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A New Approach on Orderings of Fuzzy Random Variables

D. Rajan
Associate Professor, Department of Mathematics,
Tranquebar Bishop Manickam Lutheran College, Porayar, India
C. Senthil Murugan
Assistant Professor, Department of Mathematics,
Swami Dayananda College of Arts and Science, Manjakkudi, India

Abstract:
In this paper, a new approach on the concepts of stochastic orderings, hazard rate orderings and likelihood ratio orderings of fuzzy random variables are investigated and various properties of these orderings are established.

Keywords: *Fuzzy random variables, fuzzy likelihood ratio order, fuzzy hazard rate order, fuzzy stochastic order*

1. Introduction

Fuzzy random variables can be used to describe the uncertain information when fuzziness and randomness are inseperably fused with each other. The concept of fuzzy random variable was first introduced by Kwakernaak[9]. Puri and Relescu[10] have also proposed the notion of a fuzzy random variables.

The theory of fuzzy random variables is a natural extension of that of general real valued random variables and random vectors. Owing to the topological structure of the space of closed sets and special features of set theoretic operations fuzzy random variables have many properties. This has given rise to new interpretations for the classical probability theory. As a result of the development in the study of fuzzy random variables in the past more four decades by many researchers, the theory of fuzzy random variables with many applications has become one new and active branches in probability theory.

Stochastic orderings and inequalities play an imperative role in the areas such as reliability, decision making and life testing problems. Such stochastic orderings offer new insights in the realization and application of random phenomenon. Earnest Lazarus Piriyakumar et al.,[3] have studied various stochastic orderings of fuzzy random variables. Stochastic orderings of fuzzy random variables are applicable to imprecise statistical data. If vagueness and dimness of perception is prevalent in capturing life times of components, which is a random phenomena, then the stochastic orderings of fuzzy random variables is the fittest tool to deal such situations.

The stochastic orderings of fuzzy random variables are associated with inequalities in terms of expectation of functions with respect to underlying distributions of fuzzy random variables.

In this paper, we provide the fuzzy analogue of stochastic orderings based on Kwakernaak's [9] fuzzy random variables. Now, we are introducing the α-cut concept in random variables so that the variables become the fuzzy random variables and verify that it satisfied the properties of stochastic ordering of fuzzy random variables. It is well known that the relationship between likelihood ratio order, hazard rate order and stochastic order is that likelihood ratio order is stronger than hazard rate order and the hazard rate order is stronger than the stochastic order. This relationship has also continued into fuzzy likelihood ratio order, fuzzy hazard rate order and fuzzy stochastic order of fuzzy random variables.

In this paper, we consider only continuous random variables with parameters mean μ and standard deviation σ. The range of the continuous random variables is $\left(X_\alpha^L = \mu - \sigma, X_\alpha^U = \mu + \sigma\right)$, so that the probability of this interval can be written as

$P(|X-\mu| < \sigma) = P\{(|X_\alpha^L - \mu| - \sigma \vee |X_\alpha^U - \mu| - \sigma) < 0\}$, here X_α^L and X_α^U may be $-\infty$ and ∞ respectively.

The organization of the paper is as follows. Section 2 is employed to briefly mention the α-cut concept in Kwakernaak's fuzzy random variables with parameters mean μ and standard deviation σ. Some definitions of fuzzy likelihood ratio ordering, fuzzy hazard rate ordering, fuzzy stochastic ordering of fuzzy random variables, fuzzy distribution function, related lemmas and theorems are given. To derive the fuzzy likelihood ratio ordering of fuzzy random variables by using α-cut concept in likelihood ratio order.

In section 3, we prove some propositions of fuzzy likelihood ratio order, fuzzy hazard rate order and fuzzy stochastic order.

2. Preliminaries

In this section, we provide the new approach of fuzzy stochastic order, fuzzy likelihood order and fuzzy hazard rate order and their properties of [5,6,7] related to Kwakernaak's fuzzy random variables based on [3,8].

Now, we consider every continuous random variables with parameters mean μ and standard deviation σ. In which, the probability of any continuous fuzzy random variables can be written as

$P(\mu-\sigma \le \times \le \mu + \sigma) = P(| X - \mu | < \sigma)$

That is, every fuzzy random variables has a certain continuous range and that each with an interval support which we denote by $\left(X_\alpha^L, X_\alpha^U\right)$. Here X_α^L and X_α^U may be $-\infty$ and ∞ respectively.

