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An Application of Iterative Algorithm for Fixed Point of Some Multi-Valued Mappings

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Abstract:
In general, finding a solution of some problems in some areas of mathematics may be difficult or sometimes impossible even though they exist. Iterative algorithms are used to succumb this problems. In this paper, an existing algorithm for fixed point of certain multi-valued map has been studied. Problem of variational inequality problem have been shown to be equivalent to some multi valued fixed point problem. Consequently, application of the algorithm studied was given in connection with this problem.

Keywords: *Fixed point theorem, Variational Inequality, pseudo-contractive mapping and q-uniformly smooth space*

1. Introduction

A fixed-point problem is that of finding a point $x \in X$ such that $T(x) = x$, where $T : X \to Y$ is a map and $X \subseteq Y$. The famous Banach fixed point theorem (also called the contraction mapping principle) states that every contraction mapping defined on a complete metric space into itself has a unique fixed point (see, e.g., [1]). The Brouwer fixed point theorem, [7], states that if $f : B \to B$ is a continuous function and B a closed ball in R^n, then f has a fixed point. These are some of the well-known, in fact some of the most celebrated theorems in fixed point theory. The numerous applications of fixed point theory, perhaps, are the reason why it attracts the attention of many researchers. The applications are found in such areas as, theory of Ordinary and Partial Differential Equations, Integral Equations, Integro-differential Equations, Optimization, Evolution equations and many others (see, e.g., [1], [10]). To solve many problems in these areas, one may be able to rewrite the problem as a fixed point problem for some appropriate map in some appropriate domain. For instance, to show that the equation

$$x^2 - 2 = 0 \qquad (1.1)$$

has a solution in R, one may consider the function

$Tx = x^2 + x - 2,\ x \in \mathrm{R}.$

It is immediate that x is a solution of the equation (1.1) if and only if x is a fixed point of T. A fixed-point problem normally has two parts:

- existence/uniqueness of a solution and
- obtaining a solution.

The Banach fixed point theorem addresses both existence / uniqueness and obtaining a solution questions. This, among other reasons, is what makes it famous (see, e.g., [2]). To address the question of obtaining a solution (of fixed point or other problem), the notion of iterative algorithms is developed. The study of fixed point of multi-valued maps has attracted the interest of so many mathematicians (and researchers from other fields), where a point $x \in D(T) \subseteq X$ is a fixed point of a multi-valued map $T : X \to 2^X$ if $x \in Tx$ (see, e.g., [2], [17], [9], [14], [16]). This is partly due to the fact that many problems in some areas of mathematics such as Convex Optimization, Game theory, Variational Inequality Problems (VIP), etc. can be written as fixed point problems for multi-valued maps. As in the case of single-valued, there are two questions with regard to fixed point problems of multi-valued map:

- does a solution exist?
- if a solution exists, how do we obtain it?

Definition 1.1 (Variational inequality problem (VIP)):Given a real Hilbert space H and $f : C \subseteq H \to H$ a map. A variational inequality problem *(VIP)*
with respect to f and C is of the form

$$\begin{cases} find\ x^* \in C\ such\ that \\ \langle f(x^*), x - x^* \rangle \geq 0\ \forall x \in C. \end{cases}$$ **(VIP)**

We denote by $VIP(C,f)$ the set of solutions of the variational inequality problem with respect to C and f. We now see the connection between *VIP* (Variational Inequality Problem) with fixed point problem. We first define a normal cone. Let H be a real Hilbert space and let C be a nonempty and convex subset of H. Consider the Indicator function denoted by $1_C : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$1_C(x) = \begin{cases} 0\ if\ x \in C \\ +\infty\ if\ x \notin C. \end{cases}$$ Type equation here. (*IN*)

The function $1_C$is convex. Indeed, for $x,y \in C$ and $\lambda \in (0,1)$, $1_C(\lambda x + (1-\lambda)y) = 0$ if $x,y \in C$ and $\lambda 1_C(x) + (1-\lambda)1_C(y) = +\infty$ if x and $y$are not in, C. Therefore,

$$1_C(\lambda x + (1-\lambda)y) = 0 \leq \lambda 1_C(x) + (1-\lambda)1_C(y)\ if\ x,y \in C$$

and $\lambda 1_C(x) + (1-\lambda)1_C(y) = +\infty \geq 1_C(\lambda x + (1-\lambda)y)$ *if x and y are not inC.*

Hence, for all $x,y \in H$ *and* $\lambda \in (0,1)$, $1_C(\lambda x+(1-\lambda)y) \leq \lambda 1_C(x)+(1-\lambda)1_C(y)$.

