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Analysis Of Dynamic Behavior Of A Pre-Stretched Circular Hyper Elastic Membrane With Finite Deformation

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Abstract:

This paper deals with the dynamic analysis of the linear vibration response of a pre-stretched circular hyper elastic membrane with large deformation. Geometrical as well as material nonlinearity come together due to finite deformations and a hyper elastic constitutive relationship. The membrane material is assumed to be isotropic, homogeneous, and Neo-Hookean. A commercial finite element code ANSYS is used to perform the prestressed modal analysis as well as hyper elastic material curve fitting. The results are compared to the analytical solutions obtained from literature. The effect of stretching ratio and membrane thickness on the mode frequencies, shapes and order is studied. The increase in stretching ratio increases the concentration of in plane modes in the lower frequencies, and also decreases the frequency of the same. The out of plane modes are dominant in the lower frequencies at lower stretch ratios due to the predominance bending stiffness compared to membrane stiffness. Finally, the influence of the hyper elastic constitutive laws on linear vibrations is investigated. Results show that the selection of appropriate hyper elastic material model plays a crucial part in getting the correct results.

Key words: hyper elastic, stretch ratio, Neo-Hookean, ANSYS

1.Introduction

Hyper elasticity refers to materials that can experience large strains (up to 500%) which are almost recoverable. They have highly nonlinear load-extension behavior. For most cases they are nearly incompressible except for some rubber foam materials where large volume changes can be achieved. Elastomers are usually elastically isotropic at small deformation, and then anisotropic at finite strain (as the molecule chains tend to realign to the loading direction). Under an essentially monotonic loading condition, however, a larger class of the elastomers can be approximated by an isotropic assumption, which has been historically popular in the modeling of elastomers.

Hyper elastic membranes have become a topic of interest in recent years due to their versatility and applicability in numerous engineering areas, including civil engineering structures, automotive applications (tires, belts, hoses, mounts), Aerospace applications, Biomedical/Dental Industries (artificial organs, wheelchairs, implantable surgical devices), Packaging (Styrofoam) and Sports (safety equipments, shoes, helmets). In recent years, intensive research has been conducted on the development of new membrane materials, including shape memory polymers and dielectric elastomers.

Modern developments in the mechanics of rubber-like materials started with the pioneering work of R.S. Rivlin. The first developments in this field are compiled and collected in the classical work by Green and Adkins.Treolar performed experiments using rubber and other hyper elastic materials to determine their properties. These were used by Rivlin and others to develop mathematical models. Selvadurai performed experimental study on the deflections of rubber membrane and used the available material models in literature to find the optimum model. Goncalves developed an analytical solution using Galerkin technique to find the theoretical frequencies of a pre stretched hyper elastic membrane using Neo Hookean material model.

There are only a few literatures available on the analysis of dynamic behavior of hyper elastic membranes. The aim of the present work is to study the dynamic behavior of a pre-stretched circular hyper elastic membrane using the finite element software ANSYS. The material properties taken from the experimental data published in reference and curve fitted using ANSYS. The results are compared to the analytical solution. A detailed parametric analysis is performed to predict the effect of radial stretching and membrane thickness in the mode frequencies, shape and order. The membrane material is assumed to be isotropic and incompressible, and its behavior is described by the Neo-

Hookean constitutive law. Finally different hyper elastic constitutive models are verified for the same data and the effect of the same is analyzed.

