INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH & DEVELOPMENT

ISSN 2278 – 0211 (Online)

Numerical Solution of a Nonlinear Differential Equation Governing MHD Boundary Layer Flow of Viscous Incompressible Fluid Past a Stretcing Plate

Mohammed Abdullahi
Senior Lecturer, Department of Mathematics and Statistics, Faculty of Science, University of Maiduguri, Maiduguri, Nigeria
Abba V. Mandara
Lecturer, Department of Mathematics and Statistics, Faculty of Science, University of Maiduguri, Maiduguri, Nigeria
Shuaibu G. Ngulde
Senior Lecturer, Department of Mathematics and Statistics, Faculty of Science, University of Maiduguri, Maiduguri, Nigeria
Bassi I. G.
Senior Lecturer, Department of Mathematics, Federal University of Nasarawa, Nasarawa State, Nasarawa, Nigeria

Abstract:
In the present paper, the boundary layer flow of incompressible fluid over a stretching plate has been considered. Then, shooting method is used to convert boundary value problem into an initial value problem. The popular Runge-Kutta method is then employed to solve the problem that was converted to IVP. This is done because most real life problem depends on IVP.

Keywords: *Boundary layer equation, Stretching plate, heat transfer*

1. Introduction

The flow past a stretching plate is of great importance in many industrial applications such as polymer industry to draw plastic films and artificial fibers. In the process of drawing artificial fibers the polymer solution emerges from an orifice with a speed which increases from almost at the orifice up to a plateau value at which it remains constant. The moving fiber, which is of great technical importance governs the rate at which the fiber is coded and this in turn affects the final properties of t he yarn. Crane [iii] investigated boundary layer flow past a stretching sheet whose velocity is proportional to the distance from the slit. Carragher [iv] reconsider the problem of Crane [iii] to study heat transfer and calculated Nusselt number for the entire range of Prandtl number Pr.

2. Formation of the Problem

Two dimension flow of a viscous incompressible and electrically conducting fluid past a linear stretching plate under the transversely applied magnetic field has been considered. It is assumed that induced magnetic field is negligible in comparison to applied magnetic field. Let x-axis be along the moving plate and y-axis to be perpendicular to the direction of motion of the plate. If u and v are the magnetic components along these directions, respectively, then under the usual boundary layer approximations, MHD steady flow is governed by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2.1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2 u}{\rho} \qquad (2.2)$$

where u, v are velocity components in x and y directions respectively, ν is the kinematic viscosity.
The relevant boundary conditions are

$$y=0,\ u=mx,\ v=0 \qquad m>0 \qquad (2.3)$$
$$y\to\infty,\ u=0,\ \nu=-c$$

To solve this problem, we define the following dimensionless variables:

$$\bar{y}=\frac{y}{h},\ \bar{u}=\frac{uh}{\nu},\ \bar{x}=\frac{x}{h},\ \bar{v}=\frac{vh}{\nu}$$

Substituting all these dimensionless variables in equations (2.1) and (2.2), we have the following equations in dimensionless form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.4}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}-Mu \tag{2.5}$$

where $M=\frac{\sigma B_0^2 h^2}{\mu}$, the magnetic field parameter.

and boundary conditions are as follows

$y = 0,\ u = mx,\ v = 0 \qquad m > 0$ (2.6)

$y \to \infty,\ u = 0,\ v = -c$

where dash has been dropped for convenience

setting the similarity solution of the form

$$u = mxf'(\eta) \tag{2.7}$$

where $\eta = \frac{y}{y_\infty}$

Substituting u into the equations (2.4) and using the boundary condition (2.6), we have

$$v = -m f_\infty \{f(0) - f(\eta)\}$$
$$= -m\, y_\infty f(\eta) \tag{2.8}$$

Using u and v in Equation (2.5), we have

$$m\left(f'^2(\eta)-f(\eta)f''(\eta)\right)=\frac{1}{y_\infty}f'''(\eta)-Mf'(\eta) \tag{2.9}$$

with boundary conditions

$y = 0,\ f = 0,\ f' = 1$ (2.10)

