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Dynamics in a Class of Discrete Sir Epidemic Model

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Abstract:
In this paper, we propose a class of discrete SIR epidemic model. The dynamical behaviors of a discrete-time epidemic model are investigated. The equilibrium points are obtained and stability of the equilibrium points is analyzed. The phase portraits are obtained for different sets of parameter values. Numerical simulations are performed and they capture the rich dynamics of the discrete model.

Keywords: *SIR Model, reproductive number, difference equations*

1. Introduction

The first differential equation models of infectious disease dynamics go back as far as 1766 to the work of Daniel Bernoulli [4, 6]. The study of compartmental epidemic models with Modern differential equation models, in the context of single-epidemic outbreaks, began with the work of Kermack and McKendrick [5] and later expanded by Anderson and May [1,2].

2. Model Description

Discrete epidemic models are more suitable to understand disease transmission dynamic and to prepare eradication policies because they permit arbitrary time –step units, preserving the basic features of corresponding continuous time models. Furthermore, this allows better use of statistical data for numerical simulations due to the reason that the infection data are complied at discrete given time intervals [7]. We propose a system of non-linear difference equations which models the propagation of a disease in a constant population.

$$\begin{aligned} S(n+1) &= (1-a)S(n) - bS(n)I(n) + a \\ I(n+1) &= (1-a)I(n) + bS(n)I(n) - cI(n) \\ R(n+1) &= (1-a)R(n) + cI(n) \end{aligned} \qquad \ldots(1)$$

Where $S(0) > 0, I(0) > 0, R(0) > 0, a, b, c \succ 0$.

S (n), I (n) and R (n) denote the number of a population who are susceptible to a disease, infective members and recovered at time n, respectively [3]. The system (1) has the two equilibriums, a disease free equilibrium $E_0 = (1, 0, 0)$ and a unique endemic equilibrium

$$E_1 = \left(\frac{a+c}{b}, \frac{a(b-c-a)}{b(a+c)}, \frac{c(b-c-a)}{b(a+c)}\right)$$ Provided $b > a + c$.

3. Dynamic Behavior of the Model

The linearized matrix J of for the system (1) is

$$J(S,I,R)=\begin{pmatrix}1-a-bI & -bS & 0\\ bI & 1-a+bS-c & 0\\ 0 & c & 1-a\end{pmatrix} \qquad \ldots(2)$$

Trace $J(S,I,R)=3(1-a)+b(S-I)-c$ and

Det $J(S,I,R)=(1-a)(1-a-bI)(1-a+bS-c)+b^2(1-a)SI$. For the system (1), we have the following analysis. The following lemma is needed to discuss the stability of the equilibrium points of (2)

Lemma: Let

$$p(\lambda)=\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \qquad \ldots(3)$$

Be the characteristic equation for a matrix defined by (2). Then the following statements are true:

- If every root of equation (3) has absolute value less than one, then the equilibrium point of the system (1) is locally asymptotically stable and equilibrium point is called a *sink*.
- If at least one of the roots of equation (3) has absolute value greater than one, then the equilibrium point of the system (1) is unstable and equilibrium point is called a *saddle*.
- If every root of equation (3) has absolute value greater than one, then the equilibrium point of the system (1) is a *source*.
- The equilibrium point of system (1) is called *hyperbolic* if no root of equation (3) has absolute value equal to one. If there exists a root of equation (3) with absolute value equal to one, then the equilibrium point is called *non-hyperbolic*.

Using the lemma, we have the following propositions for the system (1).

Proposition 1: The disease free equilibrium point E_0 is a

- Sink if $b-c<a<2+b-c$ and $0<a<2$.
- Source if $b-c>a>2+b-c$ and $a>2$.
- Saddle if $b-c>a>2+b-c$ and $0<a<2$.
- Non-hyperbolic if either $a=b-c$ (or) $a=2+b-c$ (or) $a=2$.

Proof: From (2), linearized matrix for E_0 is given by

$$J(E_0)=\begin{pmatrix}1-a & -b & 0\\ 0 & 1-a+b-c & 0\\ 0 & c & 1-a\end{pmatrix}$$

The Eigen values of the matrix $J(E_0)$ are $\lambda_1=1-a$ and $\lambda_2=1-a+b-c$.

In view of Lemma, we see that, E_0 is a sink if $b-c<a<2+b-c$ and $0<a<2$; E_0 is a source if $b-c>a>2+b-c$ and $a>2$; E_0 is a saddle if $b-c>a>2+b-c$ and $0<a<2$; and also E_0 is non-hyperbolic if either $a=b-c$ (or) $a=2+b-c$ (or) $a=2$. Here $R_0=\dfrac{b}{a+c}$ is the basic reproduction number.