Now, to prove some properties of fuzzy random variables such as fuzzy stochastic order, fuzzy hazard rate order, fuzzy likelihood order etc. for two fuzzy random variables R and T with means μ_1, μ_2 and standard deviations σ_1 and σ_2 respectively with $(s \le R \le t)$ and $(u \le T \le v)$, Here $s = \mu_1-\sigma_1$, $t = \mu_2-\sigma_2$, $u = \mu_1 + \sigma_1$ and $v = \mu_2 + \sigma_2$.

2.1. Definition : 2.1

If R and T are random variables. Then R is less than or equal to T in likelihood ratio order if whenever $s \le t$ and $u \le v$ one has

$P\{t < R < v\}\ P\{s < T < u\} \le P\{s < R < u\}\ P\{t < T < v\}$

We will write $R \le^{LRO} T$ if this is the case.

2.2. Definition : 2.2

If R and T are two fuzzy random variables with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively. Then R is said to be the fuzzy likelihood ratio ordering less than or equal to T, if whenever $s \le t$ and $u \le v$, where

$\mu_1 - \sigma_1 = s \qquad t = \mu_2 - \sigma_2$

$\mu_1 + \sigma_1 = u \qquad v = \mu_2 + \sigma_2$

we get $P\{((|R_\alpha^L - \mu_2| - \sigma_2) \vee (|R_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

$P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$\le P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

we will write $R \le^{FLRO} T$.

2.3. Remark : 2.3

$R \le^{FLRO} T \Leftrightarrow$

$$0 \le \begin{vmatrix} P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\} & P\{((|R_\alpha^L - \mu_2| - \sigma_2) \vee (|R_\alpha^U - \mu_2| - \sigma_2)) < 0\} \\ P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\} & P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0 \end{vmatrix}$$

whenever $s \le t$ and $u \le v$

2.4. Note : 2.4

The derivation of fuzzy likelihood ratio order, using the α-cut concept in likelihood ratio order of two fuzzy random variables R and T with parameter means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively is as follows:

By definition of likelihood ratio order,

$R \le^{LRO} T$

$$\Leftrightarrow P\{t < R < v\}\ P\{s < T < u\} \le P\{s < R < u\}\ P\{t < T < v\} \qquad (2.1)$$

whenever $s \le t$ and $u \le v$.

2.5. Proof

Given $P\{t < R < v\}\ P\{s < T < u\} \le P\{s < R < u\}\ P\{t < T < v\}$.

Now, we derive the fuzzy likelihood ratio order, using the α-cut concept in equation (2.1),

We get

$$\int_s^u f(R)dR \cdot \int_t^v f(T)dT - \int_t^v f(R)dR \cdot \int_s^u f(T)dT \geq 0 \quad (2.2)$$

we know that

$P\{f(R)\} = P\{f(R_\alpha^L, R_\alpha^U)\}$

$P\{s < R < u\} = P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$P\{s < f(R) < u\} = P\{f((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$= P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < f^{-1}(0)\}$

$= P\{s((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

Similarly,

$P\{t < f(T) < v\} = P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

$P\{t < f(R) < v\} = P\{((|R_\alpha^L - \mu_2| - \sigma_2) \vee (|R_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

$P\{s < f(T) < u\} = P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

Using these notations in equation (2.2), we get

$$\begin{aligned}&[P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\}\\ &\quad P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\}\\ &- P\{((|R_\alpha^L - \mu_2| - \sigma_2) \vee (|R_\alpha^U - \mu_2| - \sigma_2)) < 0\}\\ &\quad P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}] \geq 0 \quad (2.3)\end{aligned}$$

This is the fuzzy likelihood ratio ordering.

From equation (2.2), we get

$$\begin{aligned}P\{s < R < u\} &= P\left[\int_s^u f(R)dR\right]\\ &= P\left[\int_s^t f(R)dR + \int_t^u f(R)dR\right]\\ &= P\left[\int_{\mu_1-\sigma_1}^{\mu_2-\sigma_2} f(R)dR + \int_{\mu_2-\sigma_2}^{\mu_1+\sigma_1} f(R)dR\right]\\ &= P\{\mu_1 - \sigma_1 < R < \mu_2 - \sigma_2\} + P\{\mu_2 - \sigma_2 < R < \mu_1 + \sigma_1\}\end{aligned}$$

$$= P\left\{\left(\left|R_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2} \vee \left|R_\alpha^U - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2}\right) < 0\right\}$$

$$+ P\left\{\left(\left|R_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2} \vee \left|R_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2}\right) < 0\right\} \quad (2.4)$$

similarly,

$$P\{t<T<v\} = p\left\{\left(\left|T_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2} \vee \left|T_\alpha^U - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2}\right) < 0\right\}$$

