035 (0.17998)	2.94502 (0.15066)	3.11549 (0.19935)	2.98558 (0.16078)	3.01320 (0.16372)	2.97175 (0.15989)	3.02697 (0.16578)
200	3.02530 (0.08270)	3.01597 (0.08101)	2.99517 (0.07851)	3.03717 (0.08450)	2.97475 (0.07696)	3.05877 (0.08905)	2.99528 (0.07964)	3.00908 (0.08046)	2.98837 (0.07938)	3.01597 (0.08101)

Table 2: Average values of the different estimators and corresponding MSE based informative prior when $\alpha_2 = 3$

n	MLEs	Loss Function								
		Squared Error	LINEX				General Entropy			
			$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$	$h=-1$
25	0.45882 (0.03410)	0.42523 (0.00716)	0.42332 (0.00724)	0.42715 (0.00708)	0.42141 (0.00733)	0.42908 (0.00701)	0.41057 (0.00844)	0.42048 (0.00752)	0.40541 (0.00899)	0.42523 (0.00716)
50	0.45259 (0.01484)	0.44040 (0.00403)	0.43932 (0.00405)	0.44148 (0.00402)	0.43824 (0.00407)	0.44257 (0.00400)	0.43268 (0.00435)	0.43787 (0.00412)	0.43004 (0.00449)	0.44040 (0.00403)
75	0.45333 (0.01121)	0.43745 (0.00303)	0.43670 (0.00305)	0.43820 (0.00301)	0.43595 (0.00307)	0.43985 (0.00291)	0.43958 (0.00303)	0.43817 (0.00302)	0.43033 (0.00323)	0.43745 (0.00303)
100	0.45714 (0.00952)	0.44151 (0.00231)	0.44094 (0.00232)	0.44209 (0.00230)	0.44036 (0.00234)	0.44267 (0.00229)	0.43750 (0.00242)	0.44019 (0.00234)	0.43847 (0.00242)	0.44151 (0.00231)
200	0.45256 (0.00320)	0.44821 (0.00111)	0.44792 (0.00111)	0.44851 (0.00111)	0.44762 (0.00111)	0.44881 (0.00111)	0.44619 (0.00113)	0.44754 (0.00112)	0.44551 (0.00114)	0.44821 (0.00111)`

Table 3: Average values of the different estimators and corresponding MSE based informative prior when $p = 0.45$

n	MLEs	Loss Function								
		Squared Error	LINEX				General Entropy			
			$q=0.5$	$q=-0.5$	$q=0.9$	$q=-0.9$	$h=0.5$	$h=-0.5$	$h=0.9$	$h=-0.9$
25	2.2053 (0.57503)	2.2053 (0.57503)	2.0870 (0.41692)	2.35065 (0.89028)	2.00548 (0.34370)	2.51086 (1.73905)	2.04955 (0.45368)	2.15382 (0.52839)	2.00729 (0.43096)	2.19504 (0.56522)
50	2.07676 (0.22783)	2.07676 (0.22783)	2.02677 (0.20050)	2.1305 (0.26552)	1.98918 (0.18468)	2.17657 (0.30499)	2.02099 (0.20634)	2.05325 (0.21949)	1.98696 (0.20242)	2.07206 (0.22607)
75	2.07087 (0.13991)	2.07087 (0.13991)	2.02437 (0.12751)	2.10523 (0.15585)	2.01282 (0.11967)	2.1339 (0.17152)	2.02385 (0.12907)	2.04127 (0.13451)	2.01124 (0.12713)	2.06775 (0.13905)
100	2.02598 (0.09751)	2.02598 (0.09751)	2.01707 (0.09374)	2.04999 (0.10423)	1.98466 (0.08916)	2.06977 (0.11082)	2.01602 (0.09639)	2.0392 (0.09998)	1.98273 (0.09296)	2.02371 (0.09717)
200	2.02426 (0.04720)	2.02426 (0.04720)	2.01278 (0.04572)	2.03591 (0.04899)	2.00373 (0.04476)	2.04536 (0.05066)	2.00731 (0.04591)	2.02109 (0.04336)	2.00279 (0.04566)	2.02313 (0.04709)

