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Isometry of Riemannian Manifolds Admitting a Projective Vector Field Using Metric Semi-Symmetric Connection

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Abstract:
Purpose of this paper is to generalise integral formulas and inequalities of H. Hiramatu [1] using metric semi-symmetric connection $\overset{\circ}{\nabla}$.

Key words*: Isometry of Riemannian manifold, conformal curvature tensor and projective curvature tensor, metric semi-symmetric connection*

1. Introduction

Let M be a connected Riemannian manifold of dimension n covered by the system of coordinate neighborhoods $\{U; x^h\}$

Where the indices i,j,k,……….. Run over the range {1, 2, 3………,n}. Let $g_{ji}, \overset{\circ}{\Gamma}{}^h_{ji}, \overset{\circ}{\nabla}_j, \overset{\circ}{K}{}_{kji}{}^h, \overset{\circ}{K}_{ji}$ and $\overset{\circ}{K}$, be the covariant components of the metric tensor g, the Christoffel symbols formed by g_{ji}, the operator of the covariant differentiation with respect to Γ^h_{ji}, the components of curvature tensor and the components of Ricci tensor and the scalar curvature of M respectively. The vector field v^h is called a projective vector field if it satisfies [4]

$$L_v \overset{\circ}{\Gamma}{}^h_{ji} = \overset{\circ}{\nabla}_j \overset{\circ}{\nabla}_i v^h + v^k \overset{\circ}{K}{}^h_{kji} = \delta^h_j \rho_i + \delta^h_i \rho_j \tag{1.1}$$

for a certain covariant vector field ρ_i, called the associated vector field of v^h, where L_v denotes the operator of Lie derivation with respect to the vector field v^h .When we refer to a projective vector field v^h, we always mean ρ_i, the associated covariant vector field given in (1.1). In particular, if ρ_i is zero, then a projective vector field is called an affine vector field.

In 1980, H. Hiramatu has obtained a series of integral formulas and integral inequalities in a compact orientable Riemannian manifold assuming that scalar

Curvature of M as constant. In this paper using projective and the conformal curvature tensor field of type (1,3), we have obtained the series of integral formulas and integral inequalities on scalar curvature $\overset{\circ}{K}$ of M .we get necessary and sufficient conditions for Riemannian manifold to be isometric to a sphere of radius $\sqrt{\dfrac{n(n-1)}{\overset{\circ}{K}}}$.

2. Preliminaries

This section deals with preliminaries which are needed in the rest of the sections.
The following known results are used in this paper.(for details please see [1].)

(2.1) $\overset{\circ}{\nabla}{}^j L_v g_{ih} = 2\overset{\circ}{\rho}{}^j g_{ih} + \overset{\circ}{\rho}_i \delta^j_h + \overset{\circ}{\rho}_h \delta^j_i$

(2.2) $\overset{\circ}{\nabla}{}^j L_v g^{ih} = -2\overset{\circ}{\rho}{}^j g^{ih} - \overset{\circ}{\rho}{}^i g^{jh} - \overset{\circ}{\rho}{}^h g^{ji}$.

(2.3) $\overset{\circ}{G}_{ji} = \overset{\circ}{G}_{ij}$, $\overset{\circ}{G}_{ji} g^{ji} = 0$, $\overset{\circ}{Z}_{tji}{}^t = \overset{\circ}{G}_{ji}$.

Where Einstein's deviation tensor $\overset{\circ}{G}_{ji}$ of type (0,2) and the tensor $\overset{\circ}{Z}_{kji}{}^h$ are given by (see [2])

(2.4) $\overset{\circ}{P}_{kji}{}^h = -\overset{\circ}{P}_{jki}{}^h$

Where $\overset{\circ}{P}_{kji}{}^h$ are the components of the projective curvature tensor field of type (1,3) given by,

(2.5) $\overset{\circ}{P}_{kji}{}^h = \overset{\circ}{K}_{kji}{}^h - \frac{\overset{\circ}{K}}{n(n-1)}(\delta^h_k g_{ji} - \delta^h_j g_{ki})$

(2.6) $\overset{\circ}{P}_{kjih} g^{ji} = \frac{n}{n-1}\overset{\circ}{G}_{kh}$,

Where $\overset{\circ}{P}_{kjih} = \overset{\circ}{P}_{kji}{}^t g_{th}$.