$$+P\left\{\left(\left|T_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2} \vee \left|T_\alpha^U - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2}\right) < 0\right\} \quad (2.5)$$

$$P\{t<R<v\}=P\left\{\left(\left|R_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2} \vee \left|R_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2}\right) < 0\right\}$$

$$+P\left\{\left(\left|R_\alpha^U - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2} \vee \left|R_\alpha^U - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2}\right) < 0\right\} \quad (2.6)$$

$$P\{s<T<u\} = P\left\{\left(\left|T_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2} \vee \left|T_\alpha^L - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2}\right) < 0\right\}$$

$$+P\left\{\left(\left|T_\alpha^U - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2} \vee \left|T_\alpha^U - \frac{(\mu_1+\mu_2)}{2}\right| - \frac{|\sigma_1-\sigma_2|}{2}\right) < 0\right\} \quad (2.7)$$

using equation (2.4), (2.5), (2.6) and (2.7) in equation (2.1), we get
$P\{s < R < u\}\ P\{t < T < v\} - P\{s < T < u\}\ P\{t < R < v\} \geq 0$.

We get,
$\{P\{(s < R < t)(t < T < u)\} + P\{(s < R < t)(u < T < v)\}$
$+ P\{(t < R < u)(t < T < u)\} + P\{(t < R < u)(u < T < v)\}$
$- P\{(t < R < u)(s < T < t)\} - P\{(t < R < u)(t < T < u)\}$
$- P\{(u < R < v)(s < T < t)\} - P\{(u < R < v)(t < T < u)\}\} \geq 0$
Adding these , we get
$P\{s < R < t\} + P\{u < T < v\} + P\{t < (R, T) < u\}$
$- P\{s < T < t\} - P\{u < R < v\} - P\{t < (R, T) < u\} \geq 0$
$\therefore\ P\{s < R < t\} + P\{u < T < v\} - P\{s < T < t\} - P\{u < R < v\} \geq 0$
$\Rightarrow P\{s < R < t\} + P\{u < T < v\}$
Using equation (2.4) and (2.5), we get

$$P\left\{\left(\left|R_\alpha^L - \frac{(\mu_1 + \mu_2)}{2}\right| - \frac{|\sigma_1 - \sigma_2|}{2} \vee \left|R_\alpha^U - \frac{(\mu_1 + \mu_2)}{2}\right| - \frac{|\sigma_1 - \sigma_2|}{2}\right) < 0\right\}$$

$$+P\left\{\left(\left|T_\alpha^L - \frac{(\mu_1 + \mu_2)}{2}\right| - \frac{|\sigma_1 - \sigma_2|}{2} \vee \left|T_\alpha^U - \frac{(\mu_1 + \mu_2)}{2}\right| - \frac{|\sigma_1 - \sigma_2|}{2}\right) < 0\right\}$$

Hence the proof.

2.6. Definition : 2.5

If $R >^{st} T$. Then the collection of $U_\rangle$ measurable random variables related to R and the collection of $U_\rangle$ measurable random variables related to T are amenable to the following conditions for $a \in R$, $u \in U_\Im$ and $v \in V_\Im$.
$P(U > a) \geq P(V > a)$

2.7. Definition : 2.6

Let R and T are fuzzy random variable with means μ_1, μ_2 and standard deviations σ_1 and σ_2 respectively. Then R is said to be fuzzy stochastic orderings of T (written as $R \leq^{FSO} T$) , if whenever $s \leq t$ and $u \leq v$
Where $t = \mu_2 - \sigma_2 \to -\infty$, $s = \mu_1 - \sigma_1 \to -\infty$
$\therefore\ P\{R < v\}\ P\{T < u\} \leq P\{R < u\}\ P\{T < v\}$

$$P\{(R_\alpha^L - \mu_2 - \sigma_2 \vee R_\alpha^U - \mu_2 - \sigma_2) < 0\}\ P\{(T_\alpha^L - \mu_1 - \sigma_1 \vee T_\alpha^U - \mu_1 - \sigma_1) < 0\}$$
$$\leq P\{(R_\alpha^L - \mu_1 - \sigma_1 \vee R_\alpha^U - \mu_1 - \sigma_1) < 0\}\ P\{(T_\alpha^L - \mu_2 - \sigma_2 \vee T_\alpha^U - \mu_2 - \sigma_2) < 0\}$$

This implies that $R \leq^{FSO} T$.

2.8. Definition : 2.7

Let R and T are random variables. Then R is said to be hazard rate order of T (written as $R \leq^{HRO} T$) if whenever $s \leq t$ and $u \leq v$
where $u \to \infty$, $v \to \infty$
$\therefore\ P\{t < R\}\ P\{s < T\} \leq P\{s < R\}\ P\{t < T\}$.

