INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH & DEVELOPMENT

ISSN: 2278 – 0211 (Online)

Distribution Of The Zeros Of A Polynomial

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Abstract:
In this paper we find the bounds for the zeros of a polynomial, when the coefficients of the polynomial or their real and imaginary parts are restricted to certain conditions.

Mathematics Subject Classification: *30C10,30C15*

Key-words: *Polynomial , Unit disk, Zero.*

1.Introduction And Statement Of Results

The following elegant result on the distribution of the zeros of polynomials is due to Enestrom and Kakeya [4]:

1.1.Theorem A

Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then P(z) has all its zeros in the closed unit disk $|z| \le 1$.

In the literature there exist several generalizations and extensions of this result. Recently Y. Choo [1] proved the following result:

1.2.Theorem B

Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,

$$|\arg a_j - \beta| \le \alpha \le \beta,\ j = 0,1,2,......,n,$$

and for some $k_1 \ge 1, k_2 \ge 1$, either

$$k_1|a_n| \ge |a_{n-2}| \ge ...... \ge |a_3| \ge |a_1|$$

and $k_2|a_{n-1}| \ge |a_{n-3}| \ge ...... \ge |a_2| \ge |a_0|$, if n is odd

or

$$k_1|a_n| \ge |a_{n-2}| \ge ...... \ge |a_2| \ge |a_0|$$

and $k_2|a_{n-1}| \ge |a_{n-3}| \ge ...... \ge |a_3| \ge |a_1|$, if n is even.

Then all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \le \frac{K}{|a_n|}$$

where

$$K = (k_1 - 1)|a_n| + (k_2 - 1)|a_{n-1}| + |a_1| + |a_0| + (k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha)$$
$$+ (|a_1| + |a_0|)(\sin\alpha - \cos\alpha) + 2\sin\alpha \sum_{j=2}^{n-2} |a_j| .$$

M. H. Gulzar [3] proved the following results:

1.3.Theorem C

Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j=\alpha_j+i\beta_j,\ j=0,1,2,......,n$, such that for some $k_1\geq 1, k_2\geq 1,\ 0<\tau_1\leq 1, 0<\tau_2\leq 1$, either

$$k_1\alpha_n\geq\alpha_{n-2}\geq......\geq\alpha_3\geq\tau_1\alpha_1$$

and $k_2\alpha_{n-1}\geq\alpha_{n-3}\geq......\geq\alpha_2\geq\tau_2\alpha_0$, if n is odd

or

$$k_1\alpha_n\geq\alpha_{n-2}\geq......\geq\alpha_2\geq\tau_1\alpha_0$$

and $k_2\alpha_{n-1}\geq\alpha_{n-3}\geq......\geq\alpha_3\geq\tau_2\alpha_1$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\leq\frac{K_1}{|a_n|}$$

where

$$K_1=k_1(\alpha_n+|\alpha_n|)+k_2(\alpha_{n-1}+|\alpha_{n-1}|)+2(|\alpha_1|+|\alpha_0|)-(|\alpha_n|+|\alpha_{n-1}|)-\tau_1(\alpha_1+|\alpha_1|)$$
$$-\tau_2(\alpha_0+|\alpha_0|)+|\beta_n|+|\beta_{n-1}|+2\sum_{j=0}^{n-2}|\beta_j|,$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\leq\frac{K_2}{|a_n|}$$

where

$$K_2=k_1(\alpha_n+|\alpha_n|)+k_2(\alpha_{n-1}+|\alpha_{n-1}|)+2(|\alpha_1|+|\alpha_0|)-(|\alpha_n|+|\alpha_{n-1}|)-\tau_2(\alpha_1+|\alpha_1|)$$
$$-\tau_1(\alpha_0+|\alpha_0|)+|\beta_n|+|\beta_{n-1}|+2\sum_{j=0}^{n-2}|\beta_j|.$$