(2.7) $\overset{\circ}{C}_{kjih} = -\overset{\circ}{C}_{jkih}$, $\overset{\circ}{C}_{kjih} = -\overset{\circ}{C}_{ihkj}$

(2.8) $\overset{\circ}{C}_{tji}{}^t = 0$, $\overset{\circ}{C}_{kjt}{}^t = 0$, $\overset{\circ}{C}_{kji}{}^h g^{ji} = 0$

(2.9) $\overset{\circ}{C}_{kji}{}^h = \overset{\circ}{K}_{kji}{}^h + \delta^h_k \overset{\circ}{C}_{ji} - \delta^h_j \overset{\circ}{C}_i{}^k + \overset{\circ}{C}_k{}^h g_{ji} - \overset{\circ}{C}_j{}^h g_{ki}$

Where $\overset{\circ}{C}_{kji}{}^h$ are the components of conformal curvature tensor field of type (1,3).

(2.10) $w^h = \frac{n-1}{2}\overset{\circ}{\rho}{}^h + \frac{\overset{\circ}{K}}{n} v^h$

(2.11) $L_v \overset{\circ}{Z}_{kji}{}^h = \frac{1}{n-1}\delta^h_k L_v \overset{\circ}{G}_{ji} - \frac{1}{n-1}\delta^h_j L_v \overset{\circ}{G}_{ki})$

(2.12) $L_v \overset{\circ}{P}_{kji}{}^h = 0$

(2.13) $\overset{\circ}{C}_{kji}{}^h = \overset{\circ}{Z}_{kji}{}^h - \frac{1}{n-2}(\delta^h_k \overset{\circ}{G}_{ji} - \delta^h_j \overset{\circ}{G}_{ki} + \overset{\circ}{G}_k{}^h g_{ji} - \overset{\circ}{G}_j{}^h g_{ki})$

Where $\overset{\circ}{C}_{kji}{}^h$ is conformal curvature tensor field of type (1,3) for $n > 2$.

(2.14) $L_v \overset{\circ}{C}_{kji}{}^h = -\frac{1}{(n-1)(n-2)}(\delta^h_k L_v \overset{\circ}{G}_{ji} - \delta^h_j L_v \overset{\circ}{G}_{ki})$

$$-\frac{1}{n-2}\{(L_v G_k{}^h)g_{ji} + G_k{}^h L_v g_{ji} - (L_v G_j{}^h)g_{ki} - G_j{}^h L_v g_{ki}\}$$

(2.15) $$(L_v \overset{\circ}{G}_{ji})g^{ji} = \frac{n-1}{n}(L_v \overset{\circ}{P}_{kjihi})g^{kh}g^{ji}$$

(2.16) $$\overset{\circ}{\nabla}{}^k \overset{\circ}{G}_k{}^h = 0$$

We need the following known Lemmas which are used in rest of the sections.

LEMMA A [3]: If complete and simply connected Riemannian manifold M with positive constant scalar curvature K of dimension n.>1 admits a non affine projective vector field v^h and if the vector field w^h is a killing vector field then M is isometric to a sphere of radius $\sqrt{\frac{n(n-1)}{\overset{\circ}{K}}}$ in the Euclidean (n+1) space.

LEMMA B [3]: For Projective vector field v^h on a compact orientable Riemannian manifold M of dimension n>1, we have

(2.17) $$\int_M \left(\overset{\circ}{\nabla}_t w^t\right)^2 dV = \frac{n-1}{4(n+1)}\int_M L_v\left[\Delta\left\{(L_v \overset{\circ}{G}_{ji})g^{ji}\right\}\right.$$

$$\left.+\frac{2(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{G}_{ji})g^{ji}\right] dV$$

LEMMA C [3]: For Projective vector field v^h on a compact orientable Riemannian manifold M of manifold n>1, we have

(2.18) $$\int_M \left(\overset{\circ}{\nabla}{}^j L_v \overset{\circ}{G}_{ji}\right) w^i dV + \frac{1}{n}\int_M L_w L_v \overset{\circ}{K} dV = \frac{1}{2}\int_M \left(\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j\right)\left(\overset{\circ}{\nabla}{}^j w^i + \overset{\circ}{\nabla}{}^i w^j\right) dV$$

LEMMA D [3]: For Projective vector field v^h on a compact orientable Riemannian manifold M of dimension n>1, we have

