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Common Fixed Point Theorem In 2-Metric Space

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Abstract:
In this paper, the concept of semi-compatibility and weak compatibility has been applied to prove common fixed point theorem in 2-metric space, in which we generalize the result of sharma [13].

Mathematics Subject Classification: *47H10, 54H25.*

Keywords: *Common fixed points, 2-metric space, Semi-compatible maps, Weak compatible maps, and Compatible maps.*

1.Introduction

The concept of 2-metric space has been investigated by Gahler [2] to generalize the concept of metric i.e. distance function. A 2 metric space is one which finds its wide range of applications in the fields of military, medicine and economics. Employing various contractive conditions Iseki [4] set out the tradition of proving fixed point theorems in 2-metric spaces. Later on, Naidu and Prasad [5] contributed few fixed point theorems in 2-metric space introducing the concept of weak commutative. Cho et al. [1] introduced the notion of semi-compatible maps in d-topological space. Various authors like Saliga [7], Sharma et al. [8] and Popa [6] proved some interesting fixed point results using implicit real functions and semi-compatibility in d-complete topological spaces.

Recently, B. Singh and S. Jain [9, 10, 11, 12] introduced the concept of semi-compatibility in fuzzy metric spaces, D-metric spaces, 2 metric space and proved fixed point results using implicit relations in these spaces.

The main objective of this paper is to obtain some fixed point theorems in the setting of 2-metric spaces using weak compatibility, semi-compatibility without considering the completeness of the space X and continuity of maps. The relationship between compatible, weak - compatible and semi-compatible maps have also been established. Fisher and Murty (see [3]) proved the following result on metric space:

2.Preliminaries

- Definition2.1 A space X with a non-negative real valued function d on $X \times X \times X$ is said to be 2-metric space if it satisfies the following axioms:
  $d(x, y, z) = 0$ when at least two of x, y, z are equal,
  $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all x, y, z in X and
  $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X
  when d is a 2-metric on X, the ordered pair (X, d) is called a 2-metric space.
- Definition 2.2 A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$, if $\lim_{n\to\infty} d(x_n, x, a) = 0$.
- Definition 2.3 A sequence $\{x_n\}$ is said to be Cauchy sequence, if $\lim_{n\to\infty} d(x_n, x_m, a) = 0$
  for all $a \in X$.

- Definition 2.4 A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X
  converges to a point of X.
- Definition2.5 Two self mapping A and S of a 2-metric space(X, d) are said to be compatible if
  $\lim_{n\to\infty} d(ASx_n, SAx_n, a) = 0$ for all $a \in X$, where $\{x_n\}$ is a sequence in X such that if
  $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X.
- Definition2.6 Two self mapping A and S of a 2-metric space(X, d) are said to be weakly compatible if they commute at their coincidence points i.e., if Ax = Sx , then ASx = SAx.
- Definition2.7 Two self mapping A and S of a 2-metric space(X, d) are said to be semi-compatible if $\lim_{n\to\infty} d(ASx_n, Sx_n, a) = 0$ for all $a \in X$, where $\{x_n\}$ is a sequence in X such that if $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} S x_n = x$ for some x in X.
- Lemma2.1 Let P, Q, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (3.1.1) and (3.1.2). Then the sequence $\{y_n\}$ defined by (1.1) is a Cauchy sequence in X.

3.Main Result

- Theorem3.1: Let P, Q, S and T be mappings from a complete 2-metric space (X, d) into itself satisfying the Conditions

  3.1.1) $S(X) \subset Q(X)$, $T(X) \subset P(X)$

  3.1.2) $d(Sx, Ty, a) \leq \alpha \frac{d(Px, Sx, a]^3 ++[d(Qy, Ty, a)]^3}{[d(Px, Sx, a)]^2+[d(Qy, Ty, a)]^2} + \beta\, d(Px, Qy, a)$

  for all $x, y \in X$, where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$.

  3.1.3) one of P, Q, S and T is continuous .

  3.1.4) The pair (S, P) are semi-compatible and (T, Q) are weak compatible on X.

Then P, Q, S and T have a unique common fixed point in X.
Proof: Let x_0 be any point in X, then by condition (3.1.1) there exist $x_1, x_2 \in X$ such that $Sx_0 = Qx_1$, $Tx_1 = Sx_2$, Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n}= Sx_{2n} = Qx_{2n+1}$, $y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$, n=1, 2, 3, ... (1.1)

By lemma 1.2, $\{y_n\}$ is a Cauchy sequence and hence converges to some point u in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Px_{2n+2}\}$, $\{Tx_{2n+1}\}$ and $\{Qx_{2n+1}\}$ of sequence $\{y_n\}$ also converges to u.

$$\{Qx_{2n+1}\} \to u \quad \text{and} \quad \{Tx_{2n+1}\} \to u \qquad (1.2)$$

$$\{Px_{2n+2}\} \to u \quad \text{and} \quad \{Sx_{2n}\} \to u. \qquad (1.3)$$

- Case l. Since P is continuous and (S, P) is semi-compatible pair, we have $PSx_{2n} \to Pu$, $P^2x_{2n} \to Pu$ and $SPx_{2n} \to Pu$. Now we have to show that Pu = u.