Definition 1.2 (Normal cone) *The normal cone is defined to be the sub differential of the indicator function and it is denoted by* N_C.

1.1. Iterative Algorithms for single-valued maps

- Theorem 1.1 (Banach fixed point theorem, see, e.g., [1], [10]) Let (X,d) be a complete metric space and $T : X \to X$ be a contraction, i.e., there exist s$k \in [0,1)$ such that $d(T(x),T(y)) \leq kd(x,y)$ for all$x, y \in X$. Then T has a

unique fixed point. Moreover, the sequence$\{x_n\}$ generated iteratively by

$$\begin{cases} x_0 \in X, \\ x_{n+1} = Tx_n,\ n \geq 0 \end{cases}$$ **(BFT)**

from arbitrary$x_0 \in X$ converges to the unique fixed point of T.

The theorem above is perhaps the most applicable theorem in fixed point theory. This is, partly, due to the fact that it guarantees the existence and uniqueness of the fixed point and it gives a simple algorithm which converges to unique fixed point. Moreover, the error estimate in the convergence is 1.

Despite the simplicity and numerous applications of the Banach fixed point theorem, one may not be able to apply it if the map is not a contraction. For example if the map is non-expansive. In fact if K is closed nonempty subset of a Banach space (therefore complete), a non-expansive map $T : K \to K$ may not have a fixed point. For instance, $T : [0,\infty] \to [0,\infty]$, $Tx = 1+x$. This map has no fixed point even though $[0,\infty]$ is complete and T is non-expansive. If X is a normed linear space, $T : K \to K$ is a non-expansive map and K is convex. The iterative sequence generated by $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n$ which was given by Schaefer was used with a lot of success in approximating fixed point of non-expansive maps. See the monograph of Chidume ([1], Ch. 6). In trying to extend the result of Banach (1.1) which was given in 1922, to the setting of non-expansive maps, Browder in 1967 proved the following theorem:

- Theorem 1.2 (Browder theorem [4])Let H be a Hilbert space H and let D be abounded, closed and convex subset H. If $T : D \to D$ is a non-expansive map,$\{t_n\} \subset (0,1) : t_n \to 1^-$, then sequence $\{x_n\}$ generated by

$$\begin{cases} u \in D\ fixed, \\ x_n = t_n u + (1-t_n)Tx_n,\ n \in \mathbb{N} \end{cases}$$ (*BT*)

converges to a fixed point of T.

- Theorem 1.2 addresses both the question of existence and that of obtaining a fixed point. Although, the theorem required the domain to be bounded, which is a huge restriction and the scheme is not iterative, it has provided the chance to have existence as well as obtaining fixed point of a map which is more general than the contraction in the setting of a Hilbert space. Naturally, it is desirable to obtain a similar result in a more general Banach space. To this end, Reich in [15] obtained the following theorem in 1980.

- Theorem 1.3 (Reich, [15]) Let X be a uniformly smooth real Banach space and let D bounded, closed and convex subset of X. If $T : D \to D$ is a non-expansive map,

,

(*RT*)

$$\begin{cases} u \in D\ \text{fixed} \\ x_t = tTx_t \end{cases}$$ $+ (1-t)u, t \in (0,1)$.

Then the sequence $\{x_t\}$ converges to a fixed point of T as$t \to 1^-$.

- Reich extended the result of Browder to a setting of uniformly smooth Banach space, which is more general than Hilbert space. It is worthy of mention here that to prove both Theorem 1.2 and Theorem 1.3, Banach fixed point theorem (Theorem 1.1) has to be used. This further indicates the indispensability of the theorem. In continuation of the quest for better and sharper result, Morales and Jung extended the result of Riech to a more general one. They were able to give a profound generalization in two directions:

1. with regard to the map,
2. with regard to the space.

Precisely, they proved the following theorem:

- Theorem 1.4 (Morales and Jung, [3])Let X be a reflexive Banach space which has uniformly Gaˆteaux differentiable norm and K be a nonempty, closed and convex subset of X and T : K → K be a pseudo-contractive mapping with F(T) ≠ ∅. Suppose that every nonempty closed convex bounded subset of K has a fixed point property for non-expansive mappings. Then there exists a continuous path $t \to z_t$, satisfying

$$\begin{cases} z \in K \ , \\ y_t = tTy_t + (1-t)z, t \in [0,1) \end{cases} . \quad (MJ)$$

Then the sequence $\{y_t\}$ converges strongly to a fixed point of T.