1.4.Theorem D

Let Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,

$$|\arg a_j-\beta|\leq\alpha\leq\beta,\ j=0,1,2,......,n,$$

and for some $k_1\geq 1,\ k_2\geq 1, 0<\tau_1\leq 1, 0<\tau_2\leq 1$, either

$$k_1|a_n| \geq |a_{n-2}| \geq ...... \geq |a_3| \geq \tau_1|a_1|$$

and $k_2|a_{n-1}| \geq |a_{n-3}| \geq ...... \geq |a_2| \geq \tau_2|a_0|$, if n is odd

or

$$k_1|a_n| \geq |a_{n-2}| \geq ...... \geq |a_2| \geq \tau_1|a_0|$$

and $k_2|a_{n-1}| \geq |a_{n-3}| \geq ...... \geq |a_3| \geq \tau_2|a_1|$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq \frac{K_1}{|a_n|}$$

where

$$K_1 = (k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) - (|a_n| + |a_{n-1}|)$$

$$-(\tau_1|a_1| + \tau_2|a_0|)(\cos\alpha - \sin\alpha + 1) + 2(|a_1| + |a_0|) + 2\sin\alpha\sum_{j=2}^{n-2}|a_j|.$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq \frac{K_2}{|a_n|}$$

where

$$K_2 = (k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) - (|a_n| + |a_{n-1}|)$$

$$-(\tau_1|a_0| + \tau_2|a_1|)(\cos\alpha - \sin\alpha + 1) + 2(|a_1| + |a_0|) + 2\sin\alpha\sum_{j=2}^{n-2}|a_j|.$$

The aim of this paper is to generalize the above mentioned results. More precisely, we prove the following results:

1.5.Theorem 1

Let Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j = \alpha_j + i\beta_j, j = 0,1,2,......,n$, such that for some $\rho_1 \geq 0, \rho_2 \geq 0$, $0 < \tau_1 \leq 1, 0 < \tau_2 \leq 1$,either

$$\rho_1 + \alpha_n \geq \alpha_{n-2} \geq ...... \geq \alpha_3 \geq \tau_1\alpha_1$$

and $\rho_2 + \alpha_{n-1} \geq \alpha_{n-3} \geq ...... \geq \alpha_2 \geq \tau_2\alpha_0$, if n is odd

or

$$\rho_1 + \alpha_n \geq \alpha_{n-2} \geq ...... \geq \alpha_2 \geq \tau_1 \alpha_0$$

and $\rho_2 + \alpha_{n-1} \geq \alpha_{n-3} \geq ...... \geq \alpha_3 \geq \tau_2 \alpha_1$, if n is even.

Then for even n all the zeros of P(z) lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{K_1}{|a_n|}$$

where

$$K_1 = 2(\rho_1 + \rho_2) + (\alpha_n + \alpha_{n-1}) + 2(|a_0| + |a_1|) - \tau_1(\alpha_0 + |\alpha_0|)$$

$$- \tau_2(\alpha_1 + |\alpha_1|) + |\beta_n| + |\beta_{n-1}| + 2\sum_{j=0}^{n-2} |\beta_j|,$$

and for odd n all the zeros of P(z) lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{K_2}{|a_n|}$$

where

$$K_2 = 2(\rho_1 + \rho_2) + (\alpha_n + \alpha_{n-1}) + 2(|a_0| + |a_1|) - \tau_2(\alpha_0 + |\alpha_0|)$$

$$- \tau_1(\alpha_0 + |\alpha_0|) + |\beta_n| + |\beta_{n-1}| + 2\sum_{j=0}^{n-2} |\beta_j|.$$

1.6.Remark 1

Taking $\rho_1 = (k_1 - 1)\alpha_n$ and $\rho_2 = (k_2 - 1)\alpha_{n-1}, k_1 \geq 1, k_2 \geq 1$, Theorem 1 reduces to Theorem C.

If the coefficients a_j are real i.e. $\beta_j = 0$ for all j, Theorem 1 gives the following result:

1.7.Corollory 1

Let Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\rho_1 \geq 0, \rho_2 \geq 0$, $0 < \tau_1 \leq 1, 0 < \tau_2 \leq 1$,either

$$\rho_1 + a_n \geq a_{n-2} \geq ...... \geq a_3 \geq \tau_1 a_1$$

and $\rho_2 + a_{n-1} \geq a_{n-3} \geq ...... \geq a_2 \geq \tau_2 a_0$, if n is odd

or

$$\rho_1 + a_n \geq a_{n-2} \geq ...... \geq a_2 \geq \tau_1 a_0$$

and $\rho_2 + a_{n-1} \geq a_{n-3} \geq ...... \geq a_3 \geq \tau_2 a_1$, if n is even.

