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Minimax Non-differentiable Multiobjective Fractional Programming with Generalized (F, P) – Convexity

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Abstract:
In this paper we introduce necessary and sufficient conditions for non-differentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct one parametric dual model and also discussed duality theorems. We obtained duality theorems for two parameters-free models of a nondifferentiable minimax fractional programming problem, involving generalized convexity assumptions. we established sufficient optimality conditions and duality theorems for nondiffertiable minimax fractional programming problem under (F, α, ρ, d) convexity assumptions. we discussed the optimality conditions and duality results for nondifferentiable minimax fractional programming under α-univexity.

1. Introduction

Necessary and sufficient conditions for generalized minimax programming were developed first by Schmitendorf [12] Tanimoto [13] defined a dual problem and derived duality theorems for convex minimax programming problems using schmitendorf's results.
Yadav and Mukherjee [14] also employed the optimality conditions of Schmitendorf [12] to construct the two dual problems and derived duality theorems for differentiable fractional minimax programming problems. Chandra and Kumar [3] pointed out that the formulation of Yadav and Mukherjee [14] has some omissions and inconsistencies, and they constructed two new dual problems and proved duality theorems for differentiable fractional minimax programming. Liu et al. [10,11], Liang and Shi [9] and Yang and Hou [15] paid much attention on minimax fractional programming problem and established sufficient optimality conditions and duality results.
Lai et al. [8] derived necessary and sufficient conditions for non-differentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct one parametric dual model and also discussed duality theorems. Lai and Lee [7] obtained duality theorems for two parameters-free models of a non-differentiable minimax fractional programming problem, involving generalized convexity assumptions. Ahmad and Husain [1,2] established sufficient optimality conditions and duality theorems for nondiffertiable minimax fractional programming problem under (F, α, ρ, d) convexity assumptions, thus extending the result of Lai et al. [8] and lai and Lee [7]. Jayswal [5] discussed the optimality conditions and duality results for nondifferentiable minimax fractional programming under α-univexity. Yuan et al. [93] introduced the concept of generalized (c, α, ρ, d)-convexity and focused their study on a nondifferentiable minimax fractional programming problems. Recently, Jayswal and Kumar [4] established sufficient optimality conditions and duality theorems for a class of nondifferentiable minimax fractional programming involving (c, α, ρ, d) -convexity.
In this paper, but they not consider this is multiobjective fractional minimax under the concept of generalized convexity i.e (F, P) convexity. Hence in this chapter to fill gap by developing some theorems and duality theorems in nondifferentiable minimax fractional programming under (F, P) convexity.

1.1. Definition

A functional $F_i : X \times X \times R^n \to R$ (where $X \subseteq R^n$) is 2 said to be sublinear in its third argument, if for all $(x, x_0) \in X \times X$,

$$F_i(x, x_0; a_1 + a_2) \le F_i(x, x_0; a_1) + F_i(x, x_0; a_2), \ \forall a_1, a_2 \in R^n$$

$$F_i(x, x_0; \alpha a) = \alpha F_i(x, x_0; a) \ \forall \alpha \in R, \alpha \ge 0, \ \forall a \in R^n \qquad (2.1)$$

1.2. Formulation

Let R^n denote the η -dimensional Euclidean Space and left R_+^n be its nonnegative orthani.

In this chapter, we consider the following non differentiable minimax multiobjective fractional programming problem.

$$\min_{x\in R^n} \sup_{y\in Y} \frac{f_i(x, y) + < x, Ax >^{\frac{1}{2}}}{h_i(x, y) - < x, Bx >^{\frac{1}{2}}} \quad (FP)$$

subject to $g_j(x) \le 0,$

Where $f_i, h_i : R^n \times R^m \to R$ and $g : R^n \to R^p$ are continuous differentiable functions, Y is a compact subset of R^m, and A and B are nxn positive semidefinite matrices. The problem (FP) is nondifferentiable programming problem if either A or B is nonzero. If A and B are null matrices, then the problem (FP) is a usual minimax fractional programming problem.

Let $\tau_p = \{x \in R^n : g_j(x) \le 0\}$ be the set of all feasible solutions of (FP). For each $(x, y) \in R^n \times R^m,$ we define

$$\phi_i(x, y) = \frac{f_i(x, y) + < x, Ax >^{\frac{1}{2}}}{h_i(x, y) - < x, Bx >^{\frac{1}{2}}} \quad (3.1)$$

Assume that for each $(x, y) \in R^n x Y, f(x, y) + < x, Ax >^{\frac{1}{2}} \ge 0$ and $h_i(x, y) - < x, Bx >^{\frac{1}{2}} > 0$.

Denote

$$\left\{ \overline{y}_i(x) = \overline{y} \in Y : \frac{f_i(x, \overline{y}) + < x, Ax >^{\frac{1}{2}}}{h_i(x, \overline{y}) - < x, Bx >^{\frac{1}{2}}} = \sup_{Z\in Y} \frac{f_i(x, z) + < x, Ax >^{\frac{1}{2}}}{h_i(x, z) - < x, Bx >^{\frac{1}{2}}}, \right\}$$

$J_j = \{1, 2, \ldots\ldots p\}, \qquad J_j(x) = \{j \in J: g_j(x) = 0\} \quad (3.2)$

$$k_i(x) = \left\{ (s, t, \overline{y}) \in N \times R_+^s \times R^{ms} : 1 \le s \le n+1, t = (t_1, t_2, \ldots\ldots t_s) \in R^s + \text{ with } \sum_{i=1}^{s} t_i = 1, \ \overline{y} = \left(\overline{y}_1, \overline{y}_2, \ldots\ldots, \overline{y}_s\right), \ \overline{y}_i \in \overline{Y}_{(x)}, \ i = 1, 2, \ldots\ldots s \right\}$$

Since f_i and h_i are continuously differentiable and Y is a compact subset of R^m, it follows that for each $x^* \in \tau_p, \overline{y}_i(x^*) \ne \phi$. Thus for any $\overline{y}_i \in \overline{y}_i(x^*),$ we have a positive constant $v_i^* = \phi_i(x^*, \overline{y}_i)$.