Table 4: Average values of the different estimators and corresponding MSE based non-informative prior when $\alpha_1 = 2$

n	MLEs	Loss Function								
		Squared Error	LINEX				General Entropy			
			$q = 0.5$	$q = -0.5$	$q = 0.9$	$q = -0.9$	$h = 0.5$	$h = -0.5$	$h = 0.9$	$h = -0.9$
25	3.16876 (0.59348)	3.16876 (0.59348)	2.97964 (0.70181)	3.40181 (1.47191)	2.85042 (0.60323)	3.64276 (2.40414)	2.99029 (0.81767)	3.10966 (0.90037)	2.94207 (0.79359)	3.15697 (0.942239)
50	3.09716 (0.39426)	3.09716 (0.39426)	3.00824 (0.33717)	3.19409 (0.48261)	2.94204 (0.30891)	3.27834 (0.58287)	3.01120 (0.36243)	3.0686 (0.38194)	2.98814 (0.35657)	3.09146 (0.39167)
75	3.07964 (0.23161)	3.07964 (0.23161)	3.02189 (0.20875)	3.14055 (0.26435)	2.97778 (0.19660)	3.19177 (0.29882)	3.02333 (0.21744)	3.06091 (0.22617)	3.00826 (0.21476)	3.0759 (0.23046)
100	3.04498 (0.17650)	3.04498 (0.17650)	3.03797 (0.16831)	3.12699 (0.20412)	3.00426 (0.15949)	3.16466 (0.22478)	3.03912 (0.17358)	3.04099 (0.18709)	3.02776 (0.17154)	3.05212 (0.18844)
200	3.02941 (0.08839)	3.02941 (0.08839)	3.00845 (0.08516)	3.05077 (0.09265)	2.99196 (0.08328)	3.06815 (0.09684)	3.00866 (0.08640)	3.0225 (0.08763)	3.00312 (0.08601)	3.02803 (0.08823)

Table 5: Average values of the different estimators and corresponding MSE based non-informative prior when $\alpha_2 = 3$

n	MLEs	Loss Function								
		Squared Error	LINEX				General Entropy			
			$q = 0.5$	$q = -0.5$	$q = 0.9$	$q = -0.9$	$h = 0.5$	$h = -0.5$	$h = 0.9$	$h = -0.9$
25	0.45647 (0.03565)	0.45248 (0.00819)	0.45035 (0.00817)	0.45462 (0.00822)	0.44864 (0.00816)	0.45634 (0.00824)	0.43693 (0.00882)	0.44745 (0.00834)	0.43254 (0.00910)	0.45149 (0.00822)
50	0.45929 (0.01557)	0.45262 (0.00433)	0.45147 (0.00432)	0.45376 (0.00434)	0.45055 (0.00432)	0.45468 (0.00435)	0.44463 (0.00448)	0.44999 (0.00436)	0.442434 (0.00454)	0.45209 (0.00433)
75	0.449524 (0.01148)	0.44879 (0.00306)	0.44801 (0.00307)	0.44958(0.00307)	0.44738 (0.00307)	0.45020 (0.00307)	0.44338 (0.00317)	0.44701 (0.00309)	0.44191 (0.00321)	0.44844 (0.00307)
100	0.43375 (0.00816)	0.45230 (0.00260)	0.45171 (0.002599)	0.45290 (0.00261)	0.45123 (0.00260)	0.45338 (0.00261)	0.44825 (0.00264)	0.45096 (0.00261)	0.44716 (0.00266)	0.45204 (0.00260)
200	0.44063 (0.003997)	0.45091 (0.00118)	0.45060 (0.00118)	0.45121 (0.00118)	0.45036 (0.00118)	0.45145 (0.00118)	0.44886 (0.00119)	0.45023 (0.00118)	0.44831 (0.00119)	0.45077 (0.00118)

Table 6: Average values of the different estimators and corresponding MSE based non-informative prior when $p = 0.45$

n	Gamma prior			Jeffery's prior		
	α_1	α_2	p	α_1	α_2	p
25	1.08729 (0.14141)	2.11151 (0.34245)	0.42183 (0.00804)	1.11204 (0.17286)	2.14728 (0.38441)	0.45019 (0.00895)
50	1.03217 (0.05475)	2.0446 (0.15051)	0.43726 (0.00432)	1.04072 (0.05844)	2.05972 (0.15834)	0.45287 (0.00466)
75	1.02934 (0.03654)	2.05363 (0.11750)	0.44141 (0.00297)	1.03474 (0.03817)	2.0636 (0.12175)	0.45212 (0.00313)
100	1.01512 (0.02570)	2.03873 (0.07973)	0.44264 (0.00213)	1.0189 (0.02676)	2.04592 (0.08187)	0.45076 (0.00220)
200	1.00967 (0.01174)	2.01545 (0.03989)	0.44520 (0.00113)	1.01146 (0.01191)	2.01886 (0.04037)	0.44933 (0.00114)