2.Problem Formulation

2.1.Theory

A material is said to be hyper elastic if there exists an elastic potential function W (or strain energy density function) which is a scalar function of one of the strain or deformation tensors, whose derivative with respect to a strain component determines the corresponding stress component. This can be expressed by:

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} \equiv 2\frac{\partial W}{\partial C_{ij}} \qquad (1)$$

Where

S_{ij} = components of the second Piola-Kirchhoff stress tensor

W = strain energy function per unit un-deformed volume

E_{ij} = components of the Lagrangian strain tensor

C_{ij} = components of the right Cauchy-Green deformation tensor

The Lagrangian strain may be expressed as follows:

$$E_{ij} = \frac{1}{2}\left(C_{ij} - \delta_{ij}\right) \qquad (2)$$

where:

δ_{ij} = Kronecker delta (δ_{ij} = 1 if i = j; else δ_{ij} = 0)

The deformation tensor C_{ij} is comprised of the products of the deformation gradients F_{ij}

$$C_{ij} = F_{kj}F_{kj} \qquad (3)$$

where:

F_{ij} = components of the deformation gradient tensor

X_i = un-deformed position of a point in direction i

x_i = Xi + ui = deformed position of a point in direction i

u_i = displacement of a point in direction i

The Kirchhoff stress is defined:

$$\tau_{ij} = F_{ik} S_{kl} F_{jl}$$

And the Cauchy stress is obtained by:

$$\sigma_{ij} = \frac{1}{J}\tau_{ij} = \frac{1}{J} F_{ik} S_{kl} F_{jl} \qquad (4)$$

The Eigen values (principal stretch ratios) of C_{ij} are , λ_1^2 , λ_2^2 and λ_3^2, and exist only if:

$$\det\left[C_{ij} - \lambda_p^2 \delta_{ij}\right] = 0$$

This can be re-expressed as:

$$\lambda_p^6 - I_1\lambda_p^4 + I_2\lambda_p^2 - I_3 = 0 \qquad (5)$$

Where:

I_1, I_2, and I_3 = invariants of C_{ij},

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 = J^2 \end{aligned} \qquad (6)$$

Where J = det $[F_{ij}]$

J is also the ratio of the deformed elastic volume over the reference (un-deformed) volume of materials.

Under the assumption that material response is isotropic, it is convenient to express the strain energy function in terms of strain invariants or principal stretches.

W=W (I_1, I2, I3) =W (I_1, I_2, J)=W (λ_1 , λ_2, λ_3)

Define the volume-preserving part of the deformation gradient, as:

$$\bar{F}_{ij} = J^{-\frac{1}{3}} F_{ij} \qquad (7)$$

And thus

$$\bar{J} = \det\left[\bar{F}_{ij}\right] = 1 \qquad (8)$$

The modified principal stretch ratios and invariants are then:

$$\begin{aligned} \bar{\lambda}_p &= J^{\frac{-1}{3}} \lambda_p, (p = 1, 2, 3) \\ \bar{I}_p &= J^{\frac{-2p}{3}} I_p \end{aligned} \qquad (9)$$

2.2.Material Models

Following are several forms of strain energy potential (W) provided for the simulation of incompressible or nearly incompressible hyper elastic materials.

2.2.1.<u>Neo-Hookean</u>

The form Neo-Hookean strain energy potential is:

$$W = \frac{\mu}{2}\left(\bar{I}_1 - 3\right) + \frac{1}{d}(J-1)^2 \qquad (10)$$

Where:

μ= initial shear modulus of materials

d = material incompressibility

The initial bulk modulus is related to the material incompressibility parameter by:

$$K = \frac{2}{d} \qquad (11)$$

Where:

K = initial bulk modulus

The Neo-Hookean form of strain energy function is a special case of the Mooney–Rivlin form of strain energy function when incompressibility is accounted

2.2.2.Mooney-Rivlin

This option includes 2, 3, 5, and 9 terms Mooney-Rivlin models. The form of the strain energy potential for 2 parameter Mooney-Rivlin model is:

$$W = C_{10}\left(\bar{I}_1 - 3\right) + C_{01}\left(\bar{I}_2 - 3\right) + \frac{1}{d}(J-1)^2 \qquad (12)$$

Where C_{10}, C_{01} and d are material constants

The initial shear modulus is given by:

$M=2(C_{01}+C_{10})$

The initial bulk modulus is:

$$K = \frac{2}{d}$$

2.2.3.Ogden Potential

The Ogden form of strain energy potential is based on the principal stretches of left-Cauchy strain tensor, which has the form:

$$W = \sum_{i=1}^{N}\frac{\mu_i}{\alpha_i}\left(\lambda_1^{-\alpha_i} + \lambda_2^{-\alpha_i} + \lambda_3^{-\alpha_i} - 3\right) + \sum_{k=1}^{N}\frac{1}{d_k}(J-1)^{2k} \qquad (13)$$

Where:

N, μi, αi, dk = material constants

A higher N can provide better fit the exact solution, however, it may, on the other hand, cause numerical difficulty in fitting the material constants and also it requests to have enough data to cover the entire range of interest of the deformation. Therefore a value of N > 3 is not usually recommended.

2.2.4.Arruda-Boyce Model

The form of the strain energy potential for Arruda-Boyce model is:

$$W = \mu\left[\begin{array}{l}\frac{1}{2}\left(\bar{I}_1-3\right)+\frac{1}{20\lambda_L^2}\left(\bar{I}_1^2-9\right)+\frac{11}{1050\lambda_L^4}\left(\bar{I}_1^3-27\right)\\+\frac{19}{7000\lambda_L^6}\left(\bar{I}_1^4-81\right)+\frac{519}{673750\lambda_L^8}\left(\bar{I}_1^5-243\right)\end{array}\right]$$
$$+\frac{1}{d}\left(\bar{I}_2-3\right)+\frac{1}{d}\left(\frac{J^2-1}{2}-\ln(J)\right) \tag{14}$$

Where:

μ = initial shear modulus of material

λL = limiting network stretch

d = material incompressibility parameter

As the parameter λ_L goes to infinity, the model is converted to Neo-Hookean form.

2.2.5. Yeoh Model

The Yeoh model is also called the reduced polynomial form. The strain energy potential is:

$$W = \sum_{i=1}^{N} C_{i0}\left(\bar{I}_1-3\right)^i + \sum_{k=1}^{N}\frac{1}{d_k}\left(J-1\right)^{2k} \tag{15}$$

N = material constant

C_{i0} = material constants

d_k = material constants

The Neo-Hookean model can be obtained by setting

N = 1.

The initial shear modulus is defined:

$\mu = 2C_{10}$

The initial bulk modulus is:

$$K = \frac{2}{d_1}$$

2.3. Problem Formulation

An isotropic, circular hyper elastic membrane with deformed radius R_0, thickness h, and mass density G is considered. It is assumed that h/R_0 is very much less than one, so that the deformed membrane can be described by the theory of hyper elastic membranes under finite deformations. This also facilitates the use of shell elements for the finite element modeling of the same.

Rubber-like materials exhibit very small volume changes so incompressibility is usually assumed for simplicity. If the material is incompressible then the strain-energy density is a function of the first two strain invariants. The stress components can be determined after choosing the constitutive law. In this study, Neo-Hookean model with the second term of equation (10) neglected is utilized. The material properties and analytical formulations are taken from [787]. A circular membrane with initial radius r = 1 m, thickness h= 0.001m, and mass density Δ= 2200 kg/m^3 is considered for the numerical analysis. Stretch ratio is defined as the ratio of final radius (R) to the initial un-deformed radius (r). Unless specified otherwise the above mentioned model with a stretch ratio of 1.1 is used for analysis with Neo-Hookean material model of material constant given in table

2.4.Finite Element Theory

Geometric non linearity refers to the nonlinearities in the structure or component due to the changing geometry as it deflects. That is, the stiffness [K] is a function of the displacements {u}. The stiffness changes because the shape changes and/or the material rotation. The geometric nonlinearities accounted in this analysis are:

2.5.Large Deflection

Pure displacement formulation is considered in this paper only takes displacements or velocities as primary unknown variables. All other quantities such as strains, stresses and state variables in history-dependent material models are derived from displacements. Naturally, they are applicable to small deformations, small deformation-large rotations, and stress stiffening as particular cases. The formulations are based on principle of virtual work. It is the most widely used formulation and is able to handle most nonlinear deformation problems. Minimal assumptions are used in arriving at the slope of nonlinear force-displacement relationship, i.e., element tangent stiffness. Hence, they are also called consistent formulations 12.