2.9. Definition : 2.8

Let R and T are two fuzzy random variables with means μ_1, μ_2 and standard deviations σ_1 and σ_2 respectively. Then R is said to be fuzzy hazard rate order of T (written as $R \leq^{FHRO} T$) if whenever $s \leq t$ and $u \leq v$
where $u = \mu_1 + \sigma_1 \to \infty$, $v = \mu_2 + \sigma_2 \to \infty$

$$P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}\ P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$\leq P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}\ P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2) > 0\}$$

This implies that $R \leq^{FHRO} T$.

2.10. Definition : 2.9

The fuzzy distribution function of the fuzzy random variable R is written with μ_1 and σ_1

$F(R) = \{R, \mu_{F(R)}\}$

where $\mu^{(r)}_{F(R)} = P\{((R^L_\alpha - \mu_1 - \sigma_1) \vee (R^U_\alpha - \mu_1 - \sigma_1)) \leq 0\}$ (2.8)

clearly

$\overline{F(R)} = \{R, \mu_{F(R)}\}$

where $\mu^{(r)}_{\overline{F(R)}} = P\{((R^L_\alpha - \mu_1 - \sigma_1) \vee (R^U_\alpha - \mu_1 - \sigma_1)) > 0\}$ (2.9)

$\therefore$ By definition of fuzzy stochastic ordering,

Equations (2.8) and (2.9) together imply $R \geq^{FSO} T$, if $\overline{F(R)} \supseteq \overline{F(T)}$.

2.11. Lemma : 2.10

If $R \geq^{FSO} T$ then $ER \supseteq ET$.

2.12. Proof :

We first assume that R and T are non-negative fuzzy random variables with $s \leq t$, $u \leq v$. (A fuzzy random variable R is called non-negative. If $R_w(r) = 0$, for $r < 0$ and all $w \in \Omega$).

$ER = (Re, \mu_{ER})$

where $\mu^{(r)}_{ER} = \sup_{R' \in R'_F\, ER' = r;\, r \in Re} \mu_R(R')$

where R'_F - set of originals

R' - original of R

$$= \sup_{R' \in R'_F : \int_0^\infty p\{(R^L_\alpha - a \vee R^U_\alpha - a) > 0\} da = r;\, r \in Re} \mu_R(R')$$

$$\geq \sup_{T' \in T'_F : \int_0^\infty p\{(T^L_\alpha - a \vee T^U_\alpha - a) > 0\} da = t;\, t \in Re} \mu_T(T')$$

$$= \sup_{T' \in T'_F : ET' = t;\, t \in Re} \mu_T(T')$$

$= \mu_{ET}(t)$; $t \in Re$ ($\because$ R and T are non-negative)

which implies that $ER \supseteq ET$.

More generally one can write any fuzzy random variable Z as the difference of two non-negative fuzzy random variable as follows

$(Z, \mu_z) = (Z^+ - Z^-, \mu_{Z^+ - Z^-})$

$[(Z^L_\alpha, \mu^L_z)(Z^U_\alpha, \mu^U_z)] = [(Z^{L+}_\alpha - Z^{L-}_\alpha, \mu_{Z^{L+}_\alpha - Z^{L-}_\alpha})(Z^{U+}_\alpha - Z^{U-}_\alpha, \mu_{Z^{U+}_\alpha - Z^{U-}_\alpha})]$

$\mu_{Z^{L+}_\alpha - Z^{L+}_\alpha}(x) = \max(0, \mu_{Z^{L+}}(x) - \mu_{Z^{L-}}(x)), x \in R$

$\mu_{Z^{U+}_\alpha - Z^{U-}_\alpha}(x) = \max(0, \mu_{Z^{U+1}}(x) - \mu_{Z^{U-}}(x)), x \in R$

we define

$$Z^{L+} = \begin{cases} Z^L & ; \text{if } Z^L_w(x) = 0, \text{ for } x < 0 \text{ and all } w \in \Omega \\ 0 & ; \text{if } Z^L_w(x) = \alpha, \alpha \in [0,1], x < 0 \text{ and } w \in \Omega \end{cases}$$

$$Z^{L-} = \begin{cases} 0 & ; \text{if } Z^L_w(x) = 0, \text{ for } x < 0 \text{ and all } w \in \Omega \\ -Z^L & ; \text{if } Z^L_w(x) = \alpha, \alpha \in [0,1], x < 0 \text{ and all } w \in \Omega \end{cases}$$

$$Z^{U+} = \begin{cases} Z^U & ; \text{if } Z^U_w(x) = 0, \text{ for } x < 0 \text{ and all } w \in \Omega \\ 0 & ; \text{if } Z^U_w(x) = \alpha, \alpha \in [0,1], x < 0 \text{ and } w \in \Omega \end{cases}$$

$$Z^{U-} = \begin{cases} 0 & ;\text{ if } Z_w^U(x) = 0, \text{ for } x < 0 \text{ and all } w \in \Omega \\ -Z^U & ;\text{ if } Z_w^U(x) = \alpha, \alpha \in [0,1], x < 0 \text{ and } w \in \Omega \end{cases}$$