Table 7: Bayes estimates and MSE under Squared Error Loss Function when $(\alpha_1, \alpha_2, p) = (1,2,0.45)$

n	Gamma prior				Jeffery's prior			
	$q = 0.5$	$q = -0.5$	$q = 0.9$	$q = -0.9$	$q = 0.5$	$q = -0.5$	$q = 0.9$	$q = -0.9$
25	1.06350 (0.11633)	1.12739 (0.16986)	1.04119 (0.10293)	1.15739 (0.20567)	1.08637 (0.13988)	1.15670 (0.21588)	1.06224 (0.12184)	1.19092 (0.27440)
50	1.02764 (0.04907)	1.05323 (0.05658)	1.01790 (0.04670)	1.0640 (0.06032)	1.03598 (0.05235)	1.06231 (0.06088)	1.02597 (0.04965)	1.07341 (0.06501)
75	1.01460 (0.03014)	1.03082 (0.03294)	1.00831 (0.02923)	1.03752 (0.03429)	1.01972 (0.03135)	1.03623 (0.03444)	1.01332 (0.03033)	1.04305 (0.03592)
100	1.01230 (0.02371)	1.02425 (0.02536)	1.00762 (0.02316)	1.02915 (0.02614)	1.01604 (0.02443)	1.02815 (0.02622)	1.01131 (0.02382)	1.03311 (0.02706)
200	1.00596 (0.01170)	1.01170 (0.01207)	1.00369 (0.01157)	1.01402 (0.01225)	1.00772 (0.01186)	1.01349 (0.01226)	1.00544 (0.01172)	1.01583 (0.01245)

Table 8: Bayes estimates and MSE under LINEX Loss Function when $\alpha_1 = 1$

n	Gamma prior				Jeffery's prior			
	$q=0.5$	$q=-0.5$	$q=0.9$	$q=-0.9$	$q=0.5$	$q=-0.5$	$q=0.9$	$q=-0.9$
25	2.04716 (0.35761)	2.23836 (0.61922)	1.98388 (0.30791)	2.33523 (0.85429)	2.08133 (0.40156)	2.28421 (0.73221)	2.01497 (0.33945)	2.3909 (1.08921)
50	2.01926 (0.15874)	2.10052 (0.19609)	1.98915 (0.14927)	2.13572 (0.21774)	2.03390 (0.16575)	2.11687 (0.20750)	2.00319 (0.15488)	2.15285 (0.23136)
75	2.03009 (0.10464)	2.08292 (0.12217)	2.00997 (0.09972)	2.10514 (0.13157)	2.03966 (0.10799)	2.09319 (0.12710)	2.01929 (0.10253)	2.11571 (0.13722)
100	2.02377 (0.07111)	2.06249 (0.08018)	2.00882 (0.06851)	2.07856 (0.08495)	2.03077 (0.07283)	2.06987 (0.08265)	2.01569 (0.06997)	2.08610 (0.08775)
200	2.01547 (0.03808)	2.03434 (0.04047)	2.00806 (0.03735)	2.04203 (0.04166)	2.01888 (0.03854)	2.03784 (0.04108)	2.01143 (0.03775)	2.04556 (0.04233)

Table 9: Bayes estimates and MSEunder LINEX Loss Function when$\alpha_2 = 2$

n	Gamma prior				Jeffery's prior			
	$q=0.5$	$q=-0.5$	$q=0.9$	$q=-0.9$	$q=0.5$	$q=-0.5$	$q=0.9$	$q=-0.9$
25	0.42316 (0.00776)	0.40698 (0.00760)	0.42163 (0.00783)	0.42852 (0.00755)	0.45165 (0.00870)	0.44592 (0.00874)	0.44994 (0.00868)	0.45763 (0.00880)
50	0.43619 (0.00429)	0.43835 (0.00424)	0.43533 (0.00431)	0.43922 (0.00422)	0.45174 (0.00459)	0.45403 (0.00461)	0.45082 (0.00458)	0.45495 (0.00462)
75	0.43846 (0.00308)	0.43996 (0.00305)	0.43786 (0.00309)	0.44057 (0.00304)	0.44905 (0.00318)	0.45062 (0.00318)	0.44842 (0.00318)	0.45124 (0.00318)
100	0.44025 (0.00234)	0.44141 (0.00233)	0.43979 (0.00235)	0.44187 (0.00232)	0.44830 (0.00239)	0.44949 (0.00239)	0.44782 (0.00239)	0.44996 (0.00239)
200	0.44621 (0.00113)	0.44681 (0.00113)	0.44597 (0.00114)	0.44705 (0.00113)	0.45036 (0.00115)	0.45097 (0.00116)	0.45012 (0.00115)	0.45121 (0.00116)