1.3. Dual Formulation

To unify and extend the dual models, we need to divide {1, 2, …… p} into several parts. Let $J_\alpha (0 \le \alpha \le r)$ be a partition of {1, 2, …… p}, that is,

$$J_\alpha \cap J_\beta = \phi, \text{for } \alpha \ne \beta, \ \bigcup_{\alpha=0}^{r} J_\alpha = \{1, 2, \ldots\ldots p\} \quad (3.2.1)$$

We note that for (p)-optimal x^*,

$$\sum_{j\in J_\alpha} \mu_j^* g_j(x^*) = 0, \ \alpha = 0, 1, \ldots\ldots r \quad (3.2.2)$$

Dual Formulation is as follows

$$\max_{(s, t_i, \tilde{y}) \in k(z)} \ \sup_{(z, \mu_j, u, v) \in H(s, t_i, \tilde{y})} \left(\frac{\sum_{i=1}^{s} t_i f_i(z, \overline{y}_i) + < z, A_2 >^{\frac{1}{2}} + \sum_{j\in J_0} \mu_i g_i(z)}{\sum_{i=1}^{s} t_i \left(h_i(z, \overline{y}_i)\right) - < z, A_z >^{\frac{1}{2}}} \right)$$

Where $H(s, t, \tilde{y})$ denotes the set of all $(z, \mu_j, u, v) \in R^n \times R_+^{n^n} \times R^n \times R^n$

Satisfying

$$\left(\sum_{i=1}^{s} t_i\left(h_i(z,\overline{y}_i)-<z,Bz>^{\frac{1}{2}}\right)\right)\nabla\left(\sum_{i=1}^{s} t_i\left(f_i(z,\overline{y}_i)+Au\right)+\sum_{j=1}^{p}\mu_i g_j(z)\right)$$

$$-\left(\sum_{i=1}^{s} t_i\left(f_i(z,\overline{y}_i)+<z,Az>^{\frac{1}{2}}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i\left(h_i(z,\overline{y}_i)-BV\right)\right)=0,$$

$$\sum_{j\in J_\alpha}\mu_j g_j(z)\geq 0 \quad , \quad \alpha=1,2,\ldots.. r$$

$$J_\alpha\cap j_\beta=\phi \quad , \quad \text{for } \alpha\neq\beta, \quad \bigcup_{\alpha=0}^{r} J_\alpha=\{1,2,\ldots..p\}$$

1.4. Necessary and Sufficient Conditions for Lemma (Weak Duality)

Let x be a feasible solution for (p), and let $(z,\mu,u,v,s,t,\tilde{y})$ be a feasible solution for (3.2.12) suppose that there exist $F,\theta,\phi_0,b_0,\rho_0$ and $\phi_\alpha, b_\alpha, \rho_\alpha, \alpha=1,2,\ldots.. r$ such that

$$F\left(x,z;\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)\right)-<z,Bz>^{\frac{1}{2}}\right)\right)$$

$$\nabla\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i)+Au\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)-\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i)\right)+<z,A_z>^{\frac{1}{2}}+\sum_{j\in J_0}\mu_j g_j(z)\right)$$

$$\nabla\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)-BV\right)\right)\geq-\rho_0\left\|\theta(x,z)\right\|^2$$

$$\Rightarrow b_0(x,z)\,\phi_0\left(\left(\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)\right)-<z,Bz>^{\frac{1}{2}}\right)\right)\left(\sum_{i=1}^{s} t_i\left(f(x,\overline{y}_i)+<x,Ax>^{\frac{1}{2}}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\right.$$

$$\left.-\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i)+<z,Az>^{\frac{1}{2}}\right)+\sum_{j\in J_0}\mu_j g_i(z)\right)g\times\left(\sum_{i=1}^{s} t_i\left(h(x,\overline{y}_i)-<x,Bx>^{\frac{1}{2}}\right)\right)\right)\geq 0$$

$$-b_\alpha(x,z)\,\phi_\alpha\left(\left(\left(\sum_{i=1}^{s} t_i\left(h(x,\overline{y}_i)\right)-<z,Bz>^{\frac{1}{2}}\right)\right)\left(\sum_{j\in J_\alpha}\mu_j g_i(z)\right)\right)\leq 0 \qquad (3.2.5)$$

$$\Rightarrow F\left(x,z;\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)\right)-<z,Bz>^{\frac{1}{2}}\right)\left(\sum_{j\in J_\alpha}\mu_j g_i(z)\right) \qquad (3.2.6)$$

$$\leq-\rho_\alpha\left\|\theta(x,z)\right\|^2 \; , \; \alpha=1,2,\ldots. r$$

Further, assume that

$$a\geq 0\Rightarrow \phi_\alpha(a)\geq 0, \quad \alpha=1,2,\ldots. r \qquad (3.2.7)$$

$$\phi_0(a)\geq 0\Rightarrow a\geq 0 \qquad (3.2.8)$$

$$b_0(x,z)>0, \quad b_\alpha(x,z)\geq 0, \quad \alpha=1,2,\ldots. r \qquad (3.2.9)$$

$$\rho_0+\sum_{\alpha=1}^{r}\rho_\alpha\geq 0 \qquad (3.2.10)$$

Then

$$\sup_{y\in Y}\frac{f_i(x,y)+<x,Ax>^{\frac{1}{2}}}{h_i(x,y)-<x,Bx>^{\frac{1}{2}}}\geq\left(\frac{\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i)+<z,Az>^{\frac{1}{2}}\right)+\sum_{j\in J_0}\mu_j g_i(z)}{\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)\right)-<z,Az>^{\frac{1}{2}}}\right) \qquad (3.2.11)$$