Proposition 2: The equilibrium point E_1 is a

- Sink if $a<b-c,\ a<\dfrac{b}{2(b^2-R_0)}$ and $0<a<2$.
- Source if. $a>b-c,\ a>\dfrac{b}{2(b^2-R_0)}$ and $a>2$
- Saddle if. $a>b-c,\ a>\dfrac{b}{2(b^2-R_0)}$ and $0<a<2$
- Non- hyperbolic if either $a=b-c,\ a=\dfrac{b}{2(b^2-R_0)}$ and $a=2$

Proof: From (2), linearized matrix for E_1 is given by

$$J(E_1)=\begin{pmatrix} \dfrac{a(1-b)+c}{\frac{b}{R_0}} & -\dfrac{b}{R_0} & 0 \\ \dfrac{a(b-c-a)}{\frac{b}{R_0}} & 1 & 0 \\ 0 & c & 1-a \end{pmatrix}$$

The Eigen values of the matrix $J(E_1)$ are

$$\lambda_1 = 1-a,\ \lambda_{2,3} = \frac{(1+x)\pm\sqrt{(1+x)^2-4(x+a(b-c-a))}}{2},\ where\ x=\frac{R_0(a(1-b)+c)}{b}$$

By using Lemma, it is easy to see that, E_1 is a sink if $0<a<2, a<b-c, a<\dfrac{b}{2(b^2-R_0)}$; E_1 is a source if $a>2, a>b-c\ and\ a>\dfrac{b}{2(b^2-R_0)}$; E_1 is a saddle if $0<a<2, a>b-c\ and\ a>\dfrac{b}{2(b^2-R_0)}$;and finally non-hyperbolic if either $a=2,\ or\ a=b-c\ or\ a=\dfrac{b}{2(b^2-R_0)}$. Here $a+c=\dfrac{b}{R_0}$

4. Numerical Simulations

In this section, we carry out numerical simulations to demonstrate our theoretical results and the complex dynamics of model. We present the time plots of S (n); I (n); R (n) for the system (1). Dynamic behaviors of the system (1) about the equilibrium points under different sets of parameter values. A key parameter in epidemiology is the basic reproductive ratio, R_0 which is fundamental and widely used in the study of epidemiological models. R_0 tells us about the initial rate of spread of the disease. R_0 often serves as a threshold parameter that predicts whether an infection dies out or keeps persistence in a population. The magnitude of R_0 plays a crucial role in determining the dynamical behavior of model (1).

- **Example 1:** We shall consider the values a = 0.23; b = 0.99; c = 0.12 see figure-2.

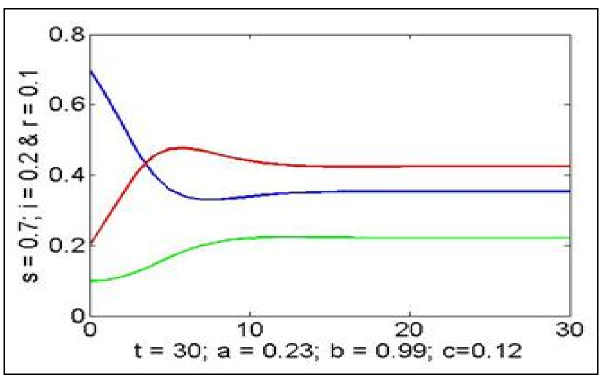

Figure 2: $R_0 \succ 1$

- **Example 2:** We shall consider the values a = 0.03; b = 0.09; c = 0.2 see figure-3.

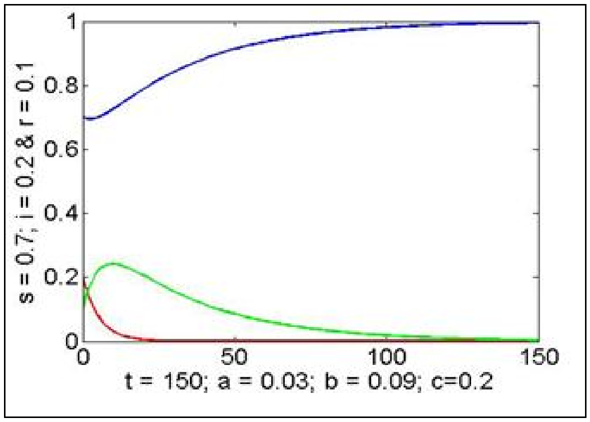

Figure 3: $R_0 \prec 1$

5. References

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