Table 10: Bayes estimates and MSEunder LINEX Loss Function when$p = 0.45$

n	Gamma prior				Jeffery's prior			
	$h=0.5$	$h=-0.5$	$h=0.9$	$h=-0.9$	$h=0.5$	$h=-0.5$	$h=0.9$	$h=-0.9$
25	1.00406 (0.09291)	1.05306 (0.10441)	0.98421 (0.08983)	1.07244 (0.11048)	1.02271 (0.10586)	1.07374 (0.12109)	1.00204 (0.10143)	1.09391 (0.12878)
50	1.00381 (0.04773)	1.02729 (0.05093)	0.99436 (0.04678)	1.03664 (0.05253)	1.01192 (0.05057)	1.03584 (0.05436)	1.00230 (0.04940)	1.04535 (0.05622)
75	0.99829 (0.03192)	1.01367 (0.03314)	0.99212 (0.03157)	1.01980 (0.03376)	1.00331 (0.03304)	1.01886 (0.03447)	0.99706 (0.03261)	1.02506 (0.03518)
100	1.00138 (0.02292)	1.01272 (0.02362)	0.99683 (0.02272)	1.01725 (0.02397)	1.00501 (0.02351)	1.01645 (0.02431)	1.00043 (0.02326)	1.02101 (0.02471)
200	0.99928 (0.01139)	1.00488 (0.01154)	0.99703 (0.01135)	1.00712 (0.01162)	1.00101 (0.01152)	1.00663 (0.01170)	0.99876 (0.01147)	1.00888 (0.01178)

Table 11: Bayes estimates and MSE under General Entropy Loss Function (GELF) when$\alpha_1 = 1$

n	Gamma prior				Jeffery's prior			
	$h=0.5$	$h=-0.5$	$h=0.9$	$h=-0.9$	$h=0.5$	$h=-0.5$	$h=0.9$	$h=-0.9$
25	1.99894 (0.33597)	2.07905 (0.37363)	1.96658 (0.32496)	2.1108 (0.39266)	2.03348 (0.37566)	2.11590 (0.42403)	2.00018 (0.36062)	2.14856 (0.44758)
50	2.00725 (0.16291)	2.04533 (0.17144)	1.99195 (0.16035)	2.06049 (0.17570)	2.02221 (0.17049)	2.06075 (0.18057)	2.00672 (0.16733)	2.07610 (0.18547)
75	2.01394 (0.11338)	2.03897 (0.11757)	2.0039 (0.11207)	2.04895 (0.11961)	2.02356 (0.11672)	2.04879 (0.12150)	2.01344 (0.11518)	2.05885 (0.12378)
100	1.98363 (0.07290)	2.00211 (0.07404)	1.97622 (0.07264)	2.00949 (0.07469)	1.99044 (0.07412)	2.00903 (0.07554)	1.98299 (0.07375)	2.01645 (0.07631)
200	1.99787 (0.03592)	2.00707 (0.03630)	1.99419 (0.03582)	2.01075 (0.03650)	2.00124 (0.03625)	2.01046 (0.03669)	1.99754 (0.03612)	2.01415 (0.03691)

Table 12: Bayes estimates and MSE underGeneral Entropy Loss Function (GELF) when$\alpha_2 = 2$

n	Gamma prior				Jeffery's prior			
	$h = 0.5$	$h = -0.5$	$h = 0.9$	$h = -0.9$	$h = 0.5$	$h = -0.5$	$h = 0.9$	$h = -0.9$
25	0.41324 (0.00866)	0.42310 (0.00779)	0.40914 (0.00908)	0.42690 (0.00751)	0.44124 (0.00915)	0.45186 (0.00874)	0.43707 (0.00938)	0.45586 (0.00865)
50	0.42904 (0.00450)	0.43425 (0.00423)	0.42691 (0.00462)	0.43629 (0.00414)	0.44439 (0.00458)	0.44976 (0.00446)	0.44220 (0.00464)	0.45186 (0.00443)
75	0.43385 (0.00340)	0.43740 (0.00326)	0.43699 (0.00302)	0.43880 (0.00321)	0.44437 (0.00343)	0.44798 (0.00336)	0.44290 (0.00346)	0.44941 (0.00333)
100	0.44126 (0.00224)	0.44393 (0.00218)	0.44018 (0.00227)	0.44499 (0.00216)	0.44940 (0.00229)	0.45210 (0.00227)	0.44830 (0.00230)	0.45317 (0.00227)
200	0.44589 (0.00121)	0.44724 (0.00120)	0.44534 (0.00122)	0.44778 (0.00119)	0.45005 (0.00123)	0.45141 (0.00123)	0.44950 (0.00124)	0.45195 (0.00123)

Table 13: Bayes estimates and MSE under General Entropy (GELF) Loss Function when $p = 0.45$