Proof: Supposed to contrary that

$$\sup_{y\in Y}\frac{f_i(x,y)+<x,Ax>^{\frac{1}{2}}}{h_i(x,y)-<x,Bx>^{\frac{1}{2}}}<\left(\frac{\sum_{i=1}^{s}t_i\left(f_i(z,\bar{y}_i)+<z,Az>^{\frac{1}{2}}\right)+\sum_{j\in J_0}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)\right)-<z,Bz>^{\frac{1}{2}}}\right)$$

(3.2.12) t

en we get

$$\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)-<z,Bz>^{\frac{1}{2}}\right)\left(f_i(x,y)+<x,Ax>^{\frac{1}{2}}\right)$$
$$<\left(\sum_{i=1}^{s}t_i\left(f_i(z,\bar{y}_i)+<z,Az>^{\frac{1}{2}}+\sum_{j\in J_0}\mu_j g_j(z)\right)\right)\left(h_i(x,y)-<x,Bx>^{\frac{1}{2}}\right),\ \forall y\in Y$$

(3.2.13)

Further, this implies

$$\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)-<z,Bz>^{\frac{1}{2}}\right)\left(\sum_{i=1}^{s}t_i\left(f_i(x,\bar{y}_i)\right)+<x,Ax>^{\frac{1}{2}}\right)$$
$$<\left(\sum_{i=1}^{s}t_i\left(f_i(z,\bar{y}_i)+<z,Az>^{\frac{1}{2}}\right)\right)+\sum_{j\in J_0}\mu_j g_j(z)\ \left(\sum_{i=1}^{s}t_i\left(h_i(x,\bar{y}_i)-<x,Bx>^{\frac{1}{2}}\right)\right)$$

(3.2.14)

Hence, we have

$$\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)-<z,Bz>^{\frac{1}{2}}\right)\sum_{i=1}^{s}t_i\left(f_i(x,\bar{y}_i)+<x,Ax>^{\frac{1}{2}}+\sum_{j\in J_0}\mu_j g_j(x)\right)$$
$$-\left(\sum_{i=1}^{s}t_i\left(f_i(z,\bar{y}_i)+<z,Az>^{\frac{1}{2}}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s}t_i\left(h_i(x,\bar{y}_i)-<x,Bx>^{\frac{1}{2}}\right)\right)$$
$$<\left(\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)-<z,Bz>^{\frac{1}{2}}\right)\right)\left(\sum_{j\in J_0}\mu_j g_j(x)\right) \quad (3.2.15)$$

Using the fact that $\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)-<z,Bz>^{\frac{1}{2}}\right)>0$ and

$\sum_{j\in J_0}\mu_j g_j(x)\le 0$ and the last inequality, we have

$$\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)-<z,Bz>^{\frac{1}{2}}\right)\left(\sum_{i=1}^{s}t_i\left(f_i(x,\bar{y}_i)+<x,Ax>^{\frac{1}{2}}\right)+\sum_{j\in J_0}\mu_j g_j(x)\right)$$
$$-\left(\sum_{i=1}^{s}t_i\left(f_i(z,\bar{y}_i)+<z,Az>^{\frac{1}{2}}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s}t_i\left(h_i(x,\bar{y}_i)-<x,Bx>^{\frac{1}{2}}\right)\right)<0$$

(3.2.16)

From (3.2.5), (3.2.8), (3.2.9) and (3.2.16), we get

$$F\left(x,z;\left(\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)\right)-<z,Bz>^{\frac{1}{2}}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i\left(f_i(z,\bar{y}_i)+Au\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)$$
$$-\left(\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)\right)+<z,A_z>^{\frac{1}{2}}+\sum_{j\in J_0}\mu_j g_j(z)\right)$$
$$\nabla\left(\sum_{i=1}^{s}t_i\left(h_i(z,\bar{y}_i)-BV\right)\right)<-\rho_0\left\|\theta(x,z)\right\|^2 \quad (3.2.17)$$

Using

$$\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) > 0 \quad (3.2.4), (3.2.7), (3.2.9),$$

We get $-b_\alpha(x, z)\, \phi_\alpha \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \left(\sum_{j \in J_0} \mu_j g_j(z) \right) \right) \leq 0 \quad \alpha = 1, 2, \ldots.. r$

(3.2.18)

From (3.2.6), we have

$$F\left(x, z; \sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \left(\sum_{j \in J_\alpha} \mu_j \nabla g_j(z) \right) \right) \quad (3.2.19)$$

$$\leq -\rho_\alpha \|\theta(x, z)\|^2 \quad , \quad \alpha = 1, 2, \ldots.. r$$

on adding (3.2.17) and (3.2.19) and making use of sub linearity of F and (3.2.10), we have

$$\left(x, z; \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \nabla \left(\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + Au \right) + \sum_{j=1}^{p} \mu_j g_j(z) \right) \right) \right)$$

$$- \left(\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} \right) + \sum_{j \in J_0} \mu_j g_j(z) \right) \nabla \left(\sum_{i=1}^{s} t_i \left(h(z, \bar{y}_i) - BV \right) \right) < 0$$

(3.2.20)

Which contradicts (3.2.3).

This completes the proof.

1.5. Theorem (Weak Duality)

Let x be a feasible solution for (FP) and let $(z, \mu, u, v, s, t, \tilde{y})$ be a feasible solution for (3.2.12). Suppose that there exists $F, \theta, \phi_0, b_0, \rho_0$ and $\phi_\alpha, b_\alpha, \rho_\alpha$, $\alpha = 1, 2 \ldots.. r$