1.2. Iterative Algorithms for Multi-valued Maps

Researchers have devoted a lot of time to see how much of the result which were obtained in the fixed point theorem of single-valued maps (see, [2], [1], [17], [9], [14], [16]) can also be obtained for the multi-valued settings. Certainly, a lot of challenges were faced and are still being faced due to the complexity of the multi-valued situation. In this direction Pietramala in [9] gave an example which shows that Browder's Theorem 1.2 cannot be extended to multi-valued settings. Very recently, Ofoedu and Zegeye (see, [17]) obtained the multi-valued version of the theorem 1.4 of Morales and Jung. They proved the following lemma:

→ Lemma 1.1 (Ofoedu and Zegeye, [17])Let D be a nonempty, open and $\overline{\text{convex}}$ subset of a real Banach space X. Assuming that $T : \overline{D} \to CB(X)$ is a multi-valued continuous (with respect to the hausdorff metric), bounded and pseudo-contractive mapping satisfying weakly inward condition and $u \in \overline{D}$ *be* fixed. Then for $t \in (0,1)$ there exists $y_t \in D$ satisfying $y_t \in tTy_t + (1-t)u$. If in addition, X is reflexive and has uniformly Ga´teaux differentiable norm and $\overline{D}$ is such that every closed, convex and bounded subset of D has the fixed-point property for non-expansive self-mapping, then T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 1$; moreover, in this case, $\{y_t\}$ converges strongly to a fixed point of T as $t \to 1$.

This marked a serious breakthrough in extending results which were known in single-valued setting to multi-valued setting. Utilizing this Lemma (1.1), Ofoedu and Zegeye were able to develop an algorithm which converges strongly to a fixed point Lipschitzs pseudo-contractive maps in the setting of reflexive real Banach space having G*a*ˆteaux differentiable norm. In fact they proved the following theorem:

- Theorem 1.5 (Ofoedu and Zegeye, [17]) Let X be a reflexive real Banach space having a uniformly Gateauxˆ differentiable norm, D be a nonempty, open and convex subset of X, such that every closed, convex, bounded and nonempty subset of D has the fixed-point property for non-expansive self-mapping. Let $T : D \to K(D)$ be a pseudo-contractive Lipschitzian mapping with constant $L > 0$ and let $u \in D$ be fixed. Let$\{x_n\}$ be a sequence generated iteratively from

arbitrary$x_0 \in D$, $w_0 \in Tx_0$ by

$$\begin{cases} w_n \in Tx_n, \\ x_{n+1} := (1-\lambda_n)x_n + \lambda_n w_n - \lambda_n\theta_n(x_n - x_0) \end{cases} . \quad (OZ)$$

Suppose that$\|w_n - w_{n-1}\| = d(w_{n-1}, Tx_n), n \geq 1$. *If* $F(T) \neq \emptyset$. Then$\{x_n\}$ converges strongly to a fixed point of T.

Even though Theorem 1.5 above has provided an algorithm that generates a sequence which converges strongly to a fixed point of a multi-valued map, Chidume *et al*. in [2] made the following observations:

➢ Remark 1.1

1. To establish convergence of the scheme (OZ) in Theorem 1.5, the authors assumed that$\|w_n - w_{n-1}\| = d(w_{n-1}, Txn)$ for all $n \geq 1$. A sufficient condition to guarantee this is to assume that for each x, the set Tx is proximal. In this addition Tx is convex and E is for example, a real Hilbert space, such w_n is characterized as follows:
$\langle w_{n-1} - w_n, w_n - u_n \rangle \geq 0 \ \forall u_n \in Tx_n$.
Consequently, this condition requires that a sub-programme be constructed to first compute w_n at each step of the iteration process.
2. Nadler remarked in [14] that requiring a multi-valued mapping to be Lipschitz is placing a strong continuity condition on the mapping. They (the authors in [2]) sought to weaken this condition. In fact, the Lipschitz condition of the map T in Theorem 2.1.5 was weakened to continuity and boundedness of the map$T$.