(2.19) $$\int_M \overset{\circ}{G}_{ji}\, \overset{\circ}{\rho}{}^j w^i dV - \frac{1}{2(n+1)}\int_M L_v\left[\Delta\left\{(L_v \overset{\circ}{G}_{ji})g^{ji}\right\}\right.$$
$$\left.+\frac{2(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{G}_{ji})g^{ji}\right] dV$$
$$= \frac{-1}{2(n-1)}\int_M \left(\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j\right)\left(\overset{\circ}{\nabla}{}^j w^i + \overset{\circ}{\nabla}{}^i w^j\right) dV$$

LEMMA E [3]: For Projective vector field v^h on a compact orientable Riemannian manifold M without of dimension n>1, we have

$$\int_M g^{kj}[L_v \overset{\circ}{\nabla}_k \overset{\circ}{G}_{ji}]w^i dV + \frac{1}{n}\int_M [w,v]\overset{\circ}{K} dV + \frac{n-4}{n-1}\int_M L_v L_w dV$$

(2.20)
$$+\frac{3}{2(n+1)}\int_M L_v\left[\Delta\left\{(L_v \overset{\circ}{G}_{ji})g^{ji}\right\} + \frac{(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{G}_{ji})g^{ji}\right.$$

dv

$$= \frac{n+2}{2(n-1)}\int_M \left(\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j\right)\left(\overset{\circ}{\nabla}^j w^i + \overset{\circ}{\nabla}^i w^j\right) dV$$

where [,] is the lie bracket.

3. Lemmas

In this section we prove series of Lemmas on the scalar curvature K of M which are needed to establish main theorems in the section 4.

LEMMA 3.1: For a projective vector field v^h on a compact orientable Riemannian manifold M of dimension n>1, we have

$$(3.1)\ \int_M (\overset{\circ}{\nabla}^k L_v \overset{\circ}{Z}_{kji}{}^h)g^{ji}w_h dV - \frac{1}{4(n+1)}\int_M L_v\left[\Delta\left\{(L_v \overset{\circ}{Z}_{kijh})g^{kh}g^{ji}\right\}\right.$$

$$\left. + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{Z}_{kijh})g^{kh}g^{ji}\right] dV$$

$$= \frac{-1}{2(n-1)}\int_M \left(\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j\right)\left(\overset{\circ}{\nabla}^j w^i + \overset{\circ}{\nabla}^i w^j\right) dV$$

Proof. From (2.11), it can be proved that

$$L_v \overset{\circ}{Z}_{kji}{}^h = \frac{1}{n-1}\delta_k^h L_v \overset{\circ}{G}_{ji} - \frac{1}{n-1}\delta_j^h L_v \overset{\circ}{G}_{ki}$$

Consider,

(3.2)
$$\nabla^k L_v \overset{\circ}{Z}_{kji}{}^h = \frac{1}{n-1}\left[\delta_k^h(\overset{\circ}{\nabla}^k L_v \overset{\circ}{G}_{ji}) - \delta_j^h(\overset{\circ}{\nabla}^k L_v \overset{\circ}{G}_{ki})\right]$$

$$= \frac{1}{n-1}\left[\delta_k^h(\overset{\circ}{\nabla}^k L_v \overset{\circ}{G}_{ji}) - \delta_j^h(\overset{\circ}{\nabla}^k L_v \overset{\circ}{G}_{ki})\right]$$

From (3.2) and after lengthy simplification, we get

(3.3) $(\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{Z}{}_{kji}{}^h) g^{ji} w_h = \frac{2}{n-1}(\overset{\circ}{\nabla}_t w^t)^2 - \frac{1}{n-1}(\overset{\circ}{\nabla}{}^j L_v \overset{\circ}{G}_{ji}) w^i$

Integrating (3.3) over M, we get

(3.4) $$\int_M (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{Z}{}_{kji}{}^h) g^{ji} w_h \, dV = \frac{2}{(n-1)} \int_M (\overset{\circ}{\nabla}_t w^t)^2 dV$$

$$- \frac{1}{(n-1)} \int_M (\overset{\circ}{\nabla}{}^j L_v \overset{\circ}{G}_{ji}) w^i \, dV$$

$$- \frac{1}{(n-1)} \int_M (\overset{\circ}{\nabla}{}^j L_v \overset{\circ}{G}_{ji}) w^i \, dV$$

Now using (2.17) of Lemma B[3] and (2.18) of Lemma C[1] in (3.4) and after simplification we get (3.1). This completes the proof of Lemma.