$$D\delta W = \int_V \delta e_{ij} C_{ijkl} De_{kl} dV + \int_V \sigma_{ij} \left[\frac{\partial \delta u_k}{\partial x_i} \frac{\partial Du_k}{\partial x_j} - \delta e_{ik} De_{kj} \right] dV \quad (16)$$

Where

$\delta W = \int_V \sigma_{ij} \delta e_{ij} dV$ is the internal virtual work

σ_{ij} = Cauchy stress tensor

C_{ijkl} = Material constitutive tensor

u_i = displacement

x_i = current coordinate

$e_{ij} = \frac{1}{2}\left\{ \frac{du_i}{dx_j} + \frac{du_j}{dx_i} \right\}$, deformation tensor

V = Volume of deformed body

D = Differential operator

The above equation is a set of linear equations of displacement change. They can be solved out by linear solvers. The stiffness has two terms: the first one is material stiffness due to straining; the second one is stiffness due to geometric nonlinearity (stress stiffness).

2.6.Stress Stiffening

Stress stiffening (also called geometric stiffening, incremental stiffening, initial stress stiffening, or differential stiffening) is the stiffening (or weakening) of a structure due to its stress state. This stiffening effect normally needs to be considered for thin structures with bending stiffness very small compared to axial stiffness, such as cables, thin beams, and shells and couples the in-plane and transverse displacements. This effect also augments the regular nonlinear stiffness matrix produced by large strain or large deflection effects. The effect of stress stiffening is accounted for by generating and then stiffness matrix is added to the regular stiffness matrix in order to give the total stiffness in the nonlinear theory of elasticity, the expressions for the strain components in an arbitrary orthogonal coordinate system are 10

$$\varepsilon_{11} = e_{11} + \frac{1}{2}\left(e^2_{11} + e^2_{12} + e^2_{31} \right) \quad (17)$$

$$\varepsilon_{22} = e_{22} + \frac{1}{2}\left(e^2_{22} + e^2_{12} + e^2_{23} \right) \quad (18)$$

$$\varepsilon_{33} = e_{33} + \frac{1}{2}\left(e^2_{33} + e^2_{31} + e^2_{23} \right) \quad (19)$$

$$\varepsilon_{12} = \left(e_{12} + e_{21}\right) + e_{11}e_{12} + e_{22}e_{21} + e_{31}e_{32} \qquad (20)$$

$$\varepsilon_{13} = \left(e_{12} + e_{21}\right) + e_{11}e_{12} + e_{22}e_{21} + e_{31}e_{32} \qquad (21)$$

$$\varepsilon_{23} = \left(e_{23} + e_{32}\right) + e_{22}e_{23} + e_{33}e_{32} + e_{12}e_{13} \qquad (22)$$

Where ε_{ij} (i=1, 2, 3) are the direct strains

ε_{ij} (i=1, 2, 3 and i≠j)) are the shear strains

e_{ij} are the functions of the displacements u, v and w along the three axes with respect to the orthogonal coordinate system under consideration. The equations are used to formulate the effect of nonlinear strain that accounts for the stress stiffening effect.

3.Finite Element Analysis

3.1.The Finite Element Model

The software package ANSYS is used for the current analysis. The quadrilateral configuration of the quadratic shell element (SHELL 281) having 8 nodes, (4 corners and 4 mid-side nodes) is employed in this analysis. These elements are well suited to model a doubly curved shell, geometric nonlinearity, pressure load stiffness and stress stiffening. Each node has 6 degrees of freedom, translations and rotations in x, y, and z direction. The degenerate form of the element (triangular configuration) should be avoided during meshing in order to maintain the accuracy level. The variation of displacement *u* can be expressed by the following polynomial in natural co-ordinates 12.