(n,m)	Y1	Y4	Ym
	(Lower,Upper) Length Percentage	(Lower,Upper) Length Percentage	(Lower,Upper) Length Percentage
(15,8)	(0.00108729,0.190035) 0.188948 0.953331	(0.0521015,0.746352) 0.694252 0.948892	(0.34242,17.0692) 16.72678 0.946772
(20,8)	(0.00109544,0.184743) 0.183648 0.954954	(0.0528216,0.687803) 0.634981 0.948147	(0.357239,7.82906) 7.471821 0.947195
(30,8)	(0.00109592,0.177005) 0.175909 0.954126	(0.0544805,0.638352) 0.583872 0.954268	(0.359171,5.2473) 4.88813 0.951337
(50,8)	(0.00109486,0.17124) 0.170145 0.95249	(0.0556378,0.60799) 0.552352 0.951025	(0.37012,4.06379) 3.69367 0.952461
(n,m)	Y1	Y5	Ym
	((Lower,Upper) Length Percentage	(Lower,Upper) Length Percentage	(Lower,Upper) Length Percentage
(15,10)	(0.000881428,0.152434) 0.151553 0.949328	(0.0642052,0.722859) 0.658654 0.954903	(0.423461,15.9465) 15.52304 0.951385
(20,10)	(0.000860132,0.142407) 0.141547 0.955442	(0.0647025,0.659665) 0.594963 0.94906	(0.431549,9.4171) 8.985551 0.948965
(30,10)	(0.000884815,0.140163) 0.139278 0.955368	(0.0661634,0.615282) 0.549119 0.949874	(0.437402,5.86762) 5.430218 0.948227
(50,10)	(0.000885094,0.135675) 0.13479 0.952345	(0.0684718,0.581582) 0.51311 0.952233	(0.449708,4.58384) 4.134132 0.946996
(n,m)	Y1	Y6	Y12
	(Lower,Upper) Length Percentage	(Lower,Upper) Length Percentage	(Lower,Upper) Length Percentage
(15,12)	(0.000739296,0.125536) 0.124797 0.954718	(0.0706617,0.684937) 0.614275 0.948462	(0.488111,16.5502) 16.06209 0.958596
(20,12)	(0.000744844,0.121148) 0.120403 0.953794	(0.0727871,0.630119) 0.557332 0.950816	(0.49713,10.7606) 10.26347 0.947293
(30,12)	(0.000731892,0.114504) 0.113772 0.95358	(0.0756898,0.590544) 0.514854 0.946848	(0.507417,6.85854) 6.351123 0.951971
(50,12)	(0.000738306,0.111423) 0.110685 0.951637	(0.0776771,0.556162) 0.478485 0.953326	(0.517379,5.04349) 4.526111 0.949503
(n,m)	Y1	Y7	Y13
	(Lower,Upper) Length Percentage	(Lower,Upper) Length Percentage	(Lower,Upper) Length Percentage
(15,13)	(0.000683231,0.114696) 0.114013 0.954547	(0.0856304,0.753906) 0.668276 0.946405	(0.513333,19.9526) 19.43927 0.948494
(20,13)	(0.000681007,0.109581) 0.1089 0.950883	(0.0892999,0.696884) 0.607584 0.945801	(0.525632,10.5694) 10.04377 0.952159
(30,13)	(0.000676344,0.105546) 0.10487 0.953432	(0.0924745,0.646672) 0.554198 0.954685	(0.538543,7.03097) 6.492427 0.951165
(50,13)	(0.000677175,0.102004) 0.101327 0.950038	(0.0954073,0.602945) 0.507538 0.95063	(0.542197,5.24309) 4.700893 0.954779

Table 14: The 95% Bayesian prediction bounds, Length of the Bayesian prediction and their Simulated coverage probability for Y_s *based informative prior.*