LEMMA 3.2: For a projective vector field v^h on a compact orientable Riemannian manifold M of dimension n>1, we have

(3.6) $$\int_M (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{Z}_{kjih}) g^{ji} w^h dV - \frac{1}{(n+1)} \int_M L_v \left[\Delta \left\{ (L_v \overset{\circ}{Z}_{kjih}) g^{kh} g^{ji} \right\} + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)} (L_v \overset{\circ}{Z}_{kjih}) g^{kh} g^{ji} \right.$$

$$= \frac{-2}{(n-1)} \int_M \left(\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j \right) \left(\overset{\circ}{\nabla}{}^j w^i + \overset{\circ}{\nabla}{}^i w^j \right) dV$$

Proof. Consider,

$$(\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{Z}_{kjih}) = \{\overset{\circ}{\nabla}{}^k L_v (\overset{\circ}{Z}{}_{kji}{}^t g_{th})\}$$

$$= \{\overset{\circ}{\nabla}{}^k [(L_v \overset{\circ}{Z}{}_{kji}{}^t) g_{th} + \overset{\circ}{Z}{}_{kji}{}^t (L_v g_{th})]\}$$

Now $$(\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{Z}_{kjih}) g^{ji} w^h = \{\overset{\circ}{\nabla}{}^k [L_v \overset{\circ}{Z}{}_{kji}{}^t] g_{th} + \overset{\circ}{Z}{}_{kji}{}^t (L_v g_{th})]\} g^{ji} w^h$$

$$= (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{Z}{}_{kji}{}^t) g^{ji} w_t + \overset{\circ}{G}{}_k{}^t (\overset{\circ}{\nabla}{}^k L_v g_{th}) w^h \}$$

Where $\overset{\circ}{G}{}^{t}_{k} = \overset{\circ}{Z}{}_{kji}{}^{t} g^{ji}$

From (2.1) and after lengthy simplification, we get

$$(3.7)\quad (\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{Z}_{kjih})g^{ji}w^{h} = (\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{Z}{}_{kji}{}^{h})g^{ji}w_{h} + 3\overset{\circ}{G}_{ji}\rho^{i}w^{i}$$

Integrating (3.11) over M, we get

$$(3.8)\quad \int_M (\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{Z}_{kjih})g^{ji}w^{h}dV = \int_M (\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{Z}{}_{kji}{}^{h})g^{ji}w_{h}\, dV + 3\int_M \overset{\circ}{G}_{ji}\rho^{i}w^{i}dV$$

Using (2.19) of Lemma D [3] and (3.1) of Lemma 3.2 in (3.8) we get (3.6). This completes the proof of Lemma.

LEMMA 3.3: For a projective vector field v^h on a compact orientable Riemannian manifold M of dimension n>1, we have

(3.9)

$$\int_M (\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{P}_{kjih})g^{ji}w^{h}dV - \frac{3}{2(n+1)}\int_M L_v\left[\Delta\left\{(L_v \overset{\circ}{P}_{kjih})g^{kh}g^{ji}\right\}\right.$$

$$\left. + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{P}_{kjih})g^{kh}g^{ji}\right]$$

$$= \frac{-3}{2}\frac{n}{(n-1)^2}\int_M \left(\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j\right)\left(\overset{\circ}{\nabla}{}^{j} w^{i} + \overset{\circ}{\nabla}{}^{i} w^{j}\right) dV$$

Proof. Consider,

$$(\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{P}_{kjih})g^{ji}w^{h} = \{\overset{\circ}{\nabla}{}^{k} L_v(\overset{\circ}{P}{}_{kji}{}^{t} g_{th})\}g^{ji}w^{h}$$

$$= \{\overset{\circ}{\nabla}{}^{k}[L_v \overset{\circ}{P}{}_{kji}{}^{t} g_{th} + \overset{\circ}{P}{}_{kji}{}^{t}(L_v g_{th})]\}g^{ji}w^{h}$$

From (2.1), (2.6), (2.12) and after lengthy simplification, we get

$$(3.10)\quad (\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{P}_{kjih})g^{ji}w^{h} = \frac{3n}{n-1}\overset{\circ}{G}_{ji}\rho^{j}w^{i}$$

Integrating (3.10) over M, we get

$$(3.11)\quad \int_M (\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{P}_{kjih})g^{ji}w^{h}dV = \frac{3n}{n-1}\int_M \overset{\circ}{G}_{ji}\rho^{i}w^{i}dV$$

Using (2.19) of Lemma D[3] in (3.11) we get (3.9). This completes the proof of Lemma.