$\eta = 1,\ f' \to 0$

Here $y_\infty > 1$ so by applying magnitude analysis, $\frac{1}{y_\infty} < 1$. Therefore, the term involving $\frac{1}{y_\infty}$ may be neglected. Thus, we have the following boundary value problem

$$m\left(f'^2(\eta)-f(\eta)f''(\eta)\right)+Mf'(\eta)=0$$

$\eta = 0,\ f = 0,\ f' = 1$ (2.11)

$\eta = 1,\ f' \to 0$

The nonlinear differential equation in boundary value problem (2.11) has singularity at $\eta = 0$. Therefore, it requires special attention. To overcome this difficulty, we solve the boundary value problem (2.9) through (2.10) by considering $m = \frac{1}{y_\infty} = M$. Now, the boundary value problem (2.9) through (2.10) reduces to

$$f'''(\eta)=f'^2(\eta)-f(\eta)f''(\eta)+f'(\eta) \tag{2.12}$$

$\eta = 0,\ f = 0,\ f' = 1$

$\eta \to 1,\ f' = 0,$

For the sake of numerical solution, we convert this nonlinear boundary value problem into its equivalent initial value problem by applying shooting method. The guess for $f''(0)$ by shooting method has been obtained by the formula

$$M_i = M_{i-2}+\frac{M_{i-1}-M_{i-2}}{f'(M_{i-1};\ 1)-f'(M_{i-2};\ 1)}\left(f'(1)-f'(M_{i-2};\ 1)\right)$$

and the results of this iterative scheme have to be listed in the following table:

i	M_i	Error
0	0	0
1	0.5	0.5
2	0	0
3	0	0
4	0	0
5	0	0

Table 1

where M_i are guesses for $M = 5f''(0) + 164f''(0) + 250$.

Thus, $f''(0) = -1.6027$ or -13.1973. Taking $f''(0) = -1.6.027$, we then follow the initial value problem equivalent to the boundary value problem (2.9) through 92.10):

$$f'''(\eta) - f'(\eta) - f(\eta)f''(\eta) + f'(\eta)$$

$$\eta = 0,\ f = 0,\ f' = 1,\ f'' = -1.6027 \tag{2.13}$$

This initial value problem is solved for $f(\eta)$, $f'(\eta)$ by Rubge-Kutta method of order four employing C+ computer programming in. Trends of $f(\eta)$ and $f'(\eta)$ ηhas been shown in Figure 1 and 2 respectively.

3. Heat Transfer

Under the usual boundary approximation, the heat transfer between stretching plate and the fluid is governed by

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} \tag{3.1}$$

with boundary conditions

$$y = 0,\ T = T_p$$

$$y \to \infty,\ T = T_\infty \tag{3.2}$$

Defining dimensionless temperature field

$$\theta = \frac{T - T_\infty}{T_p - T_\infty}$$

We have the equation (3.1) and boundary conditions (3.2) in dimensionless form as follows:

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{p_r}\frac{\partial^2 \theta}{\partial y^2} \tag{3.3}$$

where P_r is Prandtl number.

$y = 0,\ \theta = 1$

$y \to \infty,\ \theta = 0$

Using the transformation $\eta = \dfrac{y}{y_\infty}$ where y_∞ is the value of $y \to \infty$, we have,

$$\frac{d^2\theta}{d\eta^2} + y_\infty P_r f(\eta)\frac{d\theta}{d\eta} = 0$$

$$\eta = 0,\ \theta = 1 \tag{3.4}$$

$$\eta \to \infty,\ \theta \to 0$$

In the process of solving boundary value problem (3.4) numerically, we get the following tri - diagonal system of linear equations

$$\begin{bmatrix} -2 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_2 & -2 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_3 & -2 & \alpha_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_4 & -2 & \alpha_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_5 & -2 & \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_6 & -2 & \alpha_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_7 & -2 & \alpha_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_8 & -2 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_9 & -2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \theta_7 \\ \theta_8 \\ \theta_9 \end{bmatrix} = \begin{bmatrix} -\beta_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\alpha_i = 1+0.05y_\infty P_r f_i$ and $\beta_i = 1 - 0.05y_\infty P_r f_i$.