Then for even n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{K_1}{|a_n|}$$

where

$$K_1=2(\rho_1+\rho_2)+(a_n+a_{n-1})+2(|a_0|+|a_1|)-\tau_1(a_0+|a_0|)-\tau_2(a_1+|a_1|),$$ and

for odd n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{K_2}{|a_n|}$$

where

$$K_2=2(\rho_1+\rho_2)+(a_n+a_{n-1})+2(|a_0|+|a_1|)-\tau_2(a_0+|a_0|)-\tau_1(a_0+|a_0|).$$

Applying Theorem 1 to the polynomial –iP(z), we get the following result:

1.8.Corollary 2

Let Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $a_j=\alpha_j+i\beta_j,\ j=0,1,2,......,n$, such that for some $\rho_1\ge 0,\rho_2\ge 0$, $0<\tau_1\le 1, 0<\tau_2\le 1$, either

$$\rho_1+\beta_n\ge\beta_{n-2}\ge......\ge\beta_3\ge\tau_1\beta_1$$

and $\rho_2+\beta_{n-1}\ge\beta_{n-3}\ge......\ge\beta_2\ge\tau_2\beta_0$, if n is odd

or

$$\rho_1+\beta_n\ge\beta_{n-2}\ge......\ge\beta_2\ge\tau_1\beta_0$$

and $\rho_2+\beta_{n-1}\ge\beta_{n-3}\ge......\ge\beta_3\ge\tau_2\beta_1$, if n is even.

Then for even n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{K_1}{|a_n|}$$

where

$$K_1=2(\rho_1+\rho_2)+(\beta_n+\beta_{n-1})+2(|a_0|+|a_1|)-\tau_1(\beta_0+|\beta_0|)$$

$$-\tau_2(\beta_1+|\beta_1|)+|\alpha_n|+|\alpha_{n-1}|+2\sum_{j=0}^{n-2}|\alpha_j|,$$

and for odd n all the zeros of P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{K_2}{|a_n|}$$

where

$$K_2 = 2(\rho_1 + \rho_2) + (\beta_n + \beta_{n-1}) + 2(|a_0| + |a_1|) - \tau_2(\beta_0 + |\beta_0|)$$
$$- \tau_1(\beta_0 + |\beta_0|) + |\alpha_n| + |\alpha_{n-1}| + 2\sum_{j=0}^{n-2}|\alpha_j|.$$

For different choices of $\rho_1 \geq 0$, $\rho_2 \geq 0, 0 < \tau_1 \leq 1, 0 < \tau_2 \leq 1,$ we get many other interesting results from Theorem 1.

1.8.Theorem 2

Let Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,

$$|\arg a_j - \beta| \leq \alpha \leq \beta,\ j = 0,1,2,......,n,$$

and for some $\rho_1 \geq 0$, $\rho_2 \geq 0, 0 < \tau_1 \leq 1, 0 < \tau_2 \leq 1,$ either

$$|\rho_1 + a_n| \geq |a_{n-2}| \geq ...... \geq |a_3| \geq \tau_1|a_1|$$

and $|\rho_2 + a_{n-1}| \geq |a_{n-3}| \geq ...... \geq |a_2| \geq \tau_2|a_0|$, if n is odd

or

$$|\rho_1 + a_n| \geq |a_{n-2}| \geq ...... \geq |a_2| \geq \tau_1|a_0|$$

and $|\rho_2 + a_{n-1}| \geq |a_{n-3}| \geq ...... \geq |a_3| \geq \tau_2|a_1|$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq \frac{K_1}{|a_n|}$$

