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Transformation and Implementation of a Highly Efficient Fully Implicit Fourth-Order Runge-Kutta Method

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Abstract:
The essence of this work is to transform, derive and implement a fully implicit fourth-order Runge-Kutta method of the third-stage that is highly efficient with a view of adopting a new approach. This new approach is done by separating the f(y) functional derivatives and their derived equations from the f(x,y) functional derivatives and their derived equations. Efforts will be made to vary the parameters with the aim of getting a fully implicit fourth order formula that can improve results when implemented on ordinary differential equations (ODEs). Efforts will also be made to compare both functional derivatives and their derived equations to see how related they are.

Keywords: *Implicit, f(x,y) functional derivatives, Runge-Kutta Methods, Linear and non- linear equations, Taylor series, Parameters, Initial-value Problems,f(y) functional derivatives, efficient, transformation.*

1. Introduction

Implicit Runge-Kutta methods are more stable and hence, very effective for solving ordinary differential equations (ODEs).Implicit Runge-Kutta methods were earlier developed by Kuntzmann (Butcher, 1964,1988) etc. Early work in this area also include that of (Butcher and Jackiewicz, 1997), (Yakubu, 2010) did a recent work which gave good results. A more recent work is found in (Agam, Yahaya, 2014). The works of (Agbeboh, Esekhaigbe, 2015), (Agbeboh, 2013) revealed that a Runge-Kutta method can be transformed to enhance higher accuracy. The transformation done in this work came as a motivation from their work. There are at least four basic reasons for taking a serious interest in implicit Runge-Kutta methods. The reasons are:

a. Higher orders of accuracy can be obtained than for explicit methods.
b. For linear systems of differential equations, implicit methods can be implemented explicitly.
c. For stiff problems, explicit methods are never satisfactory, whereas, some implicit methods are.
d. The structure of certain high-order explicit methods can be derived directly from some related implicit methods.

Conclusively, the implicit Runge-Kutta formulas are highly more efficient and dependable than the explicit formulas because they have the potentials to improve weak stability characteristics.

2. Methods of Derivation

i. From the general R – Stage Implicit Runge – Kutta Method, get $K_r, r = 1,2,3 \ldots\ldots R$, Obtain the Taylor series expansion of the $K_r, r = 1,2,3 \ldots\ldots R$,

ii. $Since\ k_1, k_2\ and\ k_3\ are\ implicit, we\ cannot\ proceed\ by\ successive\ substitution$
$as\ in\ the\ case\ of\ explicit. we\ assume\ that\ the\ solutions\ for\ k_1, k_2, k_3\ may$
$be\ expressed\ in\ the\ form$: $K_r = \alpha_r + h\beta_r + h^2\theta_r + h^3\emptyset_r + O(h^4), r = 1,2,3$

iii. $Puting\ the\ k_1,\ k_2\ and\ k_3\ from\ (ii)\ into\ k_r\ expanded\ by\ Taylor's\ series\ above$
$from\ (i)$, equate coefficients in power of h to get α_r,β_r,θ_r and $\emptyset_r$, $r = 1,2,3 \ldots\ldots R$.

iv. Also, $Put\ the\ k_1,\ k_2\ and\ k_3\ \ from\ (ii)\ into\ \emptyset(x,y,h) = \sum_{r=1}^{3} c_r k_r = c_1k_1 + c_2k_2 + c_3k_3$ and arrange in power of h.

v. Equate $\emptyset(x,y,h) from$ (iv) above and compare with Taylor's series of the form:

$$\emptyset_T(x,y,h) = f + \frac{h}{2!}\left(f_x + ff_y\right) + \frac{h^2}{3!}\left(f_{xx} + 2ff_{xy} + f_xf_y + ff_y^2 + f^2f_{yy}\right) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + 4f^2f_yf_{yy} + f_xf_y^2 + ff_y^3 + f^3f_{yyy}) .$$

vi. as a result, a set of linear/ non-linear equations will be generated.
vii. Separate the f(y) functional derivatives and their equations from the f(x,y) functional derivatives and their equations, which are the same eventually.
viii. Solve any of the two sets of equation to derive a new fourth-order third-stage fully implicit Runge-Kutta formula.