we reduce it into initial value problem. Let

$$f(y) = f(0) + yf'(0) + \frac{y^2}{2!}f''(0) + \frac{y^3}{3!}f'''(0) + \frac{y^4}{4!}f^{(iv)}(0) + \cdots$$

$$f(y) = 0 + y + \frac{y^2}{2!}f''(0) + \frac{y^3}{3!}(m(f'^2(0) - f(0)f''(0) +$$

$$+ \frac{y^4}{4!}\{m(f'(0)f''(0) - f(0)f'''(0) - f'(0)f''(0))\} + \cdots$$

$$= y + \frac{y^2}{2!}f''(0) + \frac{y^3}{3!}m + \frac{y^4}{4!}\{m(2f''(0) - f'(0)f''(0))\} + \cdots$$

$$= y + \frac{y^2}{2!}f''(0) + \frac{y^3}{3!}m + \frac{y^4}{4!}mf''(0) + \cdots$$

$$= \left(y + \frac{y^3}{3!}m\right) + \left(\frac{y^2}{2!} + \frac{y^4}{4!}m\right)f''(0) + \cdots$$

Therefore, when y = 1, we

$$f(1) = \left(1 + \frac{1}{6}m\right) + \left(\frac{1}{2} + \frac{1}{24}m\right)f''(0) + \cdots$$

Physically, m is stretching factor of plate, taking m = 1 for our convenience, then we have

$$f(1) = \frac{7}{6} + \frac{13}{24}f''(0) + \cdots$$

according to shooting method, let $m_0 = 0.6$ and $m_1 = 0.7$ be two guesses for $f''(0)$, the next approximation m_2 for $f''(0)$ is calculated applying the following iterative formula

$$m_i = m_{i-2} + \frac{m_{i-1} - m_{i-2}}{y(m_{i-1};\ x_1) - y(m_0;\ x_1)} \times (y(x_1) - y(m_{i-2};\ x_1)); \quad i = 2, 3, 4, \cdots$$

$$m_2 = 0.6 + \frac{0.7 - 0.6}{y(m_1;\ 1) - y(m_0;\ 1)} \times (y(1) - y(m_0;\ x_1))$$

where

$y(m_1 ; 1) = f(m_1; 1)$

$$= \frac{1}{6} + \frac{13}{24} \times 0.7 = 1.167 + 0.379 = 1.564$$

and

$y(m_0 ; 1) = f(m_0; 1)$

$$= \frac{1}{6} + \frac{13}{24} \times 0.6 = 1.167 + 0.325 = 1.492$$

$$m_2 = 0.6 + \frac{0.7 - 0.6}{1.564 - 1.492} \times (0 - 1.492)$$

$$= -2297$$

Simlarly,

$$m_3 = -1.2308$$

Hence, the guess for $f''(0)$ has been given in the following table by virtue of shooting method.