Such that

$$b_0(x, z)\, \phi_0 \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \right) \left(\sum_{i=1}^{s} t_i \left(f_i(x, \bar{y}_i) + < x, Ax >^{\frac{1}{2}} \right) + \sum_{j \in J_0} \mu_j g_j(x) \right)$$

$$- \sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} + \sum_{j \in J_0} \mu_j g_j(z) \right) \times \sum_{i=1}^{s} t_i \left(h_i(x, \bar{y}_i) - < x, Bx >^{\frac{1}{2}} \right) < 0$$

$$\Rightarrow F\left(x, z; \sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) \right) - < z, Bz >^{\frac{1}{2}} \right) \nabla \left(\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + Au \right) + \sum_{j \in J_0} \mu_j g_j(x) \right)$$

$$- \sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} + \sum_{j \in J_0} \mu_j g_j(z) \right) \nabla \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - BV \right) \right) \leq -\rho_0 \|\theta(x, z)\|^2,$$

$$-b_\alpha(x, z)\, \phi_\alpha \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \left(\sum_{j \in J_\alpha} \mu_j g_j(z) \right) \right) \leq 0$$

$$\Rightarrow F\left(x, z; \sum_{i=1}^{s} t_i \left(h_I(z, \bar{y}_i) \right) - < z, Bz >^{\frac{1}{2}} \right) \left(\sum_{j \in J_\alpha} \mu_j \nabla g_j(z) \right)$$

$$\leq -\rho_\alpha \|\theta(x, z)\|^2 \quad , \quad \alpha = 1, 2, \ldots.. r \quad (3.2.21)$$

Further, assume that (3.2.8), (3.2.9) and (3.2.10) are satisfied, then

$$\underset{y \in y}{SUP} \frac{f_i(x, y) + < x, Ax >^{\frac{1}{2}}}{h_i(x, y) - < x, Bx >^{\frac{1}{2}}} \geq \left(\frac{\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} \right) + \sum_{j \in J_0} \mu_J g_J(z)}{\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) \right) - < z, Bz >^{\frac{1}{2}}} \right)$$

(3.2.22)

Proof: The proof is similar to that of the above theorem.

1.6. (Strong Duality)

Assume that x^* is an optimal solution for (P) and $\nabla g_j(x^*),\ j\in J(x^*)$ are linearly independent. Then there exist $(s^*, t^*, \tilde{y}^*)\in k(x^*)$ and $(x^*, \mu^*, u^*, v^*)\in H(s^*, t^*, \tilde{y}^*)$ such that $(x^*, \mu^*, u^*, v^*, s^*, t^*, \tilde{y}^*)$ is an optimal solution for (3.2.12). If, in addition, the hypotheses of any of the weak duality (Theorem 3.2.1 or Theorem 3.2.2) hold for a feasible point $(z, \mu, u, v, s, t, \tilde{y})$, then the problems (FP) and (4.18) have the same optimal values.

Proof: By (3.2.1) Lemma there exist $(s^*, t^*, \tilde{y}^*)\in k(x^*)$ and $(x^*, \mu^*, u^*, v^*)\in H(s^*, t^*, \tilde{y}^*)$ such that $(x^*, \mu^*, u^*, v^*, s^*, t^*, \tilde{y}^*)$ is a feasible for (3.2.12), optimality of this feasible solution for (3.2.12) follows from Theorems (3.2.1) or (3.2.2) accordingly.

Theorem: (3.2.4) (Strict converse duality). Let x^* and $(z, \mu, u, v, s, t, \tilde{y})$ be optimal solutions for (p) and (3.2.12), respectively. Suppose that $\nabla g_j(x^*),\ j\in J(x^*)$ are linearly independent and there exist $F, \theta, \phi_0, b_0, \rho_0$ and $\phi_\alpha, b_\alpha, \rho_\alpha, \alpha = 1, 2, ... r$ such that

$$F\left(x^*, z; \left(\sum_{i=1}^{s} t_i \left(h_i(z, \overline{y}_i)\right) - < z, Bz >^{\frac{1}{2}}\right)\right)\nabla\left(\sum_{i=1}^{s} t_i\left(f_i(z, \overline{y}_i)\right) + Au + \sum_{j\in J_0}\mu_j g_j(z)\right)$$

$$\times\left(\sum_{i=1}^{s} t_i\left(f_i(z, \overline{y}_i)\right) + < z, Az >^{\frac{1}{2}} + \sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i\left(h_i(z, \overline{y}_i)\right) - BV\right)$$

$$\geq -\rho_\alpha \left\|\theta(x^*, z)\right\|^2$$

$$\Rightarrow b_0(x^*, z)\,\phi_0\left(\sum_{i=1}^{s} t_i\left(h_i(z, \overline{y}_i) - < z, Bz >^{\frac{1}{2}}\right)\right)\times\left(\sum_{i=1}^{s} t_i\left(f_i(x^*, \overline{y}_i) + < x^*, Az^* >^{\frac{1}{2}}\right) + \sum_{j\in J_0}\mu_j g_i(x^*)\right)$$

$$-\sum_{i=1}^{s} t_i\left(f_i(z, \overline{y}_i) + < z, Az >^{\frac{1}{2}} + \sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i\left(h_i(x^*, \overline{y}_i) - < x^*, Bx^* >^{\frac{1}{2}}\right)\right) \geq 0$$

(3.2.23)

$$-b_\alpha(x^*, z)\,\phi_\alpha\left(\sum_{i=1}^{s} t_i\left(h_i(z, \overline{y}_i) - < z, Bz >^{\frac{1}{2}}\right)\left(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right) \leq 0$$

$$\Rightarrow F\left(x^*, z;\ \sum_{i=1}^{s} t_i\left(h_i(z, \overline{y}_i)\right) - < z, Bz >^{\frac{1}{2}}\right)\left(\sum_{j\in J_\alpha}\mu_j \nabla g_j(z)\right) \qquad (3.2.24)$$

$$\leq -\rho_\alpha\left\|\theta(x^*, z)\right\|^2, \quad \alpha = 1, 2, ... r$$

Further, assume (3.2.7), (3.2.9) and (3.2.10)

$$\phi_0(a) \geq 0 \Rightarrow a > 0 \qquad (3.2.25)$$

then $x^*=z$, that is, z is an optimal solution for (p).