LEMMA 3.4: For a projective vector field v^h on a compact orientable Riemannian manifold M of dimension n>1, we have (3.12)

$$\int_M g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{P}{}_{kji}{}^h) g^{ji} w_h dV + \frac{1}{2(n+1)} \int_M L_v \left[\Delta \left\{ (L_v \overset{\circ}{P}_{kjih}) g^{kh} g^{ji} \right\} + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)} (L_v \overset{\circ}{P}_{kjih}) g^{kh} g^{ji} \right] dV$$

$$= \frac{n}{2(n-1)^2} \int_M \left(\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j \right) \left(\overset{\circ}{\nabla}{}^j w^i + \overset{\circ}{\nabla}{}^i w^j \right) dV$$

Proof. Consider,

$$g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{P}{}_{kji}{}^h) g^{ji} w_h = (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{P}{}_{kji}{}^h) g^{ji} w_h - g^{lk}(L_v \{ {}_l{}^t{}_k \}) g^{ji} w_h \overset{\circ}{P}{}_{tji}{}^h - g^{lk}(L_v \{ {}_l{}^t{}_j \}) \overset{\circ}{P}{}_{kti}{}^h g^{ji} w_h$$

$$- g^{lk}(L_v \{ {}_l{}^t{}_i \}) \overset{\circ}{P}{}_{kjt}{}^h g^{ji} w_h + g^{lk}(L_v \{ {}_l{}^h{}_t \}) \overset{\circ}{P}{}_{kji}{}^t g^{ji} w_h$$

From (1.1)

$$= g^{lk} g^{ji} w_h \Big\{ -(\delta_l^t \overset{\circ}{\rho}_k + \delta_k^t \overset{\circ}{\rho}_l) \overset{\circ}{P}{}_{tji}{}^h - (\delta_l^t \overset{\circ}{\rho}_j + \delta_j^t \overset{\circ}{\rho}_l) \overset{\circ}{P}{}_{kti}{}^h$$

$$- (\delta_l^t \overset{\circ}{\rho}_i + \delta_i^t \overset{\circ}{\rho}_l) \overset{\circ}{P}{}_{kjt}{}^h + (\delta_l^h \overset{\circ}{\rho}_t + \delta_t^h \overset{\circ}{\rho}_l) \overset{\circ}{P}{}_{kji}{}^t \Big\}$$

$$= -\overset{\circ}{\rho}{}^l w^m g^{ji} \overset{\circ}{P}_{ljim} + \overset{\circ}{\rho}{}^i w^m g^{lk} \overset{\circ}{P}_{lkim} + \overset{\circ}{\rho}{}^j w^m g^{lk} \overset{\circ}{P}_{jlkm}$$
$$+ \overset{\circ}{\rho}{}^m w^k g^{ji} \overset{\circ}{P}_{kjim} - 2\overset{\circ}{\rho}{}^k w^m g^{ji} \overset{\circ}{P}_{kjim}$$

After lengthy simplification and from (2.6), we get

(3.13) $$g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{P}{}_{kji}{}^h) g^{ji} w_h = -\frac{n}{n-1} \overset{\circ}{G}_{ji} \overset{\circ}{\rho}{}^j w^i$$

Integrating (3.13) over M, we get

(3.14) $$\int_M g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{P}{}_{kji}{}^h) g^{ji} w_h dV = -\frac{n}{n-1} \int_M \overset{\circ}{G}_{ji} \overset{\circ}{\rho}{}^j w^i dV$$

Using (2.19) of Lemma D [3] in (3.14), we get (3.12).This completes the proof of a Lemma.