3. Derivation of the Fourth-Order Third-Stage Fully Implicit R-K Method

From Lambert (1977), the general R – Stage Implicit Runge – Kutta Method is

$$y_{n+1} - y_n = h\,\emptyset\,(x_n, y_n, h),$$

$$\emptyset(x,y,h) = \sum_{r=1}^{R} C_rK_r$$

$$K_r = f\left(x + ha_r, y + h\sum_{s=1}^{R} b_{rs}\,k_s\right), r = 1,2,3 \ldots\ldots R,$$

$a_r = \sum_{s=1}^{R} b_{rs},\ r = 1,2,3 \ldots\ldots R,$ (1)

$$For\ a\ third-stage, R = 3, \quad K_r\ becomes:$$

$K_r = f(x + ha_r, y + h\,(b_{r1}k_1 + b_{r2}k_2 + b_{r2}k_2)$ (2)

$Expanding\ (2) with\ Taylor\ series\ about\ the\ point\ (x,y), we\ obtain:$

$$K_r = \sum_{p=0}^{\infty}\frac{1}{p!}(ha_r\frac{\partial}{\partial x} + h\,(b_{r1}k_1 + b_{r2}k_2 + b_{r2}k_2)\,\frac{\partial}{\partial y})^P\,f(x,y)$$

$$K_r = f + h\left[a_rf_x + \ (b_{r1}k_1 + b_{r2}k_2 + b_{r2}k_2 + b_{r3}k_3)f_y\right] + \frac{1}{2!}h^2[a_r^2f_{xx} + 2a_r(b_{r1}k_1 + b_{r2}k_2 + b_{r2}k_2 + b_{r3}k_3)f_{xy} + (b_{r1}k_1 + b_{r2}k_2 + b_{r2}k_2 + b_{r3}k_3)^2f_{yy}] + \frac{1}{3!}h^3[a_r^3f_{xxx} + 3a_r(b_{r1}k_1 + b_{r2}k_2 + b_{r2}k_2 + b_{r3}k_3)f_{xxy} + 3a_r(b_{r1}k_1 + b_{r2}k_2 + b_{r2}k_2 + b_{r3}k_3)^2f_{xyy} + \ (b_{r1}k_1 + b_{r2}k_2 + b_{r2}k_2 + b_{r3}k_3)^3f_{yyy}] + 0(h^2), r = 1,2,3 \quad (3)$$

$Since\ k_1, k_2\ and\ k_3\ are\ implicit, we\ cannot\ proceed\ by\ successive\ substitution\ as\ in\ the\ case\ of\ explicit. we\ assume\ that\ the\ solutions\ for\ k_1, k_2, k_3\ may\ be\ expressed\ in\ the\ form:$

$K_r = \ \alpha_r + h\beta_r + h^2\theta_r + h^3\emptyset_r + \ O(h^4), r = 1,2,3$ (4)

$$Puting\ the\ k_1,\ k_2\ and\ k_3\ into\ k_r, we\ have:$$

$$K_r = f + \ ha_rf_x + hbr_1(\alpha_1 + h\beta_1 + h^2\theta_1)f_y + \ hb_{r2}(\alpha_2 + h\beta_2 + h^2\theta_2)f_y + hb_{r3}(\alpha_3 + h\beta_3 + h^2\theta_3)f_y + \frac{h^2}{2!}a_r^2f_{xx} + \ h^2arb_{r1}(\alpha_1{+}h\beta_1)f_{xy} + h^2arb_{r2}(\alpha_2{+}h\beta_2)f_{xy} + h^2arb_{r3}(\alpha_3{+}h\beta_3)f_{xy} + \frac{h^2}{2!}b_{r1}^2(\alpha_1 + 2h\alpha_1\beta_1)f_{yy} + h^2br_1br_2(\alpha_1\alpha_2{+}h\alpha_1\beta_2{+}h\beta_1\alpha_2)f_{yy} + h^2arb_{r3}(\alpha_1\alpha_3{+}h\alpha_1\beta_3{+}h\beta_1\alpha_3)f_{yy} + \frac{h^2}{2!}b_{r2}^2(\alpha_2^2{+}2h\alpha_2\beta_2)f_{yy} + h^2br_2br_3(\alpha_2\alpha_3{+}h\alpha_2\beta_3{+}h\beta_2\alpha_3)f_{yy} + \frac{h^2}{2!}b_{r3}^2(\alpha_3^2{+}2h\alpha_3\beta_3)f_{yy} + \ \frac{h^3}{3!}a_r^3f_{xxx} + \frac{h^3}{2!}a_r^2b_{r1}\alpha_1f_{xxy} + \frac{h^3}{2!}a_r^2b_{r2}\alpha_2f_{xxy} + \frac{h^3}{2!}a_r^2b_{r3}\alpha_3f_{xxy} + \frac{h^3}{2!}arb_{r1}^2\alpha_1^2f_{xyy} + h^3arb_{r1}b_{r2}\alpha_1\alpha_2f_{xyy} +$$