Proof: Supposed to contrary that $x^* \neq z$. From the strong duality theorem (3.2.3), we know that

$$\sup_{y\in Y}\frac{f_i(x^*, y) + < x^*, Ax^* >^{\frac{1}{2}}}{h_i(x^*, y) - < x^*, Bx^* >^{\frac{1}{2}}} = \frac{\sum_{i=1}^{s} t_i\left(f_i(z, \overline{y}_i) + < z, Az >^{\frac{1}{2}}\right) + \sum_{j\in J_0}\mu_j g_j(z)}{\sum_{i=1}^{s} t_i\left(h_i(z, \overline{y}_i) - < z, Bz >^{\frac{1}{2}}\right)}$$

(3.2.26)

Then, we get

$$\sum_{i=1}^{s} t_i\left(h_i(z, \overline{y}_i) - < z, Bz >^{\frac{1}{2}}\right)\left(f_i(x^*, y) + < x^*, Ax^* >^{\frac{1}{2}}\right)$$

$$\leq\left(\sum_{i=1}^{s} t_i\left(f_i(z, \overline{y}_i) + < z, Az >^{\frac{1}{2}}\right) + \sum_{j\in J_0}\mu_j g_j(z)\right)\left(h(x^*, y) - < x^*, Bx^* >^{\frac{1}{2}}\right)\forall y\in Y$$

(3.2.27)

Further, this implies

$$\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \left(\sum_{i=1}^{s} t_i \left(f_i(x^*, \bar{y}_i) + < x^*, Ax^* >^{\frac{1}{2}} \right) \right)$$

$$\leq \left(\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} \right) + \sum_{j \in J_0} \mu_j g_j(z) \right) \quad (3.2.28)$$

$$\times \left(\sum_{i=1}^{s} t_i \left(h_i(x^*, \bar{y}_i) - < x^*, Bx^* >^{\frac{1}{2}} \right) \right), \forall\ y \in Y$$

Hence, we have

$$\left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \right) \left(\sum_{i=1}^{s} t_i \left(f_i(x^*, \bar{y}_i) + < x^*, Ax^* >^{\frac{1}{2}} \right) + \sum_{j \in J_0} \mu_j g_j(x^*) \right)$$

$$- \left(\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} \right) + \left(\sum_{j \in J_0} \mu_j g_j(z) \right) \right) \left(\sum_{i=1}^{s} t_i \left(h_i(x^*, \bar{y}_i) - < x^*, Bx^* >^{\frac{1}{2}} \right) \right)$$

$$\leq \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \left(\sum_{j \in J_0} \mu_j g_j(x^*) \right) \right) \quad (3.2.29)$$

Using the fact that $\left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \right) > 0$ and $\sum_{j \in J_0} \mu_j g_j(x^*) \leq 0$ and the last inequality, we have

$$\left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) \right) \left(\sum_{i=1}^{s} t_i \left(f_i(x^*, \bar{y}_i) + < x^*, Ax^* >^{\frac{1}{2}} \right) + \sum_{j \in J_0} \mu_j g_j(x^*) \right)$$

$$- \left(\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} \right) + \sum_{j \in J_0} \mu_j g_j(z) \right) \left(\sum_{i=1}^{s} t_i \left(h_i(x^*, \bar{y}_i) - < x^*, Bx^* >^{\frac{1}{2}} \right) \right) \leq 0$$

From (3.2.9), (3.2.23), (3.2.25) and (3.2.30), we get

$$F\left(x^*, z; \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) \right) - < z, Bz >^{\frac{1}{2}} \right) \right) \nabla \left(\sum_{i=1}^{s} t_i (z, \bar{y}_i) + Au + + \sum_{j \in J_0} \mu_j g_j(z) \right)$$

$$- \left(\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) \right) + < z, Az >^{\frac{1}{2}} + \sum_{j \in J_0} \mu_j g_j(z) \right) \nabla \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) \right) - BV \right)$$

(3.2.31)

$$< -\rho_0 \left\| \theta(x^*, z) \right\|^2$$

Using $\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}} \right) > 0,$ (3.2.4), (3.2.7) and (3.2.9)

We get $-b_\alpha(x^*, Z)\, \phi_\alpha \left(\sum_{I=1}^{S} t_i \left(h_i(z, \bar{Y}_i) \right) - < z, Bz^{\frac{1}{2}} \right) \left(\sum_{j \in J_\alpha} \mu_j g_j(z) \right) \leq 0$

$$\alpha = 1, 2, ... r \quad (3.2.32)$$

$$F\left(x^*, z; \sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) \right) - < z, Bz >^{\frac{1}{2}} \right) \left(\sum_{j \in J_\alpha} \mu_j g_j(z) \right)$$

$$< -\rho_\alpha \left\| \theta(x^*, z) \right\|^2, \ \alpha = 1, 2, ... r \quad (3.2.33)$$

From (3.2.24), we have on adding (3.2.31) and (3.2.33) and making use of sub linearity of f and (3.2.10) we have

$$F\left(x^*, z; \left(\sum_{i=1}^{s} t_i \left(h_i(z, \bar{y}_i) \right) - < z, Bz >^{\frac{1}{2}} \right) \right) \nabla \left(\sum_{i=1}^{s} t_i \left(f_i(z, \bar{y}_i) + Au \right) + \sum_{j=1}^{p} \mu_j g_j(z) \right)$$

$$-\left(\sum_{i=1}^{s} t_i \left(f_i (z, \bar{y}_i)\right) + < z, Az >^{\frac{1}{2}} + \sum_{j\in J_0} \mu_j g_j (z)\right) \nabla \left(\sum_{i=1}^{s} t_i \left(h_i (z, \bar{y}_i)\right) - BV\right) < 0 \quad (3.2.34)$$

Which contradicts (3.2.3). This completes the proof.