LEMMA 3.5: For a projective vector field v^h on a compact orientable Riemannian manifold M of dimension n>2, we have

(3.15)
$$\int_M (\overset{\circ}{\nabla}{}^K L_v \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h dV + \frac{n-3}{(n-2)(n+1)} \int_M L_v [\Delta \left\{ \left(L_v \overset{\circ}{G}_{ji} \right) g^{ji} \right\} + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)} \left(L_v \overset{\circ}{G}_{ji} \right) g^{ji}$$
$$- \frac{(n^2-6n+2)}{n(n-1)(n-2)} \int_M [w,v] \overset{\circ}{K} dV$$

$$= \frac{-1}{2} \frac{(n^2-n-4)}{(n-1)(n-2)} \int_M (\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j)(\overset{\circ}{\nabla}{}^j w^i + \overset{\circ}{\nabla}{}^i w^j) dV$$

Proof. From (2.14), we have

(3.16)
$$L_v \overset{\circ}{C}{}^h_{kji} = -\frac{1}{(n-1)(n-2)} (\delta^h_k L_v \overset{\circ}{G}_{ji} - \delta^h_j L_v \overset{\circ}{G}_{ki})$$
$$- \frac{1}{n-2} \left\{ (L_v \overset{\circ}{G}{}^h_k) g_{ji} + \overset{\circ}{G}{}^h_k (L_v g_{ji}) - (L_v \overset{\circ}{G}{}^h_j) g_{ki} - \overset{\circ}{G}{}^h_j (L_v g_{ki}) \right\}$$

Applying covariant differentiation on both sides of (3.16), we get

(3.17)
$$(\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{C}{}^h_{kji}) = \frac{-1}{(n-1)(n-2)} \left(\delta^h_k (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{G}_{ji}) - \delta^h_j (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{G}_{ki}) \right)$$

$$- \frac{1}{n-2} \left\{ (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{G}{}^h_k) g_{ji} + \frac{n-2}{2n} (L_v g_{ji}) + \overset{\circ}{G}{}^h_k (\overset{\circ}{\nabla}{}^k L_v g_{ji}) \right.$$

$$\left. -(\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{G}{}^h_j) g_{ki} - (\overset{\circ}{\nabla}{}^k \overset{\circ}{G}{}^h_j)(L_v g_{ki}) - \overset{\circ}{G}{}^h_j (\overset{\circ}{\nabla}{}^k L_v g_{ki}) \right\}$$

Integrating (3.17) over M and using (2.2), (2.4), (2.6), we get
(3.18)

$$\int_M (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h dV$$

$$= \frac{-2}{(n-1)(n-2)} \int_M (\overset{\circ}{\nabla}_t w^t)^2 dV + \frac{1}{(n-1)(n-2)} \int_M (\overset{\circ}{\nabla}{}^j L_v \overset{\circ}{G}_{ji}) w^i dV$$

$$-\frac{1}{n-2}\int_M (\overset{\circ}{\nabla}{}^k \overset{\circ}{G}{}^h_k)(L_v g_{ji}) g^{ji} w^h dV + \frac{1}{n-2}\int_M g^{kj}(L_v \overset{\circ}{\nabla}{}^k \overset{\circ}{G}_{ji}) w^i dV$$

$$-\frac{n-1}{n-2}\int_M \overset{\circ}{G}_{ji}\overset{\circ}{\rho}{}^j w^i dV - \frac{n-1}{n-2}\int_M (\overset{\circ}{\nabla}{}^j L_v \overset{\circ}{G}{}^i_j) w_i dV$$

Using (2.17) of Lemma B[3], (2.18) of Lemma C[3], (2.19) of Lemma D[3] in (3.18) and after lengthy simplification we get (3.15). This completes the proof of a Lemma.

LEMMA 3.6: For a projective vector field v^h on a compact orientable Riemannian manifold M of dimension n>2, we have

$$\int_M g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h dV - \frac{n-3}{(n-2)(n+1)}\int_M L_v\left[\Delta\left\{(L_v \overset{\circ}{G}_{ji}) g^{ji}\right\}\right.$$

$$(3.19)\quad + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{G}_{ji}) g^{ji} dV$$

$$-\frac{(n^2-6n+2)}{n(n-1)(n-2)}\int_M [w,v]\overset{\circ}{K} dV$$

$$= \frac{-(n^2-n-4)}{2(n-1)(n-2)}\int_M \left(\overset{\circ}{\nabla}_j w_i + \overset{\circ}{\nabla}_i w_j\right)\left(\overset{\circ}{\nabla}{}^j w^i + \overset{\circ}{\nabla}{}^i w^j\right) dV$$

Proof. Substituting (1.1) in following equation,

$$g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h = (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h - g^{lk}(L_v\{{}_l{}^t{}_k\})\overset{\circ}{C}{}^h_{tji} g^{ji} w_h$$

$$-g^{lk}(L_v\{{}_l{}^t{}_i\}) \overset{\circ}{C}{}^h_{kjt} g^{ji} w_h + g^{lk}(L_v\{{}_l{}^h{}_t\}) \overset{\circ}{C}{}^t_{kji} g^{ji} w_h$$