$$u = a_1 + a_2 r + a_3 s + a_4 r^2 + a_5 rs + a_6 s^2 + a_7 r^2 s + a_7 r\, s^2 \qquad (4)$$

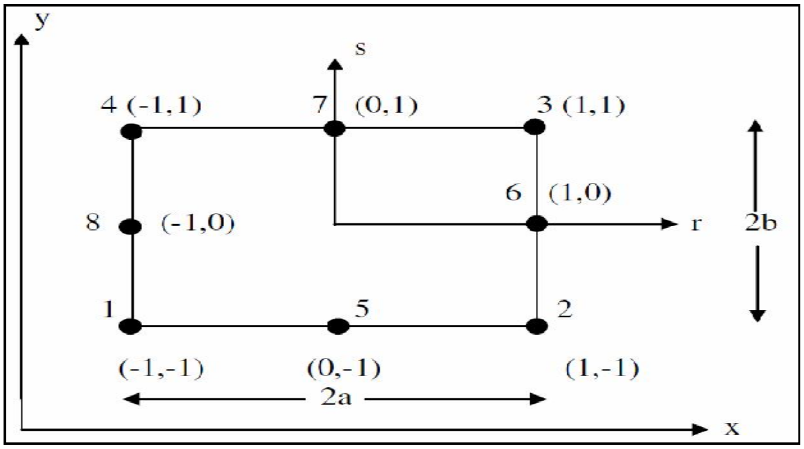

Figure 1: Eight noded rectangular element

The shape functions (N_i) for the element is given by

$$N_1 = \left[\frac{1}{4}(1-r)\ (1-s)(-r-s-1) \right] \tag{23}$$

$$N_2 = \left[\frac{1}{4}\ (1+r)\ (1-s)(r-s-1) \right] \tag{24}$$

$$N_3 = \left[\frac{1}{4}\ (1+r)\ (1+s)(r+s-1) \right] \tag{25}$$

$$N_4 = \left[\frac{1}{4}(1-r)\ (1+s)(-r+s-1) \right] \tag{26}$$

$$N_5 = \left[\frac{1}{2}\ (1+r)\ (1-r)(1-s) \right] \tag{27}$$

$$N_6 = \left[\frac{1}{2}\ (1+r)\ (1+s)(1-s) \right] \tag{28}$$

$$N_7 = \left[\frac{1}{2}\ (1+r)\ (1-s)(1+s) \right] \tag{29}$$

$$N_8 = \left[\frac{1}{2}(1-r)\ (1+s)(1-s)\ \right] \tag{30}$$

$$\{N\}^T = [N_1\ N_2\ N_3\ N_4\ N_5\ N_6\ N_7\ N_8]$$

The components of displacements and rotations are given as

$$\begin{gathered} u = \sum_{i=1}^{8} N_i u_i, v = \sum_{i=1}^{8} N_i v_i, w = \sum_{i=1}^{8} N_i w_i \\ \theta_x = \sum_{i=1}^{8} N_i \theta_{xi}, \theta_y = \sum_{i=1}^{8} N_i \theta_{yi}, \theta_z = \sum_{i=1}^{8} N_i \theta_{zi} \end{gathered} \tag{31}$$

The modal analysis is performed by initially providing a pre-stretch in the large deformation static analysis and then using the prestressed matrix with the deformed configuration in the modal analysis using fixed end conditions.

4.Results And Discussion

4.1.Material Curve Fitting

In order to establish the constants for each constitutive law, the experimental stress–strain curve given in 1 is used. The values are fed to the ANSYS material curve fitting utility, and the material constants for each constitutive model are calculated using an error minimization procedure.