Theorem 3.2.5 (Strict Converse Duality):

Let x^* and $(z, \mu, u, v, s, t, \tilde{y})$ be optimal solutions for (FP) and (3.2.12), respectively. Suppose that $\nabla g_j(x^*),\ j\in J(x^*)$ are linearly independent and there exist $F, \theta, \phi_0, b_0, \rho_0$ and $\phi_\alpha, b_\alpha, \rho_\alpha$, $\alpha = 1, 2 \ldots.. r$ such that

$$b_0(x^*, z)\, \phi_0\left(\sum_{i=1}^{s} t_i \left(h_i (z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}}\right)\right)\left(\sum_{i=1}^{s} t_i \left(f_i (x^*, \bar{y}_i) + < x^*, Ax^* >^{\frac{1}{2}}\right) + \sum_{j\in J_0} \mu_j g_i (x^*)\right)$$

$$-\sum_{i=1}^{s} t_i \left(f_i (z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} + \sum_{j\in J_0} \mu_j g_j (z)\right)\left(\sum_{i=1}^{s} t_i \left(h_i (x^*, \bar{y}_i) - < x^*, Bx^* >^{\frac{1}{2}}\right)\right) < 0$$

$$\Rightarrow F\left(x^*, z; \sum_{i=1}^{s} t_i \left(h_i (z, \bar{y}_i)\right) - < z, Bz >^{\frac{1}{2}}\right) \nabla\left(\sum_{i=1}^{s} t_i \left(f_i (z, \bar{y}_i)\right) + Au + \sum_{j\in J_0} \mu_j g_j (z)\right)$$

$$-\sum_{i=1}^{s} t_i \left(f_i (z, \bar{y}_i) + < z, Az >^{\frac{1}{2}} - \sum_{j\in J_0} \mu_j g_j (z)\right) \nabla\left(\sum_{i=1}^{s} t_i \left(h_i (z, \bar{y}_i) - BV\right)\right) \le -\rho_0 \left\|\theta(x^*, z)\right\|^2,$$

$$-b_\alpha(x^*, z)\, \phi_\alpha\left(\sum_{i=1}^{s} t_i \left(h_i (z, \bar{y}_i) - < z, Bz >^{\frac{1}{2}}\right)\left(\sum_{j\in J_\alpha} \mu_j g_j (z)\right)\right) \le 0$$

$$\Rightarrow F\left(x^*, z;\ \sum_{i=1}^{s} t_i \left(h_i (z, \bar{y}_i)\right) - < z, Bz >^{\frac{1}{2}}\right)\left(\sum_{j\in J_\alpha} \mu_j \nabla g_j (z)\right)$$

$$\le -\rho_\alpha \left\|\theta(x^*, z)\right\|^2 \ ,\ \alpha = 1, 2, \ldots.. r \quad (3.2.35)$$

Further, assume (3.2.7), (3.2.9) and (3.2.10)

$$\phi_0(a) \ge 0 \Rightarrow a > 0 \quad (3.2.36)$$

then x^*=z, that is, z is an optimal solution for (p).

1.7. Optimality Theorem

The following result from Lai and Lee [10] is needed in the sequel.

1.7.1. Necessary Optimality Theorem

Let x^* be an optimal solution for (FP) satisfying $(x^*, Ax^*)>0$, $(x^*, Bx^*)>0$ and let $\nabla gj(x^*),\ j\in J(x^*)$ be linearly independent, then there exist $(s, t^*, \tilde{y}) \in k(x^*),\ v^* \in R_+,\ u, v \in R^n$ and $\mu^* \in R_+^p$ such that

$$\sum_{i=1}^{s} t_i^* \left(\nabla f_i (x^*, \bar{y}_i) + Au - v^* \left(\nabla h_i (x^*, \bar{y}_i) - BV\right)\right) + \sum_{j=1}^{p} \mu_j^* \nabla g_j^*(x^*) = 0 \quad (4.1)$$

$$f_i \left(x^*, \bar{y}_i\right) + < x^*, Ax^* >^{\frac{1}{2}} - v^*\left(h_i \left(x^*, \bar{y}_i\right) - < x^*, Bx^* >^{\frac{1}{2}}\right) = 0,\ i = 1, 2, \ldots. s \quad (4.2)$$

$$\sum_{j=1}^{p} \mu_j^* g_j (x^*) = 0, \quad (4.3)$$

$$t_i^* \in R_+^s,\ \sum_{i=1}^{s} t_i^* = 1,\ \bar{y}_i \in Y_i (x^*),\ i = 1, 2, \ldots. s \quad (4.4)$$

$$< u, Au > \le 1, \quad < u, Bv > \le 1, \quad < x^*, Au > = < x^*, Ax^* >^{\frac{1}{2}}$$

$$< x^*, Bv > = < x^*, Bx^* >^{\frac{1}{2}} \quad (4.5)$$

It should be noted that both the matrices A and B are positive definite at the solution x^* in the above lemma. If one of $< Ax^*, x^* >$ and $< Bx^*, x^* >$ is zero, or both A and B are singular at x^*, then for $(s, t^*, \tilde{y}) \in k(x^*)$, we can take $z\overline{y}(x^*)$ .ith any one of the following (i) – (iii) holds.

(i) $< Ax^*, x^* > > 0, \quad < Bx^*, x^* > = 0$

$$\left(\sum_{i=1}^{s} t_i \nabla f_i (x^*, \overline{y}_i) + \frac{Ax^*}{< Ax^*, x^* >^{\frac{1}{2}}} - v_i^* \nabla h_i (x^*, \overline{y}_i), z \right) + < \left(v_i^{*2} B \right) z, z >^{\frac{1}{2}} < 0$$

ii) $< Ax^*, x^* >= 0, \quad < Bx^*, x^* > > 0$

$$\Rightarrow \left(\sum_{i=1}^{s} t_i^* \left(\nabla f_i (x^*, \overline{y}_i) - v_i^* \left(\nabla h_i (x^*, \overline{y}_i) - \frac{Bx^*}{< Bx^*, x^* >^{\frac{1}{2}}} \right) \right) \right) z > + < Bz, z, >^{\frac{1}{2}} < 0$$

(iii) $< Ax^*, x^* >= 0, \quad < Bx^*, x^* > = 0$

$$\left(\sum_{i=1}^{s} t_i^* \left(\nabla f_i (x^*, \overline{y}_i) - v^* \nabla h_i (x^*, \overline{y}_i) \right) \right), z + < (v^* B) z, z, >^{\frac{1}{2}} + < Bz, z >^{\frac{1}{2}} < 0$$

(4.6)

1.7.2. Sufficient Optimality Conditions

In this section, we present three sets of sufficient optimality conditions for (p) in the frame work of generalized convexity.