Moreover, in many applications, the real Banach space X is either an L_p-space, aW_p^m-space, $1 < p < \infty$, $m \geq 1$, or a Hilbert space. As has been remarked before, all these spaces are q-uniformly smooth and reflexive. With these above remarks in their mind, it was the purpose in the paper [2] to prove strong convergence theorems for fixed point of multi-valued bounded continuous pseudo contractive maps defined on q-uniformly smooth real Banach spaces. They used the recursion formula in Theorem 1.5, dispensing with the restriction that $\|w_n - w_{n-1}\| = d(w_{n-1}, Txn)$ $\forall n \geq 1$. Furthermore, their iteration process, in the setting of q-uniformly smooth real Banach spaces, is direct, much more applicable than the process in (OZ) since it does not require the creation of a sub-programme to first compute w_n at each step of the iteration process. In particular, in q-uniformly smooth real Banach spaces, their theorems extend Theorem 1.5 (of Ofoedu and Zegeye) from multi-valued lipschitz pseudo-contractive mappings to the much more general class of multi-valued continuous, bounded and pseudo-contractive mappings. They proved the following theorem:

- Theorem 1.6 (Chidume *et al.*, [2]) Let X be a q-uniformly smooth real Banach space and D be a nonempty, open and convex subset of X. Assume that

$\bar{T}:D \to \bar{CB}(D)$ is a multi-valued continuous (with respect to the hausdorff metric),bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let$\{x_n\}$ be

a sequence generated iteratively from arbitrary$x_1 \in D$ by

$$(CCDM)\quad \begin{cases} u_n \in Tx_n, \\ x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1), n \geq 1 \end{cases} .$$

Then, there exists a real constant$\gamma_0 > 0$ such that if

$\lambda_n^{q-1} < \gamma_0\theta_n \ \forall n \geq 1$,

the sequence$\{x_n\}$ converges strongly to a fixed point of T.

In this paper, motivated by the above theorem 1.6, we were able to provide an application of the theorem in convex optimization problem. However, we were able to show that finding a solution of a convex optimization problem is equivalent to finding a fixed point of some multi-valued maps.

2. Preliminaries and Results

Remark 2.1 For the purpose of both Application one and Application two, we note that every Hilbert space is 2−uniformly smooth. Indeed, the modulus of smoothness ρ_H of any Hilbert space is given by$\rho_H(\tau) = (1+\tau^2)^{\frac{1}{2}} - 1 \ , \tau > 0$which *gives* $\rho_H(\tau) < \tau^2$ *(see, e.g., [8]).*

- Remark 2.2

1. In every nonempty normed linear space X, $J_q(x) \neq \emptyset$ from one of the consequences of Hahn-Banach Theorem.
2. The generalized duality map is the identity map when X is a real Hilbert space.

Definition 2.1 (Sub-differential function) Let H be a Hilbert space and let D be a nonempty convex subset of H. Suppose $f:D \to \mathbb{R} \cup \{+\infty\}$ is a convex function. The sub-differential function *of f*, $\partial f:H \to 2^H$ is defined by

$\partial f(x) = \{y \in H : f(u) \geq f(x) + \text{h}y, u - x\text{i} \ \forall u \in H\}$

- Remark 2.3

1. Elements of the sub-differential of f are called the sub-gradients of f.
2. Sub-differential function is maximal monotone.

Lemma 2.1 Let X be a normed space and let A be an open subset of X. If f: A → R has a local minimum or local maximum at a ∈ A and f is Gaˆteaux

differentiable at a, then $D_Gf(a) = 0$.

Proof: Suppose f has a local maximum at a point a ∈ A. It follows that there exists some r > 0 such that $B(a, r) \subset A$ and $f(x) \leq f(a)$ $\forall x \in B(a,r)$.

Therefore, for all x in X y 6= 0 we have

$$f(a+ty) - f(a) \leq 0 \ \forall t \in \left(0, \frac{r}{\|y\|}\right).$$

Thus,$\lim_{t\to 0} \frac{f(a+ty) - f(a)}{t} \leq 0 \ \forall y \in X.$ Hence, $D_Gf(a)(y) \leq 0$ $\forall y \in X$. This implies $D_Gf(a)(-y) \leq 0$ $\forall y \in X$ since $D_Gf(a) \in X^*$. We have $D_Gf(a) \geq 0$ $\forall y \in X$ and therefore, $D_Gf(a)(y) = 0$ $\forall y \in Y$.

→ Lemma 2.2Let$f : A \subset X \to \mathbb{R}$ be a convex function on an open subset A of a normed space X and Gaˆteaux differentiable at x in A then $\partial f(x) = \{D_Gf(x)\}$.

- Remark 2.4

1. When f is Gaˆteaux differentiable, it is usual to write $\partial f(x) = D_Gf(x)$.
2. Also, *for* $f:A \subseteq X \to \mathbb{R}$ differentiable, it is usual to denote by ∇f the derivative of$f$.