Then $R \geq^{FSO} T$ implies that

$R^{L+} \geq^{FSO} T^{L+}$ and $R^{L-} \leq^{FSO} T^{L-}$

$R^{U+} \geq^{FSO} T^{U+}$ and $R^{U-} \leq^{FSO} T^{U-}$

Hence,

$$\mu_{E(R)}^{(x)} = \mu_{E(R^+ - R^-)}^{(x)}$$
$$= \mu_{ER^+ - ER^-}^{(x)} \geq \mu_{ET^+ - ET^-}^{(x)}$$
$$= \mu_{E(T^+ - T^-)}^{(x)} = \mu_{ET}^{(x)}$$

which implies that $ER \supseteq ET$

Hence the proof.

2.13. Theorem : 2.11

$R \geq^{FSO} T$ iff $Ef(R) \geq Ef(T)$ for all ↑ function (f) with μ_i and σ_i, i = 1, 2 and $a = \mu_1 - \sigma_1$.

2.14. Proof :

Suppose that $R \geq^{FSO} T$ and let f be ↑ function.

Let $f^{-1}(a) = \inf\{x : f(x) \geq a\}$ for $\mu \in (0,1]$ and $a \in R$.

$$P\{f((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$= P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > f^{-1}(0)\}$$
$$= P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$\geq P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}$$
$$= P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > f^{-1}(0)\}$$
$$= P\{f((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}$$

This implies that $f(R) \geq^{FSO} f(T)$ and by lemma

"If $R \geq^{FSO} T$. Then $ER \supseteq ET$"

$Ef(R) \supseteq Ef(T)$ for all ↑ functions f, for the convex part we suppose that $Ef(r) \supseteq Ef(t)$ for all ↑ function f, for any real a.

Let f_a denote the increasing function specified by

$$f_a(x) = \begin{cases} 1 & \text{if } x > a \\ 0 & \text{if } x \leq a \end{cases}$$

Then

$$Ef_a(r) = p\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$

and

$$Ef_a(t) = p\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$

By stipulation $R \geq^{FSO} T$.

Which completes the proof.

2.15 Note : 2.12

The constant (or number) in the fuzzy random variable can be written in the α-cut from (i.e.,) $C_\alpha^L = C_\alpha^U = C$ (a constant or number).

3. Some Propositions on FLRO, FHRO and FSO

In this section, we will prove some of the important propositions related to fuzzy likelihood ratio order, fuzzy hazard rate order and fuzzy stochastic order.

3.1. Proposition : 3.1

Suppose C is a number

(1) If $C \leq T$ then $C \leq^{FLR} T$

(2) If $T \leq C$ then $T \leq^{FLR} C$.

3.2. Proof

(1) If C ≤ T then $P\{((|C_\alpha^L - \mu_2| - \sigma_2) \vee (|C_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

$P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$\leq P\{((|C_\alpha^L - \mu_1| - \sigma_1) \vee (|C_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$$P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\} \quad (3.1)$$

whenever

$$P(s < c < u) = P\{((|C_\alpha^L - \mu_1| - \sigma_1) \vee (|C_\alpha^U - \mu_1| - \sigma_1)) < 0\} = 1. \quad (3.2)$$

If C ≤ s and u ≤ C then

$$P\{((|C_\alpha^L - \mu_1| - \sigma_1) \vee (|C_\alpha^U - \mu_1| - \sigma_1)) < 0\} = 0 \quad (3.3)$$

$$P(t < C < v) = P\{((|C_\alpha^L - \mu_2| - \sigma_2) \vee (|C_\alpha^U - \mu_2| - \sigma_2)) < 0\} = 1. \quad (3.4)$$