Iteration	**Approximation of f″**	**Error**
1 m_0	0.7	0.000
2 m_1	0.8	0.1
3 m_2	-1.9663	1.1663
4 m_3	-1.9708	0.0045
5 m_4	-1.9663	0.0045
6 m_5	-1.9663	0.0000

Table 2:Guess for f″(0)

Stretching factor m = 1 and magnetic field parameter M = 2 so that the differential equation (2.11) reduces to

$$f'''(y) = f'^2(y) - f(y)f''(y) + 2f'(y) \tag{2.13}$$

together with initial conditions

$y = 0,\ f = 0,\ f' = 1,\ f'' = -1.97$

To apply Runge-Kutta method, first we split the given initial value problem into the following three initial value problems of order one.

$$\frac{df}{d\eta} = f' = u(\eta, f, f', f'') \qquad \frac{du}{d\eta} = f'' = v(\eta, f, f', f'')$$

$$f(0) = f_0 = 0 \qquad u(0) = u_0 = f'(0) = 1$$

$$\frac{dv}{d\eta} = f''' = f'^2 - f''f + 2f' = w(\eta_0, f_0, f_0', f_0'')$$

$$v(0) = v_0 = f''(0) = -1.9663$$

Taking h = 0.1, we calculate the following:

$$k_1 = hu(\eta_0, f_0, f_0', f_0'') = hf_0' = 0.1 \qquad l_1 = hv(\eta_0, f_0, f_0', f_0'') = hf_0'' = -0.197 \qquad r_1 = hw(\eta_0, f_0, f_0', f_0'') = h(f_0'^2 - f_0''f_0 + 2f_0') = 0\text{-}3$$

$$k_2 = hu\left(\eta_0 + \tfrac{h}{2}, f_0 + \tfrac{k_1}{2}, f_0' + \tfrac{l_1}{2}, f_0'' + \tfrac{r_1}{2}\right) = h(f_0' + \tfrac{l_1}{2}) = 0.0902$$

$$l_2 = hv\left(\eta_0 + \tfrac{h}{2}, f_0 + \tfrac{k_1}{2}, f_0' + \tfrac{l_1}{2}, f'' + \tfrac{r_1}{2}\right) = h(f_0'' + \tfrac{r_1}{2}) = -0.182$$

$$r_2 = hw\left(\eta_0 + \tfrac{h}{2}, f_0 + \tfrac{k_1}{2}, f_0' + \tfrac{l_1}{2}, f'' + \tfrac{r_1}{2}\right)$$

$$= h\left(\left(f_0' + \tfrac{l_1}{2}\right)^2 - \left(f_0'' + \tfrac{r_1}{2}\right)\left(f_0 + \tfrac{k_1}{2}\right) + 2\left(f_0' + \tfrac{l_1}{2}\right)\right)$$

$$= 0.1(0.81270 - (-1.82)(0.05) + 1.803) = 0.27337$$

$$k_3 = hu\left(\eta_0 + \tfrac{h}{2}, f_0 + \tfrac{k_2}{2}, f_0' + \tfrac{l_2}{2}, f'' + \tfrac{r_2}{2}\right) = h(f_0' + \tfrac{l_2}{2}) = 0.0909$$

$$l_3 = hv\left(\eta_0 + \tfrac{h}{2}, f_0 + \tfrac{k_2}{2}, f_0' + \tfrac{l_2}{2}, f'' + \tfrac{r_2}{2}\right) = h(f_0'' + \tfrac{r_2}{2}) = -0.18296$$

$$r_3 = hw\left(\eta_0 + \tfrac{h}{2}, f_0 + \tfrac{k_2}{2}, f_0' + \tfrac{l_2}{2}, f_0'' + \tfrac{r_2}{2}\right)$$

$$= h\left(\left(f_0' + \tfrac{l_2}{2}\right)^2 - \left(f_0'' + \tfrac{r_2}{2}\right)\left(f_0 + \tfrac{k_2}{2}\right) + 2\left(f_0' + \tfrac{l_2}{2}\right)\right) = 0.25616$$

$$k_4 = hu\left(\eta_0 + h, f_0 + k_3, f_0' + l_3, f'' + r_3\right) = h(f_0' + l_3) = 0.081704$$

$$l_4 = hv\left(\eta_0 + h, f_0 + k_3, f_0' + l_3, f_0'' + r_3\right) = h(f_0'' + r_3) = -0.171384$$