Now, we take the following preliminaries:

→ Lemma 2.3*Let* $f : H \to \mathbb{R}$ be a convex and Gâteux differentiable function. Then for$a \in H$, a is a minimizer of f if and only iff$_G^0(a) = 0$.

✓ Proof:

(⇒) Suppose $f_G^0(x_0) = 0$. We show that x_0 is a global minimizer. Now let $x \in H$ and let $\lambda \in (0,1)$. By convexity of f,

$$\begin{aligned} f(\lambda x + (1-\lambda)x_0) &\le \lambda f(x) + (1-\lambda)f(x_0) \\ &= \lambda f(x) + f(x_0) - \lambda f(x_0). \end{aligned}$$

By rearranging this we have $f(\lambda x + (1-\lambda)x_0) - f(x_0) \le \lambda(f(x) - f(x_0))$. $f(\lambda x + (1-\lambda)x_0) - f(x_0)$

Dividing through by λ we obtain

$$\frac{f(x_0+\lambda(x-x_0))-f(x_0)}{\lambda} \le f(x) - f(x_0).$$

Taking limit as $\lambda \to 0+$ and we have,

$$f'_G(x_0)(x-x_0) = \lim_{\lambda\to 0+} \frac{f(x_0+\lambda(x-x_0))-f(x_0)}{\lambda} \le f(x)-f(x_0)\ \forall x \in H$$

From our hypothesis $f_G^0(x_0) = 0$ so that we obtain $0 \le f(x) - f(x_0)$ $\forall x \in H$. This shows that $f(x_0) \le f(x)$ $\forall x \in H$. So x_0 is a global minimizer of f. Hence the result.

(⇐) Suppose $x_0 \in C$ is a minimizer, goal is to show that $f_G^0(x_0) = 0$. This was shown in Lemma 2.1. Hence the proof.

Lemma 2.4 *Let $f:H \to$ R be a convex function. If f is bounded on bounded sets, then for all $x_0 \in H$ and for all $\rho > 0$, f is Lipschitz on $B_\rho(x_0)$.*

Let $x_0 \in H$ and let $\rho > 0$. We find $L > 0$ such that $\forall x,y \in B_\rho(x_0)$, k$f(x) - f(y)$k $\le L$k$x - y$k. Since f bounded on bounded sets, there exists some $m > 0$ such that k$f(x)$k $\le m$ $\forall x \in B_\rho(x_0)$. Now let $x, y \in B_{\frac{\rho}{4}}(x_0)$. Then

$$\begin{aligned} \|x-y\| &= \|x-x_0+x_0-y\| \\ &\le \|x-x_0\|+\|y-x_0\| \\ &\le \frac{\rho}{4}+\frac{\rho}{4} \\ &= \frac{\rho}{2}. \end{aligned}$$

Therefore, $\|x-y\| \le \frac{\rho}{2}$. We observe that

$$x = \left[\left(1-\frac{2\|x-y\|}{\rho}\right)y + \frac{2\|x-y\|}{\rho}\left(y+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right)\right].$$

Since $\|x-y\| \le \frac{\rho}{2}$, we have $0 \le 2\frac{\|x-y\|}{\rho} \le 1$. Therefore,

$$\begin{aligned} f(x) &= f\left[\left(1-\frac{2\|x-y\|}{\rho}\right)y + \frac{2\|x-y\|}{\rho}\left(y+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right)\right] \\ &\le \left(1-\frac{2\|x-y\|}{\rho}\right)f(y) + \frac{2\|x-y\|}{\rho}f\left(y+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right) \\ &\le f(y)+\frac{2\|x-y\|}{\rho}\left(f\left(y+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right)-f(y)\right). \end{aligned}$$

(by convexity of f)

Now we have

$$f(x)-f(y) \le \frac{2\|x-y\|}{\rho}\left(f\left(y+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right)-f(y)\right).$$

Observing that,

$$\begin{aligned} \left\|y+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}-x_0\right\| &= \left\|y-x_0+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right\| \\ &\le \|y-x_0\|+\left\|\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right\| \\ &\le \frac{\rho}{4}+\frac{\rho}{2} \\ &= \frac{3\rho}{4} \le \frac{4\rho}{5}, \end{aligned}$$

we have that $\left(y+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right) \in B_{\frac{4\rho}{5}}(x_0)$. Using the fact that f is bounded on bounded sets, it follows that there exists some $m \in$ R, $m > 0$ such that $|f(u)| \le m$ for all $u \in B_{\frac{4\rho}{5}}(x_0)$. So, $f\left(y+\frac{\rho}{2}\frac{(x-y)}{\|x-y\|}\right) \le m.$ Thus, $f(x) - f(y) \le \frac{2\|x-y\|}{\rho}m.$ Following similar arguments we have $f(y)-f(x) \le \frac{2\|y-x\|}{\rho}m.$

Therefore, $|f(x)-f(y)| \leq \frac{2m}{\rho}\|x-y\|$. Setting $L = \frac{2m}{\rho} > 0$, we conclude that $|f(x) - f(y)| \leq L\|x - y\|$ for all $x, y \in B_{\frac{\rho}{4}}(x_0)$. Since ρ was arbitrarily chosen, the result follows.