(n,m)	Y1 (Lower,Upper)	Y1 Length	Y1 Percentage	Y4 (Lower,Upper)	Y4 Length	Y4 Percentage	Ym (Lower,Upper)	Ym Length	Ym Percentage
(15,8)	(0.0010967,0.188683)	0.187586	0.953815	(0.0519855,0.724707)	0.672722	0.948349	(0.334753,11.0804)	10.74565	0.950549
(20,8)	(0.00110853,0.182939)	0.18183	0.94918	(0.0538305,0.685868)	0.632038	0.947108	(0.334373,6.95623)	6.621857	0.951533
(30,8)	(0.0011171,0.177669)	0.176552	0.951699	(0.054781,0.635552)	0.580771	0.949476	(0.356289,5.02081)	4.664521	0.954635
(50,8)	(0.00112416,0.174117)	0.172993	0.949533	(0.0561957,0.60466)	0.548464	0.950108	(0.367795,4.03468)	3.666885	0.952456
(n,m)	Y1 ((Lower,Upper)	Length	Percentage	Y5 (Lower,Upper)	Length	Percentage	Ym (Lower,Upper)	Length	Percentage
(15,10)	(0.000907201,0.151248)	0.150341	0.952517	(0.0608838,0.683099)	0.622215	0.947483	(0.40563,18.6495)	18.24387	0.954163
(20,10)	(0.000897151,0.145099)	0.144202	0.9475	(0.0634519,0.648762)	0.58531	0.952284	(0.420043,7.82851)	7.408467	0.952168
(30,10)	(0.000902159,0.140951)	0.140049	0.950183	(0.0658663,0.603145)	0.537279	0.950539	(0.436862,5.47448)	5.037618	0.953528
(50,10)	(0.000894121,0.135985)	0.135091	0.95356	(0.0677303,0.573281)	0.505551	0.952822	(0.445836,4.49299)	4.047154	0.949054
(n,m)	Y1 (Lower,Upper)	Length	Percentage	Y6 (Lower,Upper)	Length	Percentage	Y12 (Lower,Upper)	Length	Percentage
(15,12)	(0.000746166,0.123143)	0.122397	0.951609	(0.0708375,0.674947)	0.60411	0.954184	(0.470303,14.5989)	14.1286	0.953276
(20,12)	(0.000746669,0.119045)	0.118298	0.946863	(0.0736406,0.630087)	0.556446	0.949319	(0.490571,9.04549)	8.554919	0.950593
(30,12)	(0.000755212,0.116721)	0.115966	0.954117	(0.0756077,0.584356)	0.508748	0.949748	(0.494973,6.18009)	5.685117	0.953947
(50,12)	(0.000747158,0.112245)	0.111498	0.951032	(0.0777144,0.552466)	0.474752	0.947711	(0.511614,4.80754)	4.295926	0.946234
(n,m)	Y1 (Lower,Upper)	Length	Percentage	Y7 (Lower,Upper)	Length	Percentage	Y13 (Lower,Upper)	Length	Percentage
(15,13)	(0.000692466,0.11266)	0.111968	0.950835	(0.0878052,0.729843)	0.642038	0.947061	(0.505741,15.6126)	15.10686	0.954486
(20,13)	(0.000693234,0.110576)	0.109883	0.951589	(0.0887678,0.678769)	0.590001	0.946076	(0.51357,10.0332)	9.51963	0.946406
(30,13)	(0.000695584,0.106844)	0.106148	0.951825	(0.0924446,0.635788)	0.543343	0.95364	(0.523904,6.53862)	6.014716	0.94749
(50,13)	(0.00068577,0.102565)	0.101879	0.94953	(0.0959684,0.603029)	0.507061	0.946177	(0.541291,4.83614)	4.294849	0.950052

Table 15: The 95% Bayesian prediction bounds, Length of the Bayesian prediction and their Simulated coverage probability for Y_s based non-informative prior

6.Numerical Example

This section presents a numerical example to illustrate the methodology for the proposed estimates based on real data. The data are the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory England. This data set has been discussed by Adepoju et.al. (2014). The data has been classified into two sets using probabilistic mixing weights for $p = 0.45$, which produced $r_1 = 29$, $r_2 = 34$ are as follows:

Population-I	Population-II
0.55,1.25,1.49,1.52,1.58,1.61,0.74,1.04,1.27,1.53, 1.76,1.28,1.42,1.62,1.84,2.24,1.13,1.55,1.62,1.66, 1.77,1.84,1.48,1.61,1.63,1.67,1.7,1.78,1.89	0.93,1.36,1.64,1.68,1.73,1.81,2,1.39,1.49,1.59, 1.61,1.66,1.68,1.82,2.01,0.77,1.11,1.50,1.54,1.60, 1.66,1.69,1.76,0.81,1.29,1.48,1.5,1.61,1.7,0.86, 1.24,1.3,1.51,1.55

The estimated results presented in the below tables

parameter	MLEs	Square Error	Loss Function							
			LINEX				General Entropy			
			$q = 0.5$	$q = -0.5$	$q = 1$	$q = -1$	$h = 0.5$	$h = -0.5$	$h = 1$	$h = -1$
α_1	0.60131	0.60110	0.59802	0.60421	0.59499	0.60737	0.58568	0.59597	0.58051	0.60110
α_2	0.82613	0.82603	0.82107	0.83107	0.81619	0.83620	0.80790	0.820003	0.80182	0.82603
p	0.46032	0.44853	0.44763	0.44943	0.44674	0.45032	0.45032	0.44651	0.44030	0.44853

Table 16: Estimates incase of informative prior

parameter	MLEs	Square Error	Loss Function							
			LINEX				General Entropy			
			$q = 0.5$	$q = -0.5$	$q = 1$	$q = -1$	$h = 0.5$	$h = -0.5$	$h = 1$	$h = -1$
α_1	0.60131	0.60131	0.59822	0.60445	0.59516	0.60763	0.58578	0.59615	0.58058	0.60131
α_2	0.82613	0.82613	0.82115	0.83119	0.81625	0.83633	0.80793	0.82008	0.80183	0.82613
p	0.46032	0.46154	0.46060	0.46248	0.49568	0.46341	0.45527	0.45947	0.45313	0.46154