Let $F : X \times X \times R^n \to R$ be sublinear functional, $\phi_0, \phi_1 : R \to R$, $\theta : R^n \times R^n \to R^n$, and $b_0, b_1 : X \times X \to R_+$. Let ρ_0, ρ_1 be real numbers.

Theorem (4.3.1): Let $x^* \in \tau_p$ be a feasible solution for (FP), and there exist $v^* \in R_+, (s, t^*, \tilde{y}) \in k(x^*)$, $u, v \in R^n$ and $\mu^* \in R_+^p$ satisfying (4.1)-(4.5). Suppose that there exist $F, \theta, \phi_0, b_0, \rho_0$ and ϕ_1, b_1, ρ_1 such that

$$F\left(x, x^*; \sum_{i=1}^{s} t_i^* \left(\nabla f_i (x^*, \overline{y}_i) \right) + Au \right) - v^* \left(\nabla h_i (x^*, \overline{y}_i) - Bv \right) \geq - \rho_0 \left\| \theta (x, x^*) \right\|^2$$

$$b_0(x, x^*) \phi_0 \left(\sum_{i=1}^{s} t_i^* \left(f_i (x, \overline{y}_i) + < x, Au > - v^* \left(h_i (x, \overline{y}_i) \right) - < x, Bv > \right) \right) \quad (4.7)$$

$$- \sum_{i=1}^{s} t_i^* \left(f_i (x^*, \overline{y}_i) + < x^*, Au > - v^* \left(h_i (x^*, \overline{y}_i) - < x^*, Bv > \right) \right) \geq 0$$

$$- b_1 (x, x^*) \phi_1 \left(\sum_{j=1}^{p} \mu_j^* g_j (x^*) \right) \leq 0 \quad (4.8)$$

$$\Rightarrow F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j (x^*) \right) \leq - \rho_1 \left\| \theta (x, x^*) \right\|^2$$

Further, assume that

$$a \geq 0 \Rightarrow \phi_1 (a) \geq 0 \quad (4.9)$$

$$\phi_0 (a) \geq 0 \Rightarrow a \geq 0 \quad (4.10)$$

$$b_0(x, x^*) \geq 0, \; b_1 (x, x^*) > 0 \quad (4.11)$$

$$\rho_0 + \rho_1 \geq 0 \quad (4.12)$$

then x^* is an optimal solution of (FP).

Proof: Suppose to the contrary that x^* is not an optimal solution of (FP), then there exists $x \in \tau_\rho$ such that

$$\sup_{y \in Y} \frac{f_i (x, y) + < x, Ax >^{\frac{1}{2}}}{h_i (x, y) - < x, Bx >^{\frac{1}{2}}} < \sup_{y \in Y} \frac{f_i (x^*, y) + < x^*, Ax^* >^{\frac{1}{2}}}{h_i (x^*, y) - < x^*, Bx^* >^{\frac{1}{2}}} \quad (4.13)$$

We note that

$$\sup_{y\in Y}\frac{f_i(x^*,y)+<x^*,Ax^*>^{\frac{1}{2}}}{h_i(x^*,y)-<x^*,Bx^*>^{\frac{1}{2}}}=\frac{f_i(x^*,\overline{y}_i)+<x^*,Ax^*>^{\frac{1}{2}}}{h_i(x^*,\overline{y}_i)-<x^*,Bx^*>^{\frac{1}{2}}}=v^* \qquad (4.14)$$

for $\overline{y}_i \in \overline{Y}(x^*)$, $i = 1, 2, \ldots\ldots s$

$$\frac{f_i(x,\overline{y}_i)+<x,Ax>^{\frac{1}{2}}}{h_i(x,\overline{y}_i)-<x,Bx>^{\frac{1}{2}}}\le \sup_{y\in Y}\frac{f_i(x,y)+<x,Ax>^{\frac{1}{2}}}{h_i(x,y)-<x,Bx>^{\frac{1}{2}}} \qquad (4.15)$$

Thus, we have

$$\frac{f_i(x,\overline{y}_i)+<x,Ax>^{\frac{1}{2}}}{h_i(x,\overline{y}_i)-<x,Bx>^{\frac{1}{2}}}<v^* \quad \text{for } i=1,2,\ldots\ldots s \qquad (4.16)$$

it follows that

$$f_i(x,\overline{y}_i)+<x,Ax>^{\frac{1}{2}}-v_i^*\left(h_i(x,\overline{y}_i)-<x,bx>^{\frac{1}{2}}\right)<0 \quad \text{for } i=1,2,\ldots s \qquad (4.17)$$

From (4.2), (4.4), (4.5) and (4.17), we get

$$\sum_{i=1}^{s}t_i^*\left(f_i(x,\overline{y}_i)+<x,Au->v^*\left(h_i(x,\overline{y}_i)-<x,Bv>\right)\right)$$
$$<\sum_{i=1}^{s}t_i^*\left(f_i(x^*,\overline{y}_i)+<x^*,Au>-v^*\left(h_i(x^*,\overline{y}_i)-<x^*,Bv>\right)\right) \qquad (4.18)$$

on the other hand, from (4.3), (4.9) and (4.11), we have

$$-b_1(x,x^*)\,\phi_1\left(\sum_{j=1}^{p}\mu_j^*g_j(x^*)\right)\le 0 \qquad (4.19)$$

it follows from (4.8) that

$$F\left(x,x^*;\sum_{j=1}^{p}\mu_j^*\nabla g_j(x^*)\right)\le -\rho_1\left\|\theta(x,x^*)\right\|^2 \qquad (4.20)$$

From (4.1), the sub linearity of F, and (4.12), we get

$$F\left(x,x^*;\sum_{i=1}^{s}t_i^*\left(\nabla f_i(x^*,\overline{y}_i)\right)+Au-v^*\left(\nabla h_i(x^*,\overline{y}_i)-Bv\right)\right)\ge -\rho_0\left\|\theta(x,x^*)\right\|^2 \qquad (4.2.1)$$