$$r_4 = hw\left(\eta_0 + h, f_0 + k_3, f_0' + l_3, f'' + r_3\right)$$

$$= h((f_0' + l_3)^2 - (f_0'' + r_3)(f_0 + k_3) + 2(f_0' + l_3)) = 0.24574$$

$$f_1 = f_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \tfrac{1}{6}(1.0 + 2 \times 0.0902 + 2 \times 0.0909 + 0.081704)$$

$$= 0.24065$$

$$f_1' = f_0' + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 0 + \tfrac{1}{6}(-0.197 + 2(-0.182) + 2(-0.18296) - 0.171384)$$

$$= 1 + \tfrac{1}{6}(-1.0983)$$

$$= 0.81695$$

$$f_1'' = f_0'' + \tfrac{1}{6}(r_1 + 2r_2 + 2r_3 + r_4)$$

$$= -1.97 + \tfrac{1}{6}(0.03 + 2 \times 0.27337 + 2 \times 0.25616 + 0.24574)$$

$$= -1.97 + \tfrac{1}{6}(1.6048)$$

$$= -1.70253$$

By applying the same procedure, we solve the following initial value problem again to get the values at y = 0.2

$$f'''(y) = f'^2(y) - f(y)f''(y) + 2f'(y) \tag{2.15}$$

$$f(0.1) = 0.24065,\ f'(0.1) = 0.81695,\ f''(0.1) = -1.70253 \tag{2.16}$$

Finally, we list the solution at the points 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 in the following table:

η	f	f′	f″
0.0	0.0000	1.0000	-1.97
0.1	0.24065	0.81695	-1.70253
0.2	0.3.1554	0.65637	-1.44910
0.3	0.37554	0.52011	-1.24644
0.4	0.42260	0.40241	-1.08350
0.5	0.45850	0.29960	-1.94380
0.6	0.48470	0.20955	-1.84230
0.7	0.50230	0.12867	-0.78430
0.8	0.51210	0.0548	-0.7063
0.9	0.5171	-0.0138	-0.654
1.0	0.513	-0.881	-0.629

Table 3

Numerical solution of differential equation (2.10) when $u_0k = 0.2$

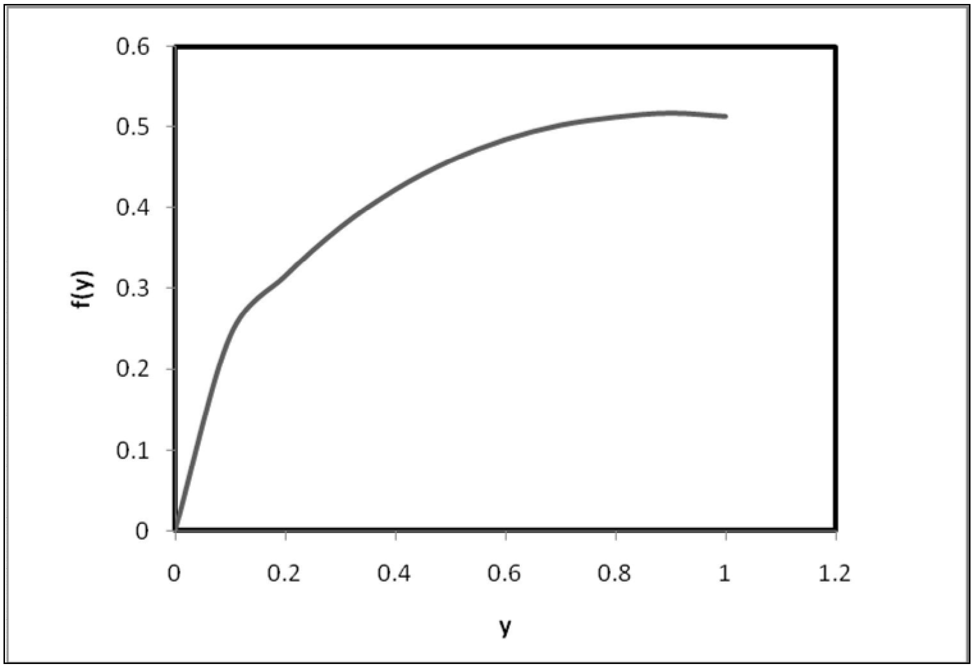

Figure 1: Velocity function f(y)