If C ≤ t and v ≤ C then

$$P\{((|C_\alpha^L - \mu_2| - \sigma_2) \vee (|C_\alpha^U - \mu_2| - \sigma_2)) < 0\} = 0. \quad (3.5)$$

from equation (3.4) in equation (3.1), we must show

$P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$\leq P\{((|C_\alpha^L - \mu_1| - \sigma_1) \vee (|C_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$$P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\} \quad (3.6)$$

from equation (3.5) in equation (3.1), we get

$0 \leq P\{((|C_\alpha^L - \mu_1| - \sigma_1) \vee (|C_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

If u ≤ C and C≤ s then

$$P(s < T < u) = P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\} = 0 \quad (3.7)$$

Using equation (3.7) in equation (3.6), we get

$0 \leq P\{((|C_\alpha^L - \mu_1| - \sigma_1) \vee (|C_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

which is true. Using equation (3.2) in equation (3.6), we get

$P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$$\leq P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\} \quad (3.8)$$

since $s = \mu_1 - \sigma_1 < C$ and $t = \mu_2 - \sigma_2 < C$

Given C ≤ T then equation (3.8) becomes

$P\{C < T < (\mu_1 + \sigma_1)\} \leq P\{C < T < (\mu_2 + \sigma_2)\}$

However, this is true

Since $(\mu_1 + \sigma_1) \leq (\mu_2 + \sigma_2)$

That is, u ≤ v

This implies that $C \leq^{FLRO} T$.

Similarly, we prove $T \leq^{FLRO} C$.

3.3. Proposition : 3.2

$R \leq^{FLR} T \Rightarrow R \leq^{FHRO} T$

3.4. Proof

Since $R \leq^{FLRO} T$

$\Rightarrow P\{((|R_\alpha^L - \mu_2| - \sigma_2) \vee (|R_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

$P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$\leq P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\}$.

Suppose, $(\mu_2 + \sigma_2) = v \rightarrow \infty$ and $(\mu_1 + \sigma_1) = u \rightarrow \infty$

Then

$P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}$

$P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$\leq P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}$

This implies that $R \leq^{FHR} T$.

3.5. Proposition : 3.3

$R \leq^{FLRO} T \quad \Rightarrow R \leq^{FSO} T$

3.6. Proof :

Since $R \leq^{FLRO} T \Rightarrow$

$P\{((|R_\alpha^L - \mu_2| - \sigma_2) \vee (|R_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

$P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$\leq P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

If we let

$s = (\mu_1 - \sigma_1) \to -\infty, \qquad t = (\mu_2 - \sigma_2) \to -\infty$

Then

$P\{((R_\alpha^L - \mu_2 - \sigma_2) \vee (R_\alpha^U - \mu_2 - \sigma_2)) < 0\}$

$P\{((T_\alpha^L - \mu_1 - \sigma_1) \vee (T_\alpha^U - \mu_1 - \sigma_1)) < 0\}$

$\leq P\{((R_\alpha^L - \mu_1 - \sigma_1) \vee (R_\alpha^U - \mu_1 - \sigma_1)) < 0\}$

$P\{((T_\alpha^L - \mu_2 - \sigma_2) \vee (T_\alpha^U - \mu_2 - \sigma_2)) < 0\}$

$\therefore R \leq^{FSO} T$.

3.7. Proposition : 3.4

For any C one has $R \leq^{FLRO} T \Leftrightarrow R + C \leq^{FLRO} T + C$.

3.8. Proof

$R \leq^{FLRO} T \Leftrightarrow$

$P\{((|R_\alpha^L - \mu_2| - \sigma_2) \vee (|R_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

$P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$\leq P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\}$

$P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\}$

when $s \leq t$ and $u \leq v$

$\Leftrightarrow [P\{((|R_\alpha^L - \mu_2| - \sigma_2) \vee (|R_\alpha^U - \mu_2| - \sigma_2)) < 0\} + C]$

$[P\{((|T_\alpha^L - \mu_1| - \sigma_1) \vee (|T_\alpha^U - \mu_1| - \sigma_1)) < 0\} + C]$

$\leq [P\{((|R_\alpha^L - \mu_1| - \sigma_1) \vee (|R_\alpha^U - \mu_1| - \sigma_1)) < 0\} + C]$

$[P\{((|T_\alpha^L - \mu_2| - \sigma_2) \vee (|T_\alpha^U - \mu_2| - \sigma_2)) < 0\} + C]$

$\Leftrightarrow P\{((|R_\alpha^L - (\mu_2 - C_\alpha^L)| - \sigma_2) \vee (|R_\alpha^U - (\mu_2 - C_\alpha^U)| - \sigma_2)) < 0\}$

$P\{((|T_\alpha^L - (\mu_1 - C_\alpha^L)| - \sigma_1) \vee (|T_\alpha^U - (\mu_1 - C_\alpha^U)| - \sigma_1)) < 0\}$

