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On Some Common Fixed Point Theorems For Occasionally Weakly Compatible Mappings In Fuzzy Metric Spaces

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Abstract:

This paper presents on some common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces.

Keywords: *Fixed point, occasionally weakly compatible mappings, fuzzy metric space.*

Introduction

Fuzzy sets and fuzzy logic were introduced by Lotfi A. Zadeh [12] in 1965. Zadeh was almost single-handedly responsible for the early development in this field. In mathematical programming, problems are expressed as optimizing some goal functions under given certain constraints, and real life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions.

Since then, there were many authors who studied the fuzzy sets with applications, especially George and Veeramani [2] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [14]. A. George and P. Veermani [2] revised the notion of fuzzy metric spaces with the help of continuous t-norm in 1994. As a result of many fixed point theorem for various forms of mapping are obtained in fuzzy metric spaces. Recently, many researchers have proved common fixed point theorems involving fuzzy sets. Pant [18] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. Vasuki [20] and Singh and Chouhan [4] also introduced some fixed point theorems in fuzzy metric spaces for R-weakly commuting and compatible mappings respectively.

Balasubramaniam et al. [15] proved the open problem of Rhoades [3] on the existence of a contractive definition which generates a fixed point but does not force the mapping to be continuous at the fixed point, posses an affirmative answer. Pant and Jha [17] proved an anologus of the result given by Balasubramaniam et al. [15]. Recent work on fixed point theorems in fuzzy metric space can be viewed in references [5, 21, 22, 23].

The idea of some fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces was used by P. Nigam and N. Malviya [16] and obtained some fruitful results. Motivated through the results obtained by P. Nigam and N. Malviya [16], we attempted to prove on some common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces.

Preliminaries

Definition 2.1

A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2

A triangular norm $*$ (shortly t− norm) is a binary operation on the unit interval [0, 1] such that for all a, b, c, d ∈ [0, 1] the following conditions are satisfied:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a * 1 = a, \quad \forall\ a \in [0,1]$;

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Definition 2.3 [2]

A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions: for all x, y, z ∈ X and s, t > 0:

(f1) $M(x,y,0) > 0$;

(f2) $M(x,y,t) = 1, \forall\ t > 0, if\ and\ only\ if\ x = y$;

(f3) $M(x,y,t) = M(y,x,t)$;

(f4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;

(f5) $M(x,y,*) : (0,\infty) \rightarrow (0,1]$ is continuous,

where $M(x,y,t)$ denote the degree of nearness between x and y with respect to t. Then M is called a fuzzy metric on X.

Example 2.4 [2]

Let (X,d) be a metric space. Define $a * b = ab, \forall\ a,b \in [0,1]$ and let M_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$M_d(x,y,t) = \frac{t}{t + d(x y)}$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.

Definition 2.5 [12]

Let $(X, M, *)$ be a fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to converges to x in X if for each $\varepsilon > 0$ $and\ eac\ \ t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon \forall\ n \geq n_0$.

(b) a sequence $\{x_n\}$ in X is said to Cauchy if for each $\varepsilon > 0$ and each $t > 0$, $t\ \ ere\ exists$

$n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon \, \forall \, n, \, m \geq n_0$.

(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.6

Two mappings f and g of a fuzzy metric space $(X, M, *)$ into itself are said to be weakly commuting if

$M(fgx, gfx, t) \geq M(fx, gx, t), \; \forall \, x \in X$ and $t > 0$.

Definition 2.7

Two mappings f and g of a fuzzy metric space $(X, M, *)$ into itself are R- weakly commuting provided there exists some positive real number R such that

$M(fgx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right), \forall \, x \in X$ and $t > 0$.

Definition 2.8 [8]

Two self maps f and g of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X.

Definition 2.9

Two self mappings A and P of a fuzzy metric space (X, M, *) are called compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X.

Definition 2.10 [10]

An element $x \in X$ is called a common fixed point of the mappings $F: X \times X \to X$ and $g: X \to X$ if $x = g(x) = F(x, x)$.

Lemma 2.11

Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0,1)$, such that $M(x, y, qt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$, $t \quad en \; x = y$.

Definition 2.12

Let X be a set f, g self maps of X. A point x in X is called coincidence point of f and g ,iff, fx = gx. We shall call w = fx = gx, a point of coincidence of f and g.

Definition 2.13 [8]

A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.