Table 17: Estimates incase of non-informative prior

(n,m)	Y1		Y4		Ym	
	(Lower,Upper)	Length	(Lower,Upper)	Length	(Lower,Upper)	Length
(63,8)	(0.003885,0.647453)	0.643568	(0.206151,2.44255)	2.236399	(1.47171,523.283)	521.8113
(n,m)	Y1		Y5		Ym	
	(Lower,Upper)	Length	(Lower,Upper)	Length	(Lower,Upper)	Length
(63,10)	(0.003121,0.504563)	0.501442	(0.252328,2.29968)	2.047352	(1.78619,715.471)	713.6848
(n,m)	Y1		Y6		Ym	
	(Lower,Upper)	Length	(Lower,Upper)	Length	(Lower,Upper)	Length
(63,12)	(0.002609,0.412524)	0.409915	(0.290711,2.19359)	1.902879	(2.06724,925.124)	923.0568
(n,m)	Y1		Y7		Ym	
	(Lower,Upper)	Length	(Lower,Upper)	Length	(Lower,Upper)	Length
(63,13)	(0.002412,0.377882)	0.37547	(0.360861,2.41261)	2.051749	(2.19827,103602)	1033.822

Table 18: Bayesian prediction boundsY_s, Length of the Bayesian prediction corresponding 95% in case informative prior for thereal data set

(n,m)	Y1		Y4		Ym	
	(Lower,Upper)	Length	(Lower,Upper)	Length	(Lower,Upper)	Length
(63,8)	(0.003954,0.65195)	0.647995	(0.208615,2.43116)	2.222545	(1.47224,506.547)	505.0748
(n,m)	Y1		Y5		Ym	
	(Lower,Upper)	Length	(Lower,Upper)	Length	(Lower,Upper)	Length
(63,10)	(0.003177,0.508787)	0.50561	(0.255198,2.29073)	2.035532	(1.7835,692.542)	690.7585
(n,m)	Y1		Y6		Ym	
	(Lower,Upper)	Length	(Lower,Upper)	Length	(Lower,Upper)	Length
(63,12)	(0.0026564,0.416387)	0.413731	(0.293882,2.18632)	1.892438	(2.06098,895.431)	893.37
(n,m)	Y1		Y7		Ym	
	(Lower,Upper)	Length	(Lower,Upper)	Length	(Lower,Upper)	Length
(63,13)	(0.002456,0.381568)	0.379112	(0.364498,2.40181)	2.037312	(2.19015,1002.75)	1000.56

Table 19: Bayesian prediction boundsY_s, Length of the Bayesian prediction corresponding 95% in case non-informative prior for the real data set

7. Summary and Conclusion

In this paper we have addressed the estimation and prediction problems of the mixture Weibull and Lomax distributions. Different estimators of the parameters are obtained using maximum likelihood and Bayesian methods, under the informative and non-informative prior distributions for the parameters, assuming. Our observations are as follows:

1- It is clear from Tables (1-3) that the performance of the estimates under LINEX loss function is smallest especially when $(q = 1)$,and also the Bayesian estimate under general entropy loss function in case of the value $(h = -1)$ are almost the same as the estimates under square error loss function .
2- From Tables(4) and (6) the performances of the MLEs and Bayes estimates under squared error loss function are quite similar for the parameters α_1 and α_2. Bayes estimation with LINEX loss function have smaller MSE value especially when $q = 0.9$.
3- We conclude from Table (7-13) that the Bayes estimates perform better under informative prior than non-informative prior for all different loss functions.
4- Estimates of the mixing weight parameter tend to converge to the true parametric value by increasing the sample size for maximum likelihood estimation and for all loss functions, this indicates that the MSE are positively skewed.
5- The estimates of α_1 and α_2 in the case of asymmetric loss function is best at the positive value of q and h than with negative values.
6- Tables (14-15) show that the lengths of the Bayesian prediction intervals decrease as the sample size increases, that the Bayesian simulated coverage probability of Y_s are close to the confidence level 95%. In general, the length of the Bayesian prediction intervals increases as$s$ increases.
7- From the analysis, estimates and Bayesian prediction for $Y_s$based onthe real data, indicate that the Bayes estimates under informative prior has a better value than the Bayes estimates under non-informative prior. We note, that general entropy loss function is best especially when$h = 1$ in terms of estimates. It is evident from Tables (18-19), ingeneral, that the length of the Bayesian prediction intervals increase as s increases based on the given set of data.