$h^3 arb_{r1}b_{r3}\alpha_1\alpha_3 f_{xyy} + \frac{h^3}{2!} arb_{r2}^2\alpha_2^2 f_{xyy} + h^3 arb_{r2}b_{r3}\alpha_2\alpha_3 f_{xyy} +$

$$\frac{h^3}{2!} arb_{r3}^2\alpha_3^2 f_{xyy} + \frac{h^3}{3!}(b_{r1}^3\alpha_1^3 + 3b_{r1}^2 b_{r2}\alpha_1^2\alpha_2 + 3b_{r1}^2 b_{r3}\alpha_1^2\alpha_3 + 3b_{r1}b_{r2}^2\alpha_1\alpha_2^2 + 6b_{r1}b_{r2}b_{r3}\alpha_1\alpha_2\alpha_3 + 3b_{r1}b_{r3}^2\alpha_1\alpha_3^2 + b_{r2}^3\alpha_2^3 + 3b_{r2}^2 b_{r3}\alpha_2^2\alpha_3 +$$

$3b_{r2}b_{r3}^2\alpha_2\alpha_3^2 + b_{r3}^3\alpha_3^3)f_{yyy} + O(h^4)$ (5)

Equating (4) *with* (5) *in powers of* h, *we obtain*:

$$\alpha_r = f$$

$$\beta_r = \alpha_r f_x + (b_{r1}\alpha_1 + b_{r2}\alpha_2 + b_{r3}\alpha_3)f_y$$

$$\theta_r = (b_{r1}\beta_1 + b_{r2}\beta_2 + b_{r3}\beta_3)f_y + \frac{1}{2!}a_r^2 f_{xx} + (a_r b_{r1}\alpha_1 + a_r b_{r2}\alpha_2 + a_r b_{r3}\alpha_3)f_{xy} + \frac{1}{2}(b_{r1}^2\alpha_1^2 + 2b_{r1}b_{r2}\alpha_1\alpha_2 + 2b_{r1}b_{r3}\alpha_1\alpha_3 + b_{r2}^2\alpha_2^2 + 2b_{r2}b_{r3}\alpha_2\alpha_3 + b_{r3}^2\alpha_3^2)f_{yy}$$

$$\emptyset_r = (b_{r1}\theta_1 + b_{r2}\theta_2 + b_{r3}\theta_3)f_y + (a_r b_{r1}\beta_1 + a_r b_{r2}\beta_2 + a_r b_{r3}\beta_3)f_{xy} + (b_{r1}^2\beta_1 + b_{r1}b_{r2}\alpha_1\beta_2 + b_{r1}b_{r2}\beta_1\alpha_2 + b_{r1}b_{r3}\alpha_1\beta_3 + b_{r1}b_{r3}\beta_1\alpha_3 + b_{r2}^2\alpha_2\beta_2 + b_{r2}b_{r3}\alpha_2\beta_3 + b_{r2}b_{r3}\beta_2\alpha_3 + b_{r3}^2\alpha_3\beta_3)f_{yy} + \frac{1}{6}a_r^3 f_{xxx} + \frac{1}{2}a_r^2(b_{r1}\alpha_1 + b_{r2}\alpha_2 + b_{r3}\alpha_3) f_{xxy} + \frac{1}{2}a_r(b_{r1}^2 a_1^2 + 2b_{r1}b_{r2}\alpha_1\alpha_2 + 2b_{r1}b_{r3}\alpha_1\alpha_3 + b_{r2}^2\alpha_2^2 + 2b_{r2}b_{r3}\alpha_2\alpha_3 + b_{r3}^2 a_3^2) f_{xyy} + \frac{1}{6}(b_{r1}^3 a_1^3 + 3b_{r1}^2 b_{r2}a_1^2\alpha_2 + 3b_{r1}^2 b_{r3}\alpha_1^2\alpha_3 + 3b_{r1}b_{r2}^2\alpha_1\alpha_2^2 + 6b_{r1}b_{r2}b_{r3}\alpha_1\alpha_2\alpha_3 + 3b_{r1}b_{r3}^2\alpha_1\alpha_3^2 + b_{r2}^3\alpha_2^3 + 3b_{r2}^2 b_{r3}\alpha_2^2\alpha_3 +$$