Definition 2.14

Two self maps f and g of a set X are occasionally weakly compatible (owc) if there is a point x in X which is a coincidence point of f and g at which f and g commute.

A. Al-Thagafi and Naseer Shahzad [1] shown that occasionally weakly is weakly compatible but converse is not true.

Example 2.15

Let R is the usual metric space. Define $f, g: R \to R$ be $f(x) = 3x$ and $g(x) = x^2, \forall x \in R$. Then $fx = gx$, for $x = 0,3$ but $f g(0) = g f(0)$, and $f g(3) \neq T g(3)$, f and g are occasionally weakly compatible self maps but not weakly compatible.

Lemma 2.16 [8]

Let X be a set, f, g owc self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

Main Results

Theorem 3.1

Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs {A, S} and {B, T} are owc. If there exists a point $q \in (0,1)$, $\forall$ $x, y \in X$ and $t > 0$. Such that

$$M(Ax, By, qt) \geq \alpha_1 \min\{M(Sx, Ty, t), M(Sx, Ax, t)\} + \alpha_2 \min\{M(By, Ty, t), M(Ax, Ty, t)\} + \alpha_3 M(By, Sx, t) \quad (3.1)$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$, and $(\alpha_1 + \alpha_2 + \alpha_3) > 1$.

Then there exists a unique point of $w \in X$, such that $Aw = Sw = w$ and a unique point $z \in X$, such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof

Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$, we claim that $Ax = By$. If not, by inequality (3.1) we have

$$M(Ax,By,qt) \geq \alpha_1 \min\{M(Sx,Ty,t), M(Sx,Ax,t)\} + \alpha_2 \min\{M(By,Ty,t), M(Ax,Ty,t)\} + \alpha_3 M(By,Sx,t)$$

$$= \alpha_1 \min\{M(Ax,By,t), M(Ax,Ax,t)\} + \alpha_2 \min\{M(By,By,t), M(Ax,By,t)\} + \alpha_3 M(By,Ax,t)$$

$$= \alpha_1 \min\{M(Ax,By,t), 1\} + \alpha_2 \min\{1, M(Ax,By,t)\} + \alpha_3 M(By,Ax,t)$$

$$= (\alpha_1 + \alpha_2 + \alpha_3)\, M(Ax,By,t).$$

A contradiction, since $(\alpha_1 + \alpha_2 + \alpha_3) > 1$. Therefore $Ax = By$, i.e., $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (3.1) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. Using Lemma 2.16, we get w is the only common fixed point of A, and S.

i.e., $w = Aw = Sw$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Assume that $w \neq z$. We have

$$M(w,z,qt) = M(Aw,Bz,qt)$$

$$\geq \alpha_1 \min\{M(Sw,Tz,t), M(Sw,Aw,t)\} + \alpha_2 \min\{M(Bz,Tz,t), M(Aw,Tz,t)\} + \alpha_3 M(Bz,Sw,t)$$

$$= \alpha_1 \min\{M(w,z,t), M(w,w,t)\} + \alpha_2 \min\{M(z,z,t), M(w,z,t)\} + \alpha_3 M(z,w,t)$$

$$= \alpha_1 \min\{M(w,z,t), 1\} + \alpha_2 \min\{1, M(w,z,t)\} + \alpha_3 M(z,w,t)$$

$$= \alpha_1 M(w,z,t) + \alpha_2 M(w,z,t) + \alpha_3 M(z,w,t)$$

$$= (\alpha_1 + \alpha_2 + \alpha_3)\, M(w,z,t).$$

Since

$(\alpha_1 + \alpha_2 + \alpha_3) > 1$. Therefore, a contradiction, we have $z = w$ by Lemma 2.16 and z is a common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (3.1).

Theorem 3.2

Let $(X,M,*)$ be a complete fuzzy metric space and let A,B,S and T be self-mappings of X. Let the pairs $\{A,S\}$ and $\{B,T\}$ are owc. If there exists a point $q \in (0,1)$, $\forall$ $x,y \in X$. Such that

$$M(Ax,By,qt) \geq \alpha\, M(Sx,Ty,t) + \beta \min\{ M(Sx,Ax,t), M(Ax,Ty,t)\} + \gamma \min\{M(By,Ty,t), M(By,Sx,t)\}$$

where $\alpha, \beta, \gamma > 0$, and $(\alpha + \beta + \gamma) > 1$ then there exist a unique point $w \in X$, such that $Aw = Sw = w$, and a unique point $z \in X$, such that $Bz = Tz = z$. Moreover $z = w$, so that it is a unique common fixed point of A,B,S and T.