8. References

i. Abu-Zinadah, H.H. (2010). A study on mixture of Exponentiated Pareto and Exponential distributions. Journal of Applied Sciences Research,.6(4),358-376.
ii. Adepoju, K.A.,Chukwu, A.U. andWang, M. (2014). The Beta Power Exponential distribution. Journal of Statistical Science and Application.2,37-46.
iii. Afaq, A., Ahmed, S.P.and Ahmed, A. (2015). Bayesian analysis of shape parameter of Lomax distribution under different loss function. International Journal of Statistics and Mathematics.2(1),055-065.
iv. Ahmed, A.O.M, Al-Kutubi, H.S. and Ibrahim, N.A. (2010). Comparison of the Bayesian and maximum likelihood estimation for Weibull distribution. Journal of Mathematics and Statistics.6 (2),100-104.
v. Ahmed, K.E., Jaheen., Z.F. and Mohammed., H.S. (2011). Bayesian estimation under a mixture of the Burr type XII distribution and its reciprocal. Journal of Statistical Computation and Simulation.81,2121-2130.
vi. Ahmed, K.E., Moustafa, H.M. and Abd-Elrahman, A.M. (1997). Approximate Bayes estimation for mixture of two Weibull distributions under type-2 censoring. Journal of Statistical Computation and Simulation. 58,269-285.
vii. Al-Hussaini, E.K., Nigm, A.M. and Jaheen, Z.F. (2001). Bayesian prediction based on finite mixtures of Lomax components model and type I censoring. Statistics. 35,259-268.
viii. Al-Noor, N.H. and Alwan, S.S. (2015). Non-Bayes, Bayes and empirical Bayes estimators for the shape parameter of Lomax distribution. Mathematical Theory and Modeling.5(2),17-27.
ix. Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992). A First Course in Order Statistics. John Wiley and Sons, New York.
x. Berger, J.O. and Sun, D. (1993). Bayesian analysis for the poly-Weibull distribution. Journal of the American Statistical Association.88,1412-1418.
xi. Daniyal, M. and Rajab, M. (2015). On some classical properties of the mixture of Burr XII and Lomax distributions. Journal of Statistics Applications and Probability.4(1),173-181.
xii. Erisoglu, U., Erisoglu, M.and Erol, H. (2011). A mixture model of two different distributions approach to the analysis of heterogeneous survival data. International Journal of Computer, Electrical, Automation, Control and Information Engineering .5(6),544-548.
xiii. Feroze., N. (2016). Bayesian inference of a finite mixture of inverse Weibull distributions with an application to doubly censoring data.Pak.j.stat.oper.res.XII (1),53-72.
xiv. Feroze., N. AndSalam. (2014). Bayesian analysis of doubly censored lifetime data using two-component mixture of Weibull distribution. Journal of National Science Foundation Sri Lanka.42(4),325-334.
xv. Ferreira, P.Gonzales, J.Tomazella, V.Ehlers, R.Louzada, F. and Silva, E. (2016). Objective Bayesian analysis for the Lomax distribution.arXiv:1602.08450v1.
xvi. Howlader, H.A. and Hossain, A.M. (2002). Bayesian survival estimation of Pareto distribution of the second kind based on failure- censored data. Computational Statistics& Data Analysis. 38,301-314.
xvii. Jaheen, Z.F. (2003). Bayesian prediction under a mixture of two- component Gompertz lifetime model. Sociedad de Estadistica e Investigacion Opertiva Test. 12(2),413-426.
xviii. Jeffery's, H. (1964). An invariant form for the prior probability in estimation problems. Proceedings of the Royal Society of London Series A: Mathematical and Physical Sciences. 186,453-461.
xix. Kaminskiy, M.P. and Krivtsov, V.V. (2005). A simple procedure for Bayesian estimation of the Weibull distribution. IEEE Transactions on Reliability.54(4),612-616.

xx. Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. American Statistical Association and the American Society for Quality. 50(2), 144-154, DOl 10.1198/004017008000000217.

xxi. Lomax, H.S. (1954). Business failure, another example of the analysis of failure data. Journal of the American Statistical Association. 49,847-852.

xxii. Mahmoud, M., Saleh.E.H. and Helmy, S.M. (2014). Bayesian prediction under a finite mixture of generalized exponential lifetime model.Pak.j.stat.oper.res.X(4),417-433.

xxiii. Nasiri, P. and Hosseini, S. (2012). Statistical inferences for Lomax distribution based on record values (Bayesian and Classical). Journal of Modern Applied Statistical Methods. 11(1),179-189.

xxiv. Shawky, A.I. and Bakoban, R.A. (2009). Onfinit mixture of two-component exponentiated gamma distribution. Journal of Applied Sciences Research.5(10),1351-1369.

xxv. Turkan, A.H.and Cahs, N. (2014). Comparison of two- component mixture distribution models for heterogeneous survival datasets: a review study. Journal of the Turkish Statistical Association. 7(2),33-42.

xxvi. Zhang, Y. and Meeker, W.Q. (2005). Bayesian life test planning for the Weibull distribution with given shape parameter. Metrika. 61,237-249.

xxvii. ZHU, T., YAN., Z. and PENG, X. (2016). A Weibull failure model to the hierarchical Bayesian reliability. Eksploatacja i Niezawodnosc-Maintenance and Reliability.18(4),501-506.