$3b_{r2}b_{r2}^3\alpha_2\alpha_3^2 + b_{r3}^3\alpha_3^3)f_{yyy} + O(h^4),\ r = 1, 2, 3$ (6)

Putting the $r_j, r = 1, 2, 3$ *into* $\alpha_r, \beta_r, \theta_r$ *and* $\emptyset_r$

$\alpha_1 = f, \alpha_2 = f, \alpha_3 = f$ (7)

$$\beta_1 = \alpha_1 f_x + (b_{11} + b_{12} + b_{13})ff_y$$

$$\beta_2 = \alpha_2 f_x + (b_{21} + b_{22} + b_{23})ff_y$$

$\beta_3 = \alpha_3 f_x + (b_{31} + b_{32} + b_{33})ff_y$ (8)

$$\theta_1 = \frac{1}{2}a_1^2 f_{xx} + (b_{11}a_1 + b_{21}a_1 + b_{31}a_1)f_x f_y + (b_{11}^2 + b_{11}b_{12} + b_{11}b_{13} + b_{12}b_{21} + b_{12}b_{23} + b_{13}b_{31} + b_{13}b_{32} + b_{13}b_{33})ff_y^2 + (a_1 b_{11} + a_1 b_{12} + a_1 b_{13})ff_{xy} + \frac{1}{2}(b_{11}^2 + 2b_{11}b_{12} + 2b_{11}b_{13} + b_{12}^2 + 2b_{12}b_{13} + b_{13}^2)f^2 f_{yy}$$

$$\theta_2 = \frac{1}{2}a_2^2 f_{xx} + (b_{21}a_1 + b_{22}a_2 + b_{23}a_3)f_x f_y + (b_{11}b_{21} + b_{12}b_{21} + b_{13}b_{21} + b_{21}b_{22} + b_{22}^2 + b_{22}b_{23} + b_{23}b_{31} + b_{23}b_{32} + b_{23}b_{33})ff_y^2 + (a_2 b_{21} + a_2 b_{22} + a_2 b_{23})ff_{xy} + \frac{1}{2}(b_{21}^2 + 2b_{21}b_{22} + 2b_{21}b_{23} + b_{22}^2 + 2b_{22}b_{23} + b_{23}^2)f^2 f_{yy}$$

$$\theta_3 = \frac{1}{2}a_3^2 f_{xx} + (b_{31}a_1 + b_{32}a_2 + b_{33}a_3)f_x f_y + (b_{31}b_{11} + b_{31}b_{12} + b_{31}b_{13} + b_{32}b_{21} + b_{32}b_{22} + b_{32}b_{23} + b_{33}b_{31} + b_{33}b_{32} + b_{33}^2)ff_y^2 + (a_3 b_{31} + a_3 b_{32} +$$

$a_3 b_{33})ff_{xy} + \frac{1}{2}(b_{21}^2 + 2b_{31}b_{32} + 2b_{31}b_{33} + b_{32}^2 + 2b_{32}b_{33} + b_{33}^2)f^2 f_{yy}$ (9)

$$\emptyset_1 = \frac{1}{6}a_1^3 f_{xxx} + \frac{1}{2}(a_1^2 b_{11} + a_2^2 b_{12} + a_3^2 b_{13})f_{xx}f_y + (a_1 b_{11}^2 + a_2 b_{11}b_{12} + a_3 b_{11}b_{13} + a_1 b_{12}b_{21} + a_2 b_{12}b_{22} + a_3 b_{12}b_{23} + a_1 b_{13}b_{31} + a_2 b_{13}b_{32} + a_3 b_{13}b_{33})f_x f_y^2 + (b_{11}^3 + b_{11}^2 b_{12} + b_{11}^2 b_{13} + b_{11}b_{12}b_{21} + b_{11}b_{12}b_{22} + b_{11}b_{12}b_{23} + b_{11}b_{13}b_{31} + b_{11}b_{13}b_{32} + b_{11}b_{13}b_{33} + b_{11}b_{12}b_{21} + b_{12}^2 b_{21} + b_{12}b_{13}b_{21} + b_{12}b_{13}b_{21} + b_{12}b_{22}^2 + b_{12}b_{22}b_{23} + b_{12}b_{23}b_{31} + b_{12}b_{23}b_{32} + b_{12}b_{23}b_{33} + b_{13}b_{31}b_{11} + b_{13}b_{31}b_{12} + b_{13}^2 b_{31}$$