→ Lemma 2.5 ([6], Ch. 16)Let H be a real Hilbert space and let $f : H \to \mathbb{R}$ *be* convex and differentiable. Suppose f is bounded on bounded set, then the gradient map $\nabla f : H \to H$ is bounded on bounded subset of *H*.

→ Lemma 2.6 Suppose H is a Hilbert space. *If* $A : H \to 2^H$ is monotone, then $(I - A)$ is pseudo-contractive.

✓ Proof:

Let $A : H \to 2^H$ be monotone. Then by definition $<u - v, x - y> \geq 0$ $\forall u \in Ax$, $v \in Ay$. Our goal here is to show that $I - A$ is pseudo-contractive. Now, define $T := I - A$, we recall from Remark 2, $J_2 = I$ (the identity map on H) for real Hilbert spaces. Therefore, for $x, y \in H$, $\bar{u} \in Tx$ and $\bar{v} \in Ty$,

$$
\begin{aligned}
<\bar{u} - v, \bar{J}(x-y)> &= <\bar{u} - v, \bar{I}(x-y)> \\
&= <\bar{u} - v, \bar{x} - y> \\
&= <x - u - y + v, x - y>, \; u \in Ax, \; v \in Ay \\
&= <x - y - (u - v), x - y>, \; u \in Ax, \; v \in Ay \\
&= <x - y, x - y> - \text{h}u - v, x - y>, \; u \in Ax, \; v \in Ay.
\end{aligned}
$$

So we have,

$$<\bar{u} - v, \bar{J}(x-y)> \leq \|x - y\|^2 - <u - v, x - y>, \; u \in Ax, \; v \in Ay.$$

From hypothesis, $<u - v, x - y> \geq 0$ $\forall u \in Ax$, $v \in Ay$. Therefore, $<\bar{u} - v, \bar{J}(x-y)> \leq \|x - y\|^2$. This shows that T is pseudo-contractive. Hence the result. Remark 2.5 The sub-differential function of the indicator function is

$$\partial 1_c(x) = \begin{cases} \{u \in H : \langle u, y - x\rangle \leq 0 \; \forall y \in C\}, \; x \in C \\ \emptyset, \; x \notin C. \end{cases}$$

(INS)

Indeed, by definition (see, Definition 2.1), $N_C(x) = \partial 1_C = \{u \in H: 1_C(y) \geq 1_C(x) + <u, y - x> \forall y \in H\}$. For $x \notin C$ *and* $u \in H$, $1_C(x) + <u, y - x> = +\infty > 0 = 1_C(y)$ $\forall y \in C$ $\forall u \in H$. Therefore, there exists no $u \in H$ such that $1_C(y) \geq 1_C(x) + <u, y - x> \forall y \in H$. Hence, $\partial 1_C(x) = \emptyset$ for x *not in C*. For $x \in C$ and $u \in H$, $1_C(y) = +\infty \geq 1_C(x) + <u, y - x> \forall y$ *not in* C. This implies that for $x \in C$, $u \in H$, $1_C(y) \geq 1_C(x) + <u, y - x> y \in H \Leftrightarrow 1_C(y) \geq 1_C(x) + <u, y - x>$ $\forall y \in C$. Therefore,

$$
\begin{aligned}
\partial 1_C(x) &= \{u \in H: 1_C(y) \geq 1_C(x) + <u, y - x> \; \forall y \in C\}. \\
&= \{u \in H: 1_C(y) \geq <u, y - x\text{i}> \forall y \in C\} \text{ (since } \partial 1_C(x) = 0). \\
&= \{u \in H: <u, y - x> \leq 0 \; \forall y \in C\} \text{ (since } \partial 1_C(y) = 0).
\end{aligned}
$$

Hence,

$$N_C(x) = \begin{cases} \{u \in H : \langle u, y - x\rangle \leq 0 \; \forall y \in C\}, \; x \in C \\ \emptyset, \; x \notin C. \end{cases}$$

→ Lemma 2.7Let H be a real Hilbert space and C be a nonempty and convex subset of H. Assume that $f: C \to H$ is a map. Then $VIP(C, f) = F(T)$, where $T = I - (f + N_C)$.