Put $x = Px_{2n}$, $y = x_{2n+1}$ in (3.1.2), we get

$$d(SPx_{2n}, Tx_{2n+1}, a) \le \alpha \frac{[d(PPx_{2n}, SPx_{2n}, a)]^3 + [d(Qx_{2n+1}, Tx_{2n+1}, a)]^3}{[d(PPx_{2n}, SPx_{2n}, a)]^2 + [d(Qx_{2n+1}, Tx_{2n+1}, a)]^2} + \beta d(PPx_{2n}, Qx_{2n+1}, a)$$

Letting $n \to \infty$, $d(Pu, u, a) \le \alpha \frac{[d(Pu, Pu, a)]^3 + [d(u, u, a)]^3}{[d(Pu, Pu, a)]^2 + [d(u, u, a)]^2} + \beta d(Pu, u, a)$

$d(Pu, u, a) \le \alpha\, [d(Pu, Pu, a)] + [d(u, u, a)] + \beta d(Pu, u, a)$

$(1 - \beta)\, d(Pu, u, a) \le 0$. So Pu = u.

Putting $x = u$ and $y = x_{2n+1}$ in (3.1.2), we get

$$d(Su, Tx_{2n+1}, a) \le \alpha \frac{[d(Pu, Su, a)]^3 + [d(Qx_{2n+1}, Tx_{2n+1}, a)]^3}{[d(Pu, Su, a)]^2 + [d(Qx_{2n+1}, Tx_{2n+1}, a)]^2} + \beta d(Pu, Qx_{2n+1}, a)$$

Letting $n \to \infty$, $d(Su, u, a) \le \alpha \frac{[d(u, Su, a)]^3 + [d(u, u, a)]^3}{[d(u, Su, a)]^2 + [d(u, u, a)]^2} + \beta\, d(u, u, a)$

$d(Su, u, a) \le \alpha\, [d(u, Su, a) + d(u, u, a)] + \beta\, d(u, u, a)$

$(1 - \alpha)\, d(Su, u, a) \le 0$. So Su = u and then Pu = Su = u.

As $S(X) \subset Q(X)$, their exists $v \in X$ such that Pu = u = Qv,

Put x = u and y = v in (3.1.2), we get

$$d(Su, Tv, a) \le \alpha \frac{[d(Pu, Su, a]^3 + [d(Qv, Tv, a)]^3}{[d(Pu, Su, a)]^2 + [d(Qv, Tv\ a)]^2} + \beta\, d(Pu, Qv, a)$$

$$d(u, Tv, a) \le \alpha \frac{[d(u, u, a]^3 + [d(u, Tv, a)]^3}{[d(u, u, a)]^2 + [d(u, Tv, a)]^2} + \beta\, d(u, u, a)$$

$d(u, Tv, a) \le \alpha\; d(u, Tv, a)$

$(1 - \alpha)\, d(Tv, u, a) \le 0$. So u = Tv. Then Qv = u = Tv.

Since (T, Q) are weak compatible, therefore, we have TQv = QTv So Tu = Qu.

Put $x = x_{2n}$ and $y = u$, in (3.1.2), we get

$$d(Sx_{2n}, Tu, a) \le \alpha \frac{[d(Px_{2n}, Sx_{2n}, a]^3 + [d(Qu, Tu, a)]^3}{[d(Px_{2n}, Sx_{2n}, a)]^2 + [d(Qu, Tu, a)]^2} + \beta\, d(Px_{2n}, Qu, a)$$

$$d(u, Tu, a) \le \alpha \frac{[d(u, u, a]^3 + [d(Tu, Tu, a)]^3}{[d(u, u, a)]^2 + [d(Tu, Tu, a)]^2} + \beta\, d(u, Tu, a)$$

$d(u, Tu, a) \le \beta\, d(u, Tu, a)$

$(1 - \beta)\, d(Tu, u, a) \le 0$. So that Tu = u, which implies Tu = Qu = u.

Therefore u, is common fixed point of P, Q, S and T.