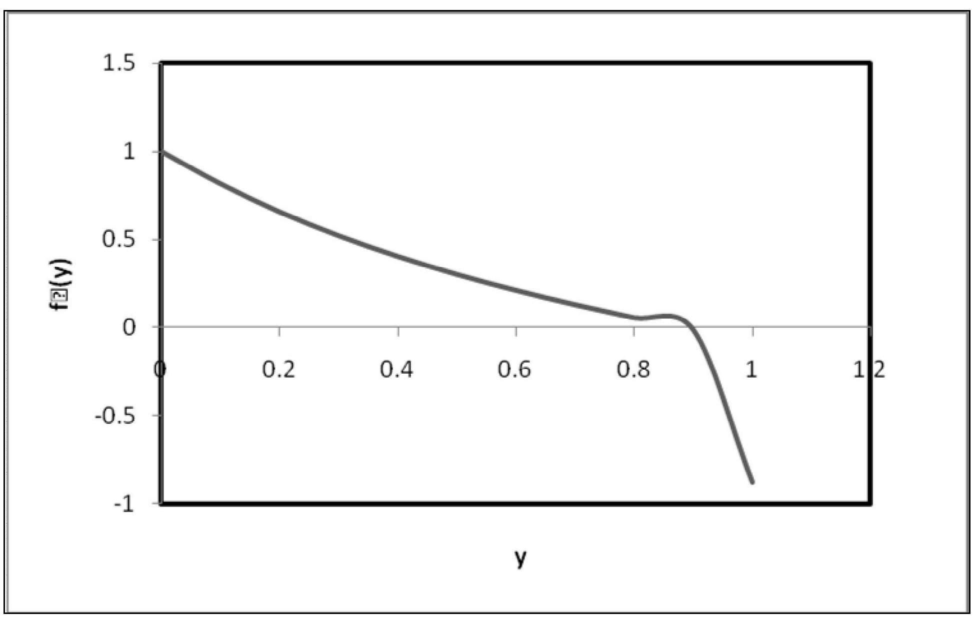

Figure 2: Graph of f′(y)

4. Discussion and Conclusions

In this paper, we solved a non linear boundary value problem numerically. This problem arises in the manufacturing of plastic film and sheet, and in paper making industries. This problem has been solved to get closed form solution in several variants, Naseem [v] and [vi].

First we convert this non linear boundary value problem into two point boundary initial value boundary value problem and then by application of fourth-order Runge-Kutta method, we obtained the solution. We observed that the problem has been solved easily. The horizontal and vertical components can be seen in Fig. 1 and Fig 2 respectively

5. References

i. Ian Jacques and Colin Judd, (1987) Numerical Analysis, Chapman and Hall Pub., London
ii. B. S. Grewel,(2001), Higher Engineering Mathematics, Khanna Pub., India.
iii. Crane L. J, (91970), Flow the horizontal Past a Stretching Plate, ZAMP, Vol. 21, pp645 – 647.
iv. Carragher, P.,(1978), Boundary Layer Flow and Heat Transfer for Stretching Plate, PhD Thesis, University of Dublin.
v. Naseem Ahmad, (1993), MHD Boundary Layer Flow of Viscoelastic fluid past a stretching plate and heat transfer, Asian J. of physics, Vol. 4, pp229 – 234.
vi. Naseem Ahmad, (1998), Hydro magnetic boundary layer flow past a stretching porous plate and heat transfer, IJEMS, Vol. 5, pp441 -444