We get,

$$(3.20)\quad g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h = (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h - g^{lk}\left\{(\delta^t_l \overset{\circ}{\rho}_k + \delta^t_k \overset{\circ}{\rho}_l)\overset{\circ}{C}{}^h_{tji}\right.$$

$$\left. + (\delta^t_l \overset{\circ}{\rho}_j + \delta^t_j \overset{\circ}{\rho}_l)\overset{\circ}{C}{}^h_{kti} + (\delta^t_l \overset{\circ}{\rho}_i + \delta^t_i \overset{\circ}{\rho}_l)\overset{\circ}{C}{}^h_{kjt} - (\delta^h_l \overset{\circ}{\rho}_t + \delta^h_t \overset{\circ}{\rho}_l)\overset{\circ}{C}{}^t_{kji}\right\} g^{ji} w_h$$

After simplification and using (2.7) and (2.8), we get

$$(3.21)\quad g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h = \overset{\circ}{\nabla}{}^k L_v \overset{\circ}{C}{}^h_{kji}) g^{ji} w_h$$

Integrating (3.25) over M, we get

$$(3.22)\quad \int_M g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{C}_{kji}{}^h) g^{ji} w_h dV = \int_M (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{C}_{kji}{}^h) g^{ji} w_h dV$$

Using (3.14) of Lemma 3.5 we get (3.19). This completes the proof of Lemma.

4. Theorems

In this section we prove that series of integral inequalities without putting any restriction on the scalar curvature of a Riemannian manifold M and obtain the necessary and sufficient conditions for M to be isometric to a sphere.

THEOREM 4.1: Suppose that a compact, orientable Riemannian manifold M of dimension n>1 admits a projective vector field v^h. Then we have,

(4.1)

$$\int_M (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{Z}_{kji}{}^h) g^{ji} w_h dV - \frac{1}{4(n+1)} \int_M L_v \left[\Delta \left\{ (L_v \overset{\circ}{Z}_{kjih}) g^{kh} g^{ji} \right\} + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)} (L_v \overset{\circ}{Z}_{kjih}) g^{kh} g^{ji} \right] dV \le 0$$

Where w^h is defined by (2.10). Equality in (4.1) holds if w^h is a killing vector field.

Proof. Follows from Lemma A [3] and (3.1) of Lemma 3.1

If in the Theorem 4.1 $\overset{\circ}{Z}_{kji}{}^h = 0$ for n>2 then $\overset{\circ}{K}$ is necessarily a constant and consequently we have following corollary from Theorem 4.1.

COROLLARY 4.1: Suppose that a compact orientable and simply connected Riemannian manifold M of dimension n>2 admits a non-affine projective vector field v^h then $\overset{\circ}{Z}_{kji}{}^h = 0$ if and only if M is isometric to a sphere of Radius $\sqrt{\dfrac{n(n-1)}{\overset{\circ}{K}}}$ which is the corollary 4.1 due to H. Hiramatu [1] .

THEOREM 4.2: Suppose that a compact orientable Riemannian manifold M of dimension n>1 admits a projective vector field v^h, then we have

(4.2)

$$\int_M (\overset{\circ}{\nabla}{}^k L_v \overset{\circ}{P}_{kjih}) g^{ji} w^h dV - \frac{3}{2(n+1)} \int_M L_v \left[\Delta \left\{ (L_v \overset{\circ}{P}_{kjih}) g^{kh} g^{ji} \right\} + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)} (L_v \overset{\circ}{P}_{kjih}) g^{kh} g^{ji} \right] \le 0$$

Where w^h is defined by (2.10). Equality in (4.2) holds if if w^h is a Killing vector field

Proof. Follows from Lemma A [3] and (3.110) of Lemma 3.3

If $\overset{\circ}{P}_{kji}{}^{h}=0$ for n>2, that is M is projectively flat for n>2, then from (2.6), $\overset{\circ}{K}$ is necessarily a constant and consequently we have following corollary
from Therorem 4.2.