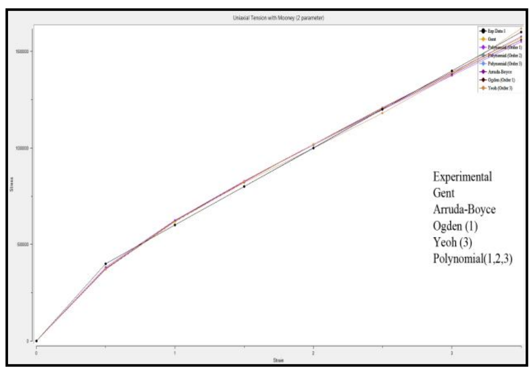

Figure 2: Curve fitting g of hyper elastic models (set 1)

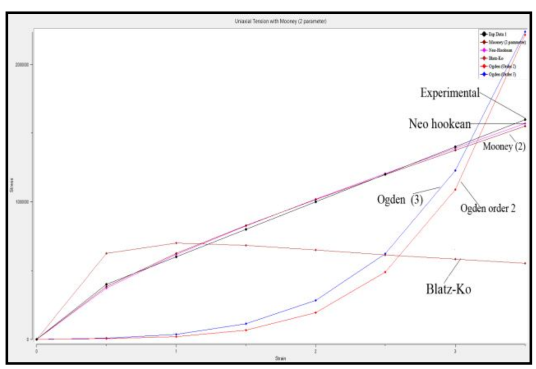

Figure 3: Curve fitting g of hyper elastic models (set 2)

The material constants are tabulated in

Neo Hookean	Gent
μ=352942.68	μ = 352942.677
	J_m= -58987280255.8
Mooney-Rivlin (2 parameter)	
C_{10}=170878.24 C_{01}=14868.89	**Ogden (1)**
	μ = 359986.137
	A_1= 1.98
Arruda Boyce	
μ = 351719.998 λ_L= 24.95077	**Blatz-ko foam**
	μ = 1201376.769

Table 1: Hyper elastic material constants

Modes		Neo-Hookean	Ogden 1st	Mooney 2 parameter	Arruda Boyce	Yeoh order 3
No	**Type (m,n)**					
1	1,0	3.199	3.218	3.312	3.195	3.244
2	1,1	5.098	5.127	5.273	5.091	5.169
3	0,ipt	6.534	6.568	6.714	6.526	6.621
4	1,2	6.833	6.873	7.065	6.824	6.921

Table 2: Frequencies of the first 4 modes using different hyper elastic models

The results obtained using the material models are verified with the analytically computed frequencies given in 3. It is found that that Arruda Boyce, Ogden (1), Gent, Polynomial (1), Neo-Hookean, Mooney Rivlin (2 parameter) and Yeoh models resulted in almost similar frequencies with very low deviations. Mode order as well as shape was the same. Deviations were noticed in case of Blatz-Ko foam, Ogden hyper foam, Polynomial (2, 3), Ogden (2,3) and Mooney Rivlin (5,9 parameter). This is due to the more number of terms in these formulations for modeling compressibility along with incompressibility. All the material constants are not displayed for the sake of brevity.

4.2.Modal Analysis

Modal analysis of the membrane structure yielded mode shapes which can be identified in terms of wave numbers m and n.