$\leq P\{((|R_\alpha^L - (\mu_1 - C_\alpha^L)| - \sigma_1) \vee (|R_\alpha^U - (\mu_1 - C_\alpha^U)| - \sigma_1)) < 0\}$

$P\{((|T_\alpha^L - (\mu_2 - C_\alpha^L)| - \sigma_2) \vee (|T_\alpha^U - (\mu_2 - C_\alpha^U)| - \sigma_2)) < 0\}$

$P\{((|(R + C)_\alpha^L - \mu_2 + C_\alpha^L| - \sigma_2) \vee (|(R + C)_\alpha^U - \mu_2 + C_\alpha^U| - \sigma_2)) < 0\}$

$P\{((|(T + C)_\alpha^L - \mu_1 + C_\alpha^L| - \sigma_1) \vee (|(T + C)_\alpha^L - \mu_1 + C_\alpha^U| - \sigma_1)) < 0\}$

$\leq p\{((|(R + C)_\alpha^L - \mu_1 + C_\alpha^L| - \sigma_1) \vee (|(R + C)_\alpha^U - \mu_1 + C_\alpha^U| - \sigma_1)) < 0\}$

$p\{((|(T + C)_\alpha^L - \mu_2 + C_\alpha^L| - \sigma_2) \vee (|(T + C)_\alpha^U - \mu_2 + C_\alpha^U| - \sigma_2)) < 0\}$

$\Leftrightarrow$ R + C $\leq^{FLRO}$ T + C.

3.9. Proposition : 3.5

If R $\leq^{FHRO}$ T Then R $\leq^{FSO}$ T

3.10. Proof

Since R $\leq^{FHRO}$ T

$$P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}$$
$$P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$\leq P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\} \qquad (3.9)$$

If we let s = $(\mu_1 - \sigma_1) \to -\infty$ then we get,

$$P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\} = 1$$
$$P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\} = 1 \qquad (3.10)$$

use equation (3.10) in equation (3.9), we get

$$\therefore P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}$$
$$\leq P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}$$

which implies that R $\leq^{FSO}$ T.

3.11. Proposition : 3.6

For any C one has R $\leq^{FHRO}$ T $\Leftrightarrow$ R + C $\leq^{FHRO}$ T + C

3.12. Proof

Since R $\leq^{FHRO}$ T

$$P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}$$
$$P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$\leq P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\} \text{ for } s \leq t.$$
$$\Leftrightarrow [P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\} + C]$$
$$[P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\} + C]$$
$$\leq [P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\} + C]$$
$$[P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\} + C]$$

$\Leftrightarrow$

$$P\{(((R+C)_\alpha^L - \mu_2 + \sigma_2 + C_\alpha^L) \vee ((R+C)_\alpha^U - \mu_2 + \sigma_2 + C_\alpha^U))) > 0\}$$
$$P\{(((T+C)_\alpha^L - \mu_1 + \sigma_1 + C_\alpha^L) \vee ((T+C)_\alpha^U - \mu_1 + \sigma_1 + C_\alpha^U))) > 0\}$$
$$\leq P\{(((R+C)_\alpha^L - \mu_1 + \sigma_1 + C_\alpha^L) \vee ((R+C)_\alpha^U - \mu_1 + \sigma_1 + C_\alpha^U))) > 0\}$$
$$P\{(((T+C)_\alpha^L - \mu_2 + \sigma_2 + C_\alpha^L) \vee ((T+C)_\alpha^U - \mu_2 + \sigma_2 + C_\alpha^U))) > 0\}$$

$\Leftrightarrow$ R + C $\leq^{FHRO}$ T + C

3.13. Proposition : 3.7

$$R \leq^{FHRO} T \Leftrightarrow \frac{p\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{p\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}}$$

non-decreasing for t < sup R.

3.14. Proof :

Since R $\leq^{FHRO}$ T

$P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}$

$P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$\leq P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}$

$\Leftrightarrow$ we get

$$\frac{P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}}{P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

$$\leq \frac{P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}} \qquad (3.11)$$

Here RHS is non-decreasing for $t < \sup R$.

Conversely, If RHS of equation (3.11) is non-decreasing for $t < \sup R$.

$\Leftrightarrow$

Then

$P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}$

$P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$\leq P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}$

for $s \leq t < \sup R$.

However, it automatically holds for $t \geq \sup R$.

since $p\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\} = 0$, for $t \geq \sup R$.