Proof

Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc, so there are points $x,y \in X$, such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (3.2)

$$\begin{aligned} M(Ax,By,qt) &\geq \alpha\, M(Sx,Ty,t) + \beta \min\{ M(Sx,Ax,t), M(Ax,Ty,t)\} + \gamma \min\{M(By,Ty,t), M(By,Sx,t)\} \\ &= \alpha\, M(Ax,By,t) + \beta \min\{ M(Ax,Ax,t), M(Ax,By,t)\} + \gamma \min\{ M(By,By,t), M(By,Ax,t)\} \\ &= \alpha\, M(Ax,By,t) + \beta \min\{ 1, M(Ax,By,t)\} + \gamma \min\{ 1, M(By,Ax,t)\} \\ &= \alpha\, M(Ax,By,t) + \beta M(Ax,By,t)) + \gamma\, M(Ax,By,t) \\ &= (\alpha + \beta + \gamma)\, M(Ax,By,t) \end{aligned}$$

a contradiction, since $(\alpha + \beta + \gamma) > 1$. Therefore $Ax = By$, i.e., $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (3.2) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. Using Lemma 2.16, we get w is the only common fixed point of A and S, i.e., $w = Aw = Sw$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Assume that $w \neq z$. We have

$$M(w,z,qt) = M(Aw,Bz,qt)$$

$$\begin{aligned} &\geq \alpha\, M(Sw,Tz,t) + \beta \min\{M(Sw,Aw,t), M(Aw,Tz,t)\} + \gamma \min\{M(Bz,Tz,t), M\{Bz,Sw,t)\} \\ &= \alpha\, M(w,z,t) + \beta \min\{M(w,w,t), M(w,z,t)\} + \gamma \min\{M(z,z,t), M(z,w,t)\} \\ &= \alpha\, M(w,z,t) + \beta \min\{1, M(w,z,t)\} + \gamma \min\{1, M(z,w,t)\} \end{aligned}$$

$= \alpha M(w,z,t) + \beta M(w,z,t) + \gamma M(w,z,t)$

$= (\alpha + \beta + \gamma) M(w,z,t)$

a contradicition, since $(\alpha + \beta + \gamma) > 1$. Therefore $z = w$ by Lemma 2.16, $z = w$ is common fixed point of A,B,S and T. Therefore the unqueness of the fixed point holds form (3.2).

Theorem 3.3

Let $(X,M,*)$ be a complete fuzzy metric space and let A, and S be selfmap of X. Let the pairs A and S are owc. If there exists a point $q \in (0,1)$, $\forall x,y \in X$ and $t > 0$, such that

$$M(Sx,Sy,qt) \geq \alpha_1 M(Ax,Ay,t) + \alpha_2 M(Ax,Ay,t) + \alpha_3 \min\{M(Sy,Ay,t), M(Sx,Ay,t)\} + \alpha_4 M(Sy,Ax,t), \quad (3.3)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, and $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$. Then A and S have a unique common fixed point.

Proof

Let the pair {A,S}be owc, so there exist a points $x \in X$ such that $Ax = Sx$. $Ax = Sx$. Suppose that there exist another point $y \in X$ for which $Ay = Sy$. We claim that $Sx = Sy$. If not, by inequality (3.3)

$$M(Sx,Sy,qt) \geq \alpha_1 M(Ax,Ay,t) + \alpha_2 M(Ax,Ay,t) + \alpha_3 \min\{M(Sy,Ay,t), M(Sx,Ay,t)\} + Z\, M(Sy,Ax,t)$$

$= \alpha_1 M(Sx,Sy,t) + \alpha_2 M(Sx,Sy,t) + \alpha_3 \min\{M(Sy,Sy,t), M(Sx,Sy,t)\} + \alpha\_4\ M(Sy,Sx,t)$

$= \alpha_1 M(Sx,Sy,t) + \alpha_2 M(Sx,Sy,t) + \alpha_3 \min\{1, M(Sx,Sy,t)\} + \alpha_4 M(Sy,Sx,t)$

$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) M(Sx,Sy,t)$

a contraction, since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$. Therefore Sx = Sy. Therefore Ax = Ay and Ax is unique. From lemma 2.16, A and S have a unique fixed point.

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