where

$$K_1 = [(\rho_1 + \rho_2)(\cos\alpha + \sin\alpha + 1) + (|a_n| + |a_{n-1}|)(\cos\alpha + \sin\alpha) + 2(|a_0| + |a_1|)$$
$$- (\tau_1|a_1| + \tau_2|a_0|)(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha\sum_{j=2}^{n-2}|a_j|]$$

and for even n all the zeros of P(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq \frac{K_2}{|a_n|}$$

where

$$K_2 = [(\rho_1+\rho_2)(\cos\alpha+\sin\alpha+1)+(|a_n|+|a_{n-1}|)(\cos\alpha+\sin\alpha)+2(|a_0|+|a_1|)$$

$$-(\tau_1|a_0|+\tau_2|a_1|)(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=2}^{n-2}|a_j|]$$

.

1.9.Remark 2

Taking $\rho_1=(k_1-1)\alpha_n$ and $\rho_2=(k_2-1)\alpha_{n-1}, k_1\geq 1, k_2\geq 1$, Theorem 2 reduces to Theorem D.

For different choices of $\rho_1\geq 0$, $\rho_2\geq 0, 0<\tau_1\leq 1, 0<\tau_2\leq 1$, we get many other interesting results from Theorem 2.

2. Lemmas

For the proofs of the above theorems, we need the following result:

2.1.Lemma 1

Let P(z)=$\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that

$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2}, j=0,1,...n,$ *for* some real β and

$$|a_j|\geq|a_{j-1}|, j=1,2,.......,n,$$

then for some t>0,

$$|ta_j-a_{j-1}|\leq\left[t|a_j|-|a_{j-1}|\right]\cos\alpha+\left[t|a_j|+|a_{j-1}|\right]\sin\alpha.$$

The proof of lemma 1 follows from a lemma due to Govil and Rahman [2].

3. Proofs of Theorems

3.1.Proof Of Theorem 1

Let n be odd. Consider the polynomial

$$F(z)=(1-z^2)P(z)$$

$$=-a_n z^{n+2}-a_{n-1}z^{n+1}+(a_n-a_{n-2})z^n+(a_{n-1}-a_{n-3})z^{n-1}+(a_{n-2}-a_{n-4})z^{n-2}$$

$$+......+(a_4-a_2)z^4+(a_3-a_1)z^3+(a_2-a_0)z^2+a_1z+a_0$$

$$=-a_n z^{n+2}-a_{n-1}z^{n+1}+(\alpha_n-\alpha_{n-2})z^n+(\alpha_{n-1}-\alpha_{n-3})z^{n-1}+(\alpha_{n-2}-\alpha_{n-4})z^{n-2}$$

$$+\ldots\ldots+(\alpha_4-\alpha_2)z^4+(\alpha_3-\alpha_1)z^3+(\alpha_2-\alpha_0)z^2+\alpha_1 z+\alpha_0$$

$$+i\begin{Bmatrix}(\beta_n-\beta_{n-2})z^n+(\beta_{n-1}-\beta_{n-3})z^{n-1}+(\beta_{n-2}-\beta_{n-4})z^{n-2}\\+\ldots\ldots+(\beta_4-\beta_2)z^4+(\beta_3-\beta_1)z^3+(\beta_2-\beta_0)z^2+\beta_1 z+\beta_0\end{Bmatrix}$$

$$=-a_n z^{n+2}-a_{n-1}z^{n+1}-\rho_1 z^n-\rho_2 z^{n-1}+(\rho_1+\alpha_n-\alpha_{n-2})z^n$$

$$+(\rho_2+\alpha_{n-1}-\alpha_{n-3})z^{n-1}+(\alpha_{n-2}-\alpha_{n-4})z^{n-2}+\ldots\ldots$$

$$+(\alpha_4-\alpha_2)z^4+(\alpha_3-\tau_1\alpha_1)z^3+(\tau_1\alpha_1-\alpha_1)z^3$$

$$+(\alpha_2-\tau_2\alpha_0)z^2+(\tau_2\alpha_0-\alpha_0)z^2+\alpha_1 z+\alpha_0$$

$$+i\begin{Bmatrix}(\beta_n-\beta_{n-2})z^n+(\beta_{n-1}-\beta_{n-3})z^{n-1}+(\beta_{n-2}-\beta_{n-4})z^{n-2}\\+\ldots\ldots+(\beta_4-\beta_2)z^4+(\beta_3-\beta_1)z^3+(\beta_2-\beta_0)z^2+\beta_1 z+\beta_0\end{Bmatrix}$$