Casell. Since S is continuous and (S, P) is semi-compatible pair, we have $SPx_{2n} \rightarrow Su$, $S^2x_{2n} \rightarrow Su$ and $PSx_{2n} \rightarrow Su$. Now we have to show that $Su = u$, Put $x = Sx_{2n}$ and $y = x_{2n+1}$ in (3.1.2), we get

$$d(SSx_{2n}, Tx_{2n+1}, a) \leq \alpha \frac{[d(PSx_{2n}, SSx_{2n}, a)]^3+[d(Qx_{2n+1}, Tx_{2n+1}, a)]^3}{[d(PSx_{2n}, SSx_{2n}, a)]^2+[d(Qx_{2n+1}, Tx_{2n+1}, a)]^2} + \beta\, d(PSx_{2n}, Qx_{2n+1}, a)$$

Letting $n\rightarrow\infty$, $d(Su, u, a) \leq \alpha \frac{[d(Su, Su, a]^3+[d(u, u, a)]^3}{[d(Su, Su, a)]^2+[d(u, u, a)]^2} + \beta d(Su, u, a)$

$d(Su, u, a) \leq \alpha\, [d(Su, Su, a] + [d(u, u, a)] + \beta d(Su, u, a)$

$(1 - \beta)\, d(Su, u, a) \leq 0$. Then $Su = u$.

$S(X) \subset Q(X)$, their exists a point $w \in X$ such that $u = Su = Qw$

Putting $x = Sx_{2n}$ and $y = w$ in (3.1.2), we get

$$d(SSx_{2n}, Tw, a) \leq \alpha \frac{[d(PSx_{2n}, SSx_{2n}, a)]^3+[d(Qw, Tw, a)]^3}{[d(PSx_{2n}, SSx_{2n}, a)]^2+[d(Qw, Tw\ a)]^2} + \beta d(PSx_{2n}, Qw, a)$$

Letting $n\rightarrow\infty$, $d(u, Tw, a) \leq \alpha \frac{[d(u, u, a)]^3+[d(u, Tw, a)]^3}{[d(u, u, a)]^2+[d(u, Tw, a)]^2} + \beta d(u, u, a)$

$d(Tw, u, a) \leq \alpha\, d(u, Tw, a)$

$(1-\alpha)\, d(Tw, u, a) \leq 0$. So that $Tw = u$. i,e., $Su = Tw = u$.

Put $x = x_{2n}$ and $y = w$ in (3.1.2), we get

$$d(Sx_{2n}, Tw, a) \leq \alpha \frac{[d(Px_{2n}, Sx_{2n}, a)]^3+[d(Qw, Tw, a)]^3}{[d(Px_{2n}, Sx_{2n}, a)]^2+[d(Qw, Tw\ a)]^2} + \beta d(Px_{2n}, Qw, a)$$

Letting $n\rightarrow\infty$, $d(u, Tw, a) \leq \alpha \frac{[d(u, u, a)]^3+[d(u, Tw, a)]^3}{[d(u, u, a)]^2+[d(u, Tw, a)]^2} + \beta d(u, u, a)$

$d(u, Tw, a) \leq \alpha\ [d(u, u, a) + d(u, Tw, a)]$

$(1 - \alpha)\, d(u, Tw, a) \leq 0$. So that $Tw = u$. i,e., $Tw = u = Qw$.

Since (T, Q) are weak compatible, therefore, we have $TQw = QTw$ so that $Tu = Qu$.

Put $x = x_{2n}$ and $y = u$, in (3.1.2), we get

$$d(Sx_{2n}, Tu, a) \leq \alpha \frac{[d(Px_{2n}, Sx_{2n}, a)]^3+[d(Qu, Tu, a)]^3}{[d(Px_{2n}, Sx_{2n}, a)]^2+[d(Qu, Tu, a)]^2} + \beta d(Px_{2n}, Qu, a)$$

Letting $n\rightarrow\infty$, $d(u, Tu, a) \leq \alpha \frac{[d(u, u, a)]^3+[d(Tu, Tu, a)]^3}{[d(u, u, a)]^2+[d(Tu, Tu, a)]^2} + \beta\, d(u, Tu, a)$

$d(u, Tu, a) \leq \beta\, d(u, Tu, a)$

$(1 - \beta)\, d(Tu, u, a) \leq 0$. So that $Tu = u$, which implies $Tu = Qu = u$.Therefore u, is common fixed point of P, Q, S and T.

Uniqueness Let z be another common fixed point of P, Q, S and T. So $Pz = Qz = Sz = Tz = z$.

Put $x = u$ and $y = z$ in (3.1.2), we get

$d(Su, Tz, a) \leq \alpha \frac{[d(Pu, Su, a)]^3+[d(Qz, Tz, a)]^3}{[d(Pu, Su, a)]^2+[d(Qz, Tz, a)]^2} + \beta d(Pu, Tz, a)$

$d(u, z, a) \leq \alpha \frac{[d(u, u, a)]^3+[d(z, z, a)]^3}{[d(u, u, a)]^2+[d(z, z, a)]^2} + \beta d(u, z, a)$

$d(u, z, a) \leq \alpha [d(u, u, a)+d(z, z, a)] + \beta d(u, z, a)$

$(1 - \beta) d(u, z, a) \leq 0$, which is a contradiction, Hence u = z. Therefore u, is a unique common fixed point of P, Q, S and T.

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