3.15. Proposition : 3.8

$R \leq^{FHRO} T$

$\Leftrightarrow$

(1) $$\frac{P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

$$\leq \frac{P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

for $s = \mu_1 - \sigma_1 < \min\{\sup R, \sup T\}$

(2) $$\frac{P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

$$\leq \frac{P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

3.16. Proof :

By definition of $R \leq^{FHRO} T$, statement (1) is exists and statement (2) also exists from statement (1). Now, we prove statement (2) implies $R \leq^{FHRO} T$.

Note that

$$\frac{P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

$$\leq \frac{P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

for $s < t \leq \min\{\sup R, \sup T\}$.

implies

$P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee R_\alpha^U - \mu_2 + \sigma_2)) > 0\}$

$\leq P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}$

for $s < t \le \min\{\sup R, \sup T\}$ (3.12)

If sup T < sup R. Then we can put $t = \mu_2 - \sigma_2 = \sup T$ in equation (3.12) to get
$P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\} = 0$

$\therefore P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$

$P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\} = 0$

Then we pick s so $P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\} > 0$
to get $P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\} = 0$
$\therefore$ we get, $t \ge \sup R$, which is a contradiction.
$\therefore \sup R \le \sup T$.
we get equation (3.12) holds for $s \le t \le \sup R$.
which implies $\dfrac{P\{((T_\alpha^L - \mu_2 + \sigma_2) \vee (T_\alpha^U - \mu_2 + \sigma_2)) > 0\}}{P\{((R_\alpha^L - \mu_2 + \sigma_2) \vee (R_\alpha^U - \mu_2 + \sigma_2)) > 0\}}$ is non-decreasing for
$t < \sup R$ (by using proposition 3.7)
Which implies $R \le^{FHRO} T$.

3.17. Proposition : 3.9

$R \le^{FHRO} T \Leftrightarrow$ for $s = \mu_1 - \sigma_1 < \min\{\sup R, \sup T\}$
and $t = (\mu_2 - \sigma_2) > 0$ one has

$$P\{(((R_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (R_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2))) > 0) / ((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$\le P\{(((T_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (T_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2))) > 0) / ((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$

$\Leftrightarrow$ for $s = (\mu_1 - \sigma_1) < \min\{\sup R, \sup T\}$ and
$t = (\mu_2 - \sigma_2) > 0$ one has

$$P\{(((R_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (R_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2))) \ge 0) / ((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) \ge 0\}$$
$$\le P\{(((T_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (T_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2))) \ge 0) / ((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) \ge 0\}$$

3.18. Proof :

Let $s = (\mu_1 - \sigma_1) < \sup R$, $t = (\mu_2 - \sigma_2) > 0$

$$\Rightarrow P\{(((R_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (R_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2))) > 0 / ((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$= \frac{P\{((R_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (R_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) > 0\}}{P\{((R_\alpha^L - \mu_1 + \sigma_1) \vee (R_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

since $R > (\mu_1 + \mu_2) - (\sigma_1 + \sigma_2)$

$\Rightarrow R > (\mu_1 - \sigma_1) = s$.
Similarly,

$$P\{(((T_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (T_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2))) > 0) / ((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}$$
$$= \frac{P\{(((T_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (T_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2))) > 0\}}{P\{((T_\alpha^L - \mu_1 + \sigma_1) \vee (T_\alpha^U - \mu_1 + \sigma_1)) > 0\}}$$

since $T > (\mu_1 + \mu_2) - (\sigma_1 + \sigma_2)$
Therefore, the first $\Leftrightarrow$ follows from proposition (3.8).
The second $\Leftrightarrow$ follows from the first $\Leftrightarrow$ and the fact that
$p\{(((R_\alpha^L - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2)) \vee (R_\alpha^U - (\mu_1 + \mu_2) + (\sigma_1 + \sigma_2))) \ge 0\}$

$= \lim_{n \to \infty} p\{(((R_\alpha^L - (\mu_1+\mu_2)+(\sigma_1+\sigma_2)+(1/n)) \vee (R_\alpha^U - (\mu_1+\mu_2)+(\sigma_1+\sigma_2)+(1/n))) > 0\}$

and $p\{((R_\alpha^L - (\mu_1+\mu_2)+(\sigma_1+\sigma_2)) \vee (R_\alpha^U - (\mu_1+\mu_2)+(\sigma_1+\sigma_2)) > 0\}$

$= \lim_{n \to \infty} p\{((R_\alpha^L - (\mu_1+\mu_2)+(\sigma_1+\sigma_2)-(1/n)) \vee (R_\alpha^U - (\mu_1+\mu_2)+(\sigma_1+\sigma_2)-(1/n))) \geq 0\}$

which implies $R \leq^{FHRO} T$.

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