For $|z|>1$, we have, by using the hypothesis,

$$|F(z)|\geq|z|^{n+1}\Bigg[|a_n z+a_{n-1}|-\Bigg\{\frac{|\rho_1|}{|z|}+\frac{|\rho_2|}{|z|^2}+\frac{\rho_1+\alpha_n-\alpha_{n-2}}{|z|}+\frac{\rho_2+\alpha_{n-1}-\alpha_{n-3}}{|z|^2}$$

$$+\frac{\alpha_{n-2}-\alpha_{n-4}}{|z|^3}+\ldots\ldots+\frac{\alpha_4-\alpha_2}{|z|^{n-3}}+\frac{\alpha_3-\tau_1\alpha_1}{|z|^3}+\frac{(1-\tau_1)|\alpha_1|}{|z|^3}$$

$$+\frac{\alpha_2-\tau_2\alpha_0}{|z|^{n-1}}+\frac{(1-\tau_2)|\alpha_0|}{|z|^{n-1}}+\frac{|\alpha_1|}{|z|^n}+\frac{|\alpha_0|}{|z|^{n+1}}+\frac{|\beta_n-\beta_{n-1}|}{|z|}+\ldots\ldots$$

$$+\frac{|\beta_2-\beta_0|}{|z|^{n-1}}+\frac{|\beta_1|}{|z|^n}+\frac{|\beta_0|}{|z|^{n+1}}\ \Bigg\}\Bigg]$$

$$>|z|^{n+1}\big[|a_n z+a_{n-1}|-\{\rho_1+\rho_2+\rho_1+\alpha_n-\alpha_{n-2}+\rho_2+\alpha_{n-1}-\alpha_{n-3}$$

$$+\alpha_{n-2}-\alpha_{n-4}+\ldots\ldots+\alpha_4-\alpha_2+\alpha_3-\tau_1\alpha_1+(1-\tau_1)|\alpha_1|$$

$$+\alpha_2-\tau_2\alpha_0+(1-\tau_2)|\alpha_0|+|\alpha_1|+|\alpha_0|+|\beta_n|+|\beta_{n-2}|+\ldots\ldots$$

$$+|\beta_2|+|\beta_0|+|\beta_1|+|\beta_0|\ \}\ \big]$$

$$=|z|^{n+1}\big[|a_n z+a_{n-1}|-\{2(\rho_1+\rho_2)+\alpha_n+\alpha_{n-1}+2(|\alpha_0|+|\alpha_1|)-\tau_1(\alpha_1+|\alpha_1|)$$

$$-\tau_2(\alpha_0+|\alpha_0|)+2(|\beta_0|+|\beta_1|)+\sum_{j=2}^{n}|\beta_j|\ \}\ \big]$$

$$>0 \text{ if}$$

$$\left|z+\frac{a_{n-1}}{a_n}\right|>\frac{K_1}{|a_n|}.$$

This shows that all the zeros of F(z) of modulus greater than 1 lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{K_1}{|a_n|}.$$

Since the zeros of F(z) of modulus less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) and hence P(z) lie in the disk

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{K_1}{|a_n|}.$$

The proof for the case when n is even is similar.

3.2.Proof Of Theorem 2

Let n be odd. Consider the polynomial

$$F(z)=(1-z^2)P(z)$$

$$=-a_n z^{n+2}-a_{n-1}z^{n+1}+(a_n-a_{n-2})z^n+(a_{n-1}-a_{n-3})z^{n-1}+(a_{n-2}-a_{n-4})z^{n-2}$$

$$+......+(a_4-a_2)z^4+(a_3-a_1)z^3+(a_2-a_0)z^2+a_1 z+a_0$$

$$=-a_n z^{n+2}-a_{n-1}z^{n+1}-\rho_1 z^n-\rho_2 z^{n-1}+(\rho_1+a_n-a_{n-2})z^n$$