COROLLARY 4.3: Suppose that a compact orientable and simply connected Riemannian manifold M of dimension n>2 admits a non-affine projective vector field v^h. Then M is projectively flat if and only if M is isometric to a sphere of radius $\sqrt{\dfrac{n(n-1)}{\overset{\circ}{K}}}$, which is the Corollary 4.2 due to H. Hiramatu[1].

Since $\overset{\circ}{P}_{kji}{}^{h}=0$ for n=2, we have the following Corollary.

COROLLARY 4.4: Suppose that a compact orientable and simply connected Riemannian manifold M with constant scalar curvature $\overset{\circ}{K}$ of dimension n=2 admits a non affine projective vector field v^h then M is isometric to a sphere of radius $\sqrt{\dfrac{n(n-1)}{\overset{\circ}{K}}}$, which is the Corollary 4.3 page No.513 due to H. Hiramatu[1].

THEOREM 4.3: Suppose that a compact orientable Riemannian manifold M of dimension n>1 admits a projective vector field v^h. Then we have

$$(4.3)\quad \int_M g^{lk}(L_v \overset{\circ}{\nabla}_l \overset{\circ}{P}_{kji}{}^{h})g^{ji}w_h dV + \frac{1}{2(n+1)}\int_M L_v\left[\Delta\left\{(L_v \overset{\circ}{P}_{kjih})g^{kh}g^{ji}\right\}\right. \\ \left. + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{P}_{kjih})g^{kh}g^{ji} \geq 0\right.$$

Where w^h is defined by (2.10). Equality in (4.3) holds if w^h is a killing vector field.

Proof. Follows from Lemma A [3] and (3.18) of Lemma 3.5.

THEOREM 4.4: Suppose that a compact orientable Riemannian manifold M of dimension n>1 admits a projective vector field v^h. Then we have
(4.4)

$$\int_M (\overset{\circ}{\nabla}{}^{k} L_v \overset{\circ}{C}_{kji}{}^{h})g^{ji}w_h dV + \frac{n-3}{(n-2)(n+1)}\int_M L_v\left[\Delta\left\{(L_v \overset{\circ}{G}_{ji})g^{ji}\right\}\right. \\ \left. + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{G}_{ji})g^{ji} \leq 0\right.$$

Where w^h is defined by (2.10). Equality in (4.4) holds if w^h is a Killing vector field.

Proof. Follows from Lemma A [3] and (3.22) of Lemma 3.6

COROLLARY 4.5: Suppose that a compact orientable and simply connected Riemannian manifold M with constant scalar curvature $\overset{\circ}{K}$ of dimension n>3 admits a non affine projective vector field v^h then M is conformally flat and $(L_v G_{ji})g^{ji}=0$ if and only if M is isometric to a sphere of radius , $\sqrt{\frac{n(n-1)}{\overset{\circ}{K}}}$., which is the Corollary 4.4 due to H. Hiramatu[1].

Since $\overset{\circ}{C}_{kji}{}^{h}=0$ for n=3 , we have the following Corollary from Theorem 4.4

COROLLARY 4.6: Suppose that a compact orientable and simply connected

Riemannian manifold M with constant scalar curvature $\overset{\circ}{K}$ of dimension n=3 admits a

Non-affine projective vector field v^h. then $(L_v \overset{\circ}{G}_{ji})g^{ji}=0$ if and only if M is isometric to a sphere of radius $\sqrt{\frac{n(n-1)}{\overset{\circ}{K}}}$., which is the Corollary 4.5 page No.515 due to H. Hiramatu[1].

THEOREM 4.5: Suppose that a compact orientable Riemannian manifold M of dimension n>1 admits a projective vector field v^h. Then we have

$$\int_M g^{lk}(\overset{\circ}{\nabla}_l L_v \overset{\circ}{C}_{kji}{}^{h})g^{ji}w_h dV - \frac{n-3}{(n-2)(n+1)}\int_M L_v\left[\left\{(L_v \overset{\circ}{G}_{ji})g^{ji}\right\} + \frac{2(n+1)\overset{\circ}{K}}{n(n-1)}(L_v \overset{\circ}{G}_{ji})g^{ji}\right] dV - \frac{(n^2-6n+2)}{n(n-1)(n-2)}\int_M [w,v]\overset{\circ}{K}\, dV \leq 0 \tag{4.5}$$

Where w^h is defined by (2.10). Equality in (4.5) holds if w^h is a Killing vector field.

Proof. Follows from Lemma A [3] and (3.19) of Lemma 3.6

5. References

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