✓ Proof: Now,

$$
\begin{aligned}
x \in VIP(C,f) \quad &\Leftrightarrow \quad x \in C \text{ and h}f(x), y - x\text{i} \geq 0 \;\; \forall y \in C \\
&\Leftrightarrow \quad x \in C \text{ and h}{-f(x)}, y - x\text{i} \leq 0 \;\; \forall y \in C \\
&\Leftrightarrow \quad -f(x) \in N_C(x) \\
&\Leftrightarrow \quad 0 \in f(x) + N_C(x) \\
&\Leftrightarrow \quad x \in x - f(x) - N_C(x) \\
&\Leftrightarrow \quad x \in (I - f - N_C)(x) \\
&\Leftrightarrow \quad x \in (I - (f + N_C))(x) \\
&\Leftrightarrow \quad x \in Tx.
\end{aligned}
$$

Hence the result.

→ Lemma 2.8 ([6], Ch.1)Let X be a normed linear space. Suppose A is a nonempty subset *of X. A map* $f: A \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous if and only if for every $x \in A$ *and* $\{x_n\} \subset A$, $x_n \to x$ implies $f(x) \leq \liminf f(x_n)$ *as* $n \to \infty$

→ Lemma 2.9 ([13]) Let X be a real Banach space. Suppose $f: X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper *(i.e.*, $f(x_0) < \infty$ for some $x_0 \in X$*)* and lower semi-continuous. Then the sub-differential of f is maximal monotone.

→ Lemma 2.10The sub-differential of the indicator function (N_C) is maximal monotone.

✓ Proof:

Since 1_C is convex and proper, it suffices to show that N_C is lower semi- continuous. Now let $x \in H$ and $1_C : H \to \mathbb{R} \cup \{+\infty\}$.
Let $x \in H$ and $\{x_n\} \subset H$ such that $x_n \to x$. If $x \in C$, $1_C(x) = 0 \le 1_C(x_n)$ for all n. Therefore,

$$1_C(x) \le \liminf 1_C(x_n) \text{ as } n\to\infty$$

For x *not in* C, since C^0 is open and $x_n \to x \in C$ then, there exists some Nin N such that $x_n \in C^0$ for each $n \ge N$. So, $1_C(x_n) = +\infty$ for all $n \ge N$. Therefore, $\liminf 1_C(x_n) = +\infty = 1_C(x)\ \forall x \in H$. Now we have $1_C(x) \le \liminf 1_C(x_n)$ *as* $n\to\infty. n\to\infty$
Thus 1_C is lower semi-continuous and so proper and N_C, the sub-differential of 1_C is maximal monotone.

- → Lemma 2.11 ([11])Let X be a reflexive Banach space. Let $T_1 : X \to 2^{X*}$ and $T_2 : X \to 2^{X*}$ be two maximal monotone maps. Suppose that $D(T_1)^T int(DT_2)$ 6= ∅ . Then$T_1 + T_2$ *is* maximal monotone.

3. Main Result

Application

- Theorem 3.1Let H be a real Hilbert space and let D be a nonempty, open and convex subset of H. Suppose $f : H \to H$is a continuous monotone map and $N_C$is continuous (with respect to Hausdorff metric), $I - (f + N_C)$is bounded on bounded set and$(I - (f + N_C))(C) \subseteq C$, where $C = D$. Define a sequence iteratively by

$$\begin{cases} x_1 \in C \ arbitrary, \\ x_{n+1} := x_n - \lambda_n f(x_n) - \lambda_n u_n - \lambda_n \theta_n (x_n - x_1),\ u_n \in N_C(x_n) \end{cases}, \quad (A2)$$

where $\{\lambda_n\}$ *and* $\{\theta_n\}$ are sequences in(0,1) satisfying the following conditions:
(i) $\lambda_n(1 + \theta_n) < 1$; (ii) $\lim \theta_n = 0$; $n\to\infty$

$$(iii)\ \sum_{n=1}^{\infty} \lambda_n\theta_n = \infty,\ \lambda_n = o(\theta_n);$$

$$(iv)\ \lim_{n\to\infty}\left(\frac{\frac{\theta_n-1}{\theta_n} - 1}{\lambda_n\theta_n}\right) = 0,\ \sum_{n=1}^{\infty}\lambda_n^2 < \infty.$$

Suppose $V\,IP(C,f)$ 6= ∅ . Then, there exists a real constant$\gamma_0>0$ such that$\lambda^{q}_{n}{}^{-1} < \gamma_0\theta_n\ \forall n \ge 1$, the sequence$\{x_n\}$ converges strongly to$x^* \in V\,IP(f,C)$.