$$+(\rho_2+a_{n-1}-a_{n-3})z^{n-1}+(a_{n-2}-a_{n-4})z^{n-2}+......$$

$$+(a_4-a_2)z^4+(a_3-\tau_1 a_1)z^3+(\tau_1 a_1-a_1)z^3$$

$$+(a_2-\tau_2 a_0)z^2+(\tau_2 a_0-a_0)z^2+a_1 z+a_0 .$$

For $|z|>1$, we have, by using Lemma 1,

$$|F(z)|\ge |z|^{n+1}\Bigg[|a_n z+a_{n-1}|-\{\frac{|\rho_1|}{|z|}+\frac{|\rho_2|}{|z|^2}+\frac{|\rho_1+a_n-a_{n-1}|}{|z|}+\frac{|\rho_2+a_{n-1}-a_{n-2}|}{|z|^2}$$

$$+\frac{|a_{n-2}-a_{n-4}|}{|z|^3}+......+\frac{|a_4-a_2|}{|z|^{n-3}}+\frac{|a_3-\tau_1 a_1|}{|z|^3}+\frac{(1-\tau_1)|a_1|}{|z|^3}$$

$$+\frac{|a_2-\tau_2 a_0|}{|z|^{n-1}}+\frac{(1-\tau_2)|a_0|}{|z|^{n-1}}+\frac{|a_1|}{|z|^n}+\frac{|a_0|}{|z|^{n+1}}\ \}\ ]$$

$$>|z|^{n+1}\big[|a_n z+a_{n-1}|-\{\rho_1+\rho_2+|\rho_1+a_n-a_{n-2}|+|\rho_2+a_{n-1}-a_{n-3}|$$

$$+|a_{n-2}-a_{n-4}|+......+|a_4-a_2|+|a_3-\tau_1 a_1|+(1-\tau_1)|a_1|$$

$$+ |a_2 - \tau_2 a_0| + (1-\tau_2)|a_0| + |a_1| + |a_0| \ \} \ ]$$

$$\geq |z|^{n+1} [|a_n z + a_{n-1}| - \{\rho_1 + \rho_2 + (|\rho_1 + a_n| - |a_{n-1}|)\cos\alpha$$

$$+ (|\rho_1 + a_n| + |a_{n-2}|)\sin\alpha + (|\rho_2 + a_{n-1}| - |a_{n-3}|)\cos\alpha$$
$$+ (|\rho_2 + a_{n-1}| + |a_{n-3}|)\sin\alpha + (|a_{n-2}| - |a_{n-4}|)\cos\alpha$$
$$+ (|a_{n-2}| + |a_{n-4}|)\sin\alpha + ... + (|a_4| - |a_2|)\cos\alpha$$
$$+ (|a_4| + |a_2|)\sin\alpha + (|a_3| - \tau_1|a_1|)\cos\alpha + (|a_3| - \tau_1|a_1|)\sin\alpha$$
$$+ (1-\tau_1)|a_1| + (|a_2| - \tau_2|a_0|)\cos\alpha + (|a_2| + \tau_2|a_0|)\sin\alpha$$
$$+ (1-\tau_2)|a_0| + |a_1| + |a_0| \ \} \ ]$$

$$\geq |z|^{n+1} [|a_n z + a_{n-1}| - \{\rho_1 + \rho_2)(\cos\alpha + \sin\alpha + 1)$$

$$+ (|a_n| + |a_{n-1}|)(\cos\alpha + \sin\alpha) + 2(|a_0| + |a_1|$$

$$- (\tau_1|a_1| + \tau_2|a_0|)(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha \sum_{j=2}^{n-2} |a_j| \ \} \ ]$$

> 0 if

$$\left| z + \frac{a_{n-1}}{a_n} \right| > \frac{K_1}{|a_n|}$$

This shows that all the zeros of F(z) of modulus greater than 1 lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{K_1}{|a_n|}.$$

Since the zeros of F(z) of modulus less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) and hence P(z) lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{K_1}{|a_n|}.$$

That proves the result in case n is odd. The proof is similar in case of even n.

4.Reference

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