Then by (4.7), we have

$$b_0(x,x^*)\,\phi_0\left(\sum_{i=1}^{s}t_i^*\left(f_i(x,\overline{y}_i)+<x,Au>-v^*\left(h_i(x,\overline{y}_i)\right)-<x,Bv>\right)\right)$$
$$-\sum_{i=1}^{s}t_i^*\left(f_i(x^*,\overline{y}_i)+<x^*,Au>-v^*\left(h_i(x^*,\overline{y}_i)-<x^*,Bv>\right)\right)\ge 0 \qquad (4.22)$$

From (4.10), (4.11) and the above inequality, we obtain

$$\sum_{i=1}^{s}t_i^*\left(f_i(x,\overline{y}_i)+<x,Au>-v^*\left(h_i(x,\overline{y}_i)\right)-<x,Bv>\right)$$
$$-\sum_{i=1}^{s}t_i^*\left(f_i(x^*,\overline{y}_i)+<x^*,Au>-v^*\left(h_i(x^*,\overline{y}_i)-<x^*,Bv>\right)\right)\ge 0 \qquad (4.23)$$

which contradicts (4.18). Therefore, x^* is an optimal solution for (FP). This completes the proof.

3.2. Theorem: Let $x^*\in\tau_\rho$ be a feasible solution for (FP), and there exist $v^*\in R_+,(s,t_i^*,\tilde{y})\in K_i(x^*)$, $u_i, v_i \in R^n$ and $\mu_j^*\in R_+^p$ satisfying (4.1)-(4.5). Suppose that there exist $F,\theta,\phi_0,b_0,\rho_0$ and ϕ_1,b_1,ϕ_1,ρ_1 such that

$$F_i\left(x,x^*;\sum_{i=1}^{s}t_i^*\left(\nabla f_i(x^*,\overline{y}_i)+Au-v^*\left(\nabla h_i(x^*,\overline{y}_i)-Bv\right)\right)\right)\ge -\rho_0\left\|\theta(x,x^*)\right\|^2$$

$$\Rightarrow b_0(x, x^*)\,\phi_0\left(\sum_{i=1}^{s} t_i^*\left(f_i(x, \bar{y}_i) + \langle x, Au\rangle - v^*\left(h_i(x, \bar{y}_i)\right) - \langle x, Bv\rangle\right)\right) \quad (4.7)$$

$$-\sum_{i=1}^{s} t_i^*\left(f_i(x^*, \bar{y}_i) + \langle x^*, Au\rangle - v^*\left(h_i(x^*, \bar{y}_i) - \langle x^*, Bv\rangle\right)\right) \geq 0 \qquad (4.24)$$

or equivalently,

$$b_0(x, x^*)\,\phi_0\left(\sum_{i=1}^{s} t_i^*\left(f_i(x, \bar{y}_i) + \langle x, Au\rangle - v^*\left(h_i(x, \bar{y}_i)\right) - \langle x, Bv\rangle\right)\right)$$

$$-\sum_{i=1}^{s} t_i^*\left(f_i(x^*, \bar{y}_i) + \langle x^*, Au\rangle - v^*\left(h_i(x^*, \bar{y}_i) - \langle x^*, Bv\rangle\right)\right) \leq 0 \qquad (4.25)$$

$$\Rightarrow F_i\left(x, x^*; \sum_{i=1}^{s} t_i^*\left(\nabla f_i(x^*, \bar{y}_i) + Au - v^*\left(\nabla h_i(x^*, \bar{y}_i) - Bv_i\right)\right)\right) \leq -\rho_0 \left\|\theta(x, x^*)\right\|^2$$

$$-b_1(x, x^*)\,\phi_1\left(\sum_{j=1}^{p} \mu_j^* g_j(x^*)\right) \leq 0 \qquad (4.26)$$

$$\Rightarrow F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right) \leq -\rho_1 \left\|\theta(x, x^*)\right\|^2$$

Further, assume that (4.9), (4.11), (4.12) and

$$a \leq 0 \Rightarrow \phi_0(a) \leq 0 \qquad (4.27)$$

are satisfied, then x^* is an optimal solution of (FP).

Proof: Suppose to the contrary that x^* is not an optimal solution of (P). Following the proof of theorem (4.7), we get

$$\sum_{i=1}^{s} t_i^*\left(f_i(x, \bar{y}_i) + \langle x, Au\rangle - v_i^*\left(h_i(x, \bar{y}_i) - \langle x, Bv\rangle\right)\right)$$

$$< \sum_{i=1}^{s} f_i\left((x^*, \bar{y}_i) + \langle x^*, Au\rangle - v_i^*\left(h_i(x^*, \bar{y}_i) - \langle x^*, Bv\rangle\right)\right)$$

$$(4.28)$$

Using (4.11), (4.25), (4.27) and (4.28), we have

$$F_i\left(x, x^*; \sum_{i=1}^{s} t_i^*\left(\nabla f_i(x^*, \bar{y}_i) + Au - v^*\left(\nabla h_i(x^*, \bar{y}_i) - Bv\right)\right)\right) < -\rho_0 \left\|\theta(x, x^*)\right\|^2$$

$$(4.29)$$

$$\Rightarrow F_i\left(x, x^*; \sum_{i=1}^{s} t_i^*\left(\nabla f_i(x^*, \bar{y}_i) + Au - v^*\left(\nabla h_i(x^*, \bar{y}_i) - Bv\right)\right)\right) + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*) < 0$$

On the other hand, (4.1) implies

$$F_i\left(x, x^*; \sum_{i=1}^{s} t_i^*\left(\nabla f_i(x^*, \bar{y}_i) + Au - v^*\left(\nabla h_i(x^*, \bar{y}_i) - Bv\right) + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right)\right) = 0$$

$$(4.30)$$

Hence we have a contradiction to inequality (4.29). Therefore, x^* is an optimal solution for (P). This completes the proof.

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