- ✓ Proof:

The scheme is well defined using the facts D is convex, Remark 1.6.4, $(I -(f + N_C))(C) \subset C$ and the condition $\lambda_n(1 + \theta_n) <1$. Indeed, for $\lambda,\theta \in (0,1) : 0 < \lambda(1 + \theta) <1$, $x,y,z \in C$, we have

$$(1-\lambda)x - \lambda y - \lambda\theta(x-z) = (1-\lambda(1+\theta))x + (1+\theta)\left[\frac{1}{1+\theta}y + \frac{\theta}{1+\theta}z\right] \in C.$$

Now, using the scheme (A2) we have

$$\begin{aligned} x_{n+1} &= x_n - \lambda_n f(x_n) - \lambda_n u_n - \lambda_n\theta_n(x_n - x_1) \text{ with } u_n \in N_C(x_n) \\ &= x_n - \lambda_n x_n + \lambda_n x_n - \lambda_n f(x_n) - \lambda_n N_C(x_n) - \lambda_n\theta_n(x_n - x_1) \\ &= x_n - \lambda_n x_n + \lambda_n(x_n - f(x_n) - N_C(x_n)) - \lambda_n\theta_n(x_n - x_1) \\ &= (1 - \lambda_n)x_n + \lambda_n(I - (f + N_C))(x_n) - \lambda_n\theta_n(x_n - x_1). \end{aligned}$$

Setting $T := I-(f+N_C)$, we see that $x_{n+1} := (1-\lambda_n)x_n+\lambda_n v_n-\lambda_n\theta_n(x_n-x_1)$ for some $v_n \in Tx_n$, which is exactly the scheme in theorem 1.6 We note that $x^* \in Tx^* \Leftrightarrow x^* \in V\,IP(C,f)$ by Lemma 2.7. Therefore, every $x^* \in F(T)$ is a solution of the variational inequality problem. It is enough, therefore, to show that $\{x_n\}$ converges to a fixed point of T. To do this, we employ Theorem 1.6.
Space requirement: The authors in [2] worked on a nonempty, open and convex subset D of a q-uniformly smooth real Banach space X. We have a real Hilbert space H, i.e., $X = H$. From our assumption D is a nonempty open and convex subset of H. It is well known that every Hilbert space is 2-uniformly smooth space (see Remark 2.1). So, the space requirements of Theorem 1.6 are satisfied. Map requirements: In Theorem 1.6, the map used is a multivalued continuous, bounded and pseudo-contractive mapping. We need to show that the map used in the application also satisfies all these conditions. The set $Tx = \{x-f(x)\}-N_C(x)$ is closed since $N_C(x)$ is closed. This is because $\partial f(x)$ is closed for any convex function f. Also, Tx is bounded from hypothesis. So Tx is closed and bounded. Also, $Tx \subseteq T(C) \subseteq CB(C)$. Thus, $Tx \in CB(C)$. To show pseudo-contractiveness of the map, we have $D(f)\cap int(D(N_C)) \supseteq H\cap int(C) \neq \emptyset$

(since $C = D$ and D is open and nonempty). Therefore using Lemma 4.2.8, $f + N_C$ is maximal monotone. Also by Lemma 4.1.10, $I - (f + N_C)$ is pseudo contractive. For continuity, from our hypothesis, f is continuous. The identity function I is also continuous. From hypothesis, N_C is continuous with respect to the Hausdorff metric. So, we have $f + N_C$ to be continuous with respect to the Hausdorff metric. The fact that the difference of two continuous functions is also a continuous function, we obtained $I - (f + N_C)$ to be continuous. We also have $I - (f + N_C)$ map bounded sets to bounded sets. Thus, all the map requirements of Theorem 1.6 are satisfied.
In Theorem 1.6, it was assumed that (the set of fixed points) $F(T)$ is not empty. C is nonempty closed convex subset of a real Hilbert space H, by lemma 2.7, $F(T) = V\,IP(C,f)$. Since $V\,IP(C, f)$ 6= ∅ from hypothesis, $F(T)$ is not empty. We therefore conclude that $\{x_n\}$ converges to a fixed point of T which is a solution of the variational inequality problem (VIP) with respect to C and f.

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