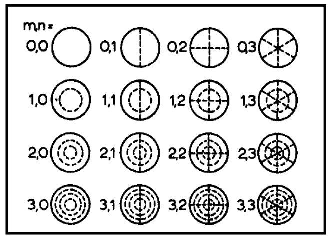

Figure 4: Wavenumbers of circular palte vibration

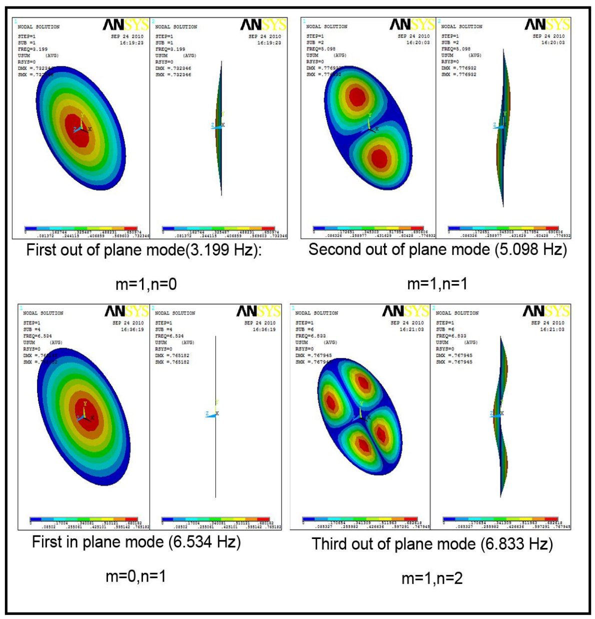

Figure 5: Mode shapes for first 4 modes

4.3.Parametric Study

4.3.1.<u>Effect Of Radial Stretching</u>

The equation for the natural frequency shows that there is a considerable effect of stretch ratio on the frequency of vibration. The analysis results also confirm the same along with certain interesting findings. The stretch ratio is found to affect not only the mode frequencies of in plane as well as out of plane modes but also the order of modes. At a stretching ratio of unity, bending stiffness of the membrane dominates over the membrane stiffness due to zero stretching. Only out of plane modes are found even up to first twenty frequencies. Also the frequencies are clustered together to a great extent. The in plane modes are almost absent.

As the stretch ratio increases we see that the order of modes also changes. More are more in-plane modes starts appearing, first in the lower frequency domain and then

Moving to the lower frequencies form the back pushing down the out o plane modes.

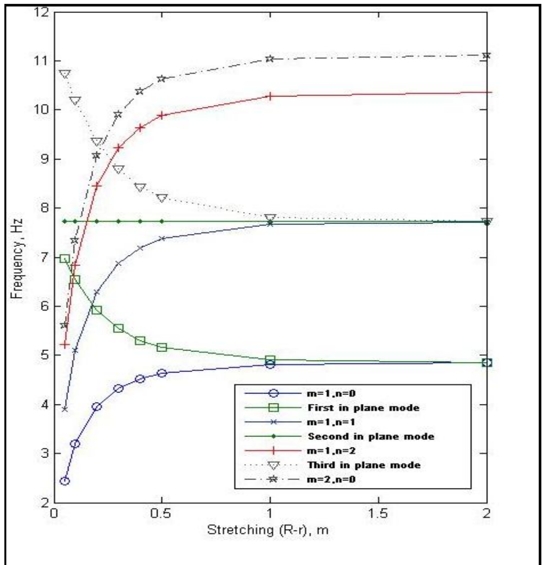

Figure 6: Variation of frequencies with stretching

5.Conclusion

The development of an accurate hyper elastic model has been a highly intriguing work due to the highly nonlinear response over a wide range of strains. This makes the static as well as the dynamic behavior of hyper elastic structures highly complex. The paper highlights the importance of the same and suggests a finite element procedure to model hyper elastics based on the experimental data obtained. Due to the relative importance of membrane structures in diverse fields and their complex behavior under different loading as well as geometric configuration, proper analysis of static as well as dynamic response is essential especially in aerospace applications, vibration control and biomedical devices. The modal analysis of pre-stretched hyper elastic circular membrane showed that the increase in stretching increases the frequencies of out of plane modes till a saturation level is attained. Majority of in plane modes exhibit decrease in frequencies with increase in stretching till a saturation level is attained. The changes in frequencies are drastic in the initial stretching zone compared to the later stages. At very high stretching values (>2) we can see that the in plane and out of plane modes form clusters of same frequency depending on the wave number. The increase in stretching also produces a change in mode order which is of great significance in vibration control. The mode order plays a crucial role in determining the placement of actuators and to avoid spillover effects. Variation in membrane thickness had no effect on mode frequencies, order or shapes. The selection of material model was found to be crucial. The incompressible behavior of rubber was best modeled using Neo-Hookean and Arruda Boyce models. Other models were also found to give very good match. The work has significant contributions owing to the limited studies on statics and dynamics of hyper elastic structures. The work may be further extended to modeling of complex structures incorporating effects of compressibility, nonlinear vibration and different loading/boundary conditions.

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