$$+b_{13}b_{32}b_{21} + b_{13}b_{32}b_{22} + b_{13}b_{32}b_{23} + b_{13}b_{33}b_{31} + b_{13}b_{33}b_{32} + b_{13}b_{33}^2)ff_y^3 +$$
$$a_1b_{11}^2 + a_1b_{11}b_{12} + a_1b_{11}b_{13} + a_2b_{12}b_{21} + a_2b_{12}b_{22} + a_2b_{12}b_{23} + a_3b_{13}b_{31}$$
$$+a_3b_{13}b_{32} + a_3b_{13}b_{33} + a_1b_{11}^2 + a_1b_{11}b_{12} + a_1b_{11}b_{13} + a_1b_{12}b_{21} + a_1b_{12}b_{22} +$$
$$a_1b_{12}b_{23} + a_1b_{13}b_{31} + a_1b_{13}b_{32} + a_1b_{13}b_{33})ff_yf_{xy} + \frac{1}{2}(b_{11}^3 + 2b_{11}^2b_{12} +$$
$$2b_{11}^2b_{13} + b_{11}b_{12}^2 + 2b_{11}b_{13}b_{12} + b_{11}b_{13}^2 + b_{12}b_{21}^2 + 2b_{12}b_{21}b_{22} + 2b_{12}b_{21}b_{23}$$
$$+b_{12}b_{22}^2 + 2b_{12}b_{22}b_{23} + b_{12}b_{23}^2 + b_{13}b_{31}^2 + 2b_{13}b_{31}b_{32} + 2b_{13}b_{31}b_{33} + b_{13}b_{32}^2$$
$$+2b_{13}b_{32}b_{33} + b_{13}b_{33}^2 + 2b_{11}^3 + 2b_{11}^2b_{12} + 2b_{11}^2b_{13} + 2b_{11}b_{12}b_{21} + 2b_{11}b_{12}b_{22}$$
$$+2b_{11}b_{12}b_{23} + 2b_{11}^2b_{12} + 2b_{11}b_{12}^2 + 2b_{11}b_{12}b_{13} + 2b_{11}b_{13}b_{31} + 2b_{11}b_{13}b_{32} +$$
$$2b_{11}b_{13}b_{33} + 2b_{11}^2b_{13} + 2b_{11}b_{12}b_{13} + 2b_{11}b_{13}^2 + 2b_{12}^2b_{21} + 2b_{12}^2b_{22} + 2b_{12}^2b_{23} +$$
$$2b_{12}b_{13}b_{31} + 2b_{12}b_{13}b_{32} + 2b_{12}b_{13}b_{33} + 2b_{12}b_{13}b_{21} + 2b_{12}b_{13}b_{22} + 2b_{12}b_{13}b_{23}$$
$$+2b_{13}^2b_{31} + 2b_{13}^2b_{32} + 2b_{13}^2b_{33})f^2f_yf_{yy} + (a_1^2b_{11} + a_1a_2b_{12} + a_1a_3b_{13})f_xf_{xy} +$$
$$(a_1b_{11}^2 + a_2b_{11}b_{12} + a_1b_{11}b_{12} + a_3b_{11}b_{13} + a_1b_{11}b_{13} + a_2b_{12}^2 + a_3b_{12}b_{13} +$$
$$a_2b_{12}b_{13} + a_3b_{13}^2)ff_xf_{yy} + \frac{1}{2}a_1^2(b_{11} + b_{12} + b_{13})ff_{xxy} + \frac{1}{2}a_1(b_{11}^2 +$$
$$2b_{11}b_{12} + 2b_{11}b_{13} + b_{12}^2 + 2b_{12}b_{13} + b_{13}^2)f^2f_{xyy} + \frac{1}{6}(b_{11}^3 + 3b_{11}^2b_{12} +$$
$$3b_{11}^2b_{13} + 3b_{11}b_{12}^2 + 6b_{11}b_{12}b_{13} + 3b_{11}b_{13}^2 + b_{12}^3 + 3b_{12}^2b_{13} + 3b_{12}b_{13}^2 + b_{13}^3)f^3f_{yyy}$$

$$\emptyset_2 = \frac{1}{6}a_2^3f_{xxx} + \frac{1}{2}(a_1^2b_{21} + a_2^2b_{22} + a_3^2b_{23})f_{xx}f_y + (a_1b_{11}b_{12} + a_2b_{21}b_{12} + a_3b_{21}b_{13} +$$
$$a_1b_{21}b_{22} + a_2b_{22}^2 + a_3b_{22}b_{23} + a_1b_{23}b_{31} + a_2b_{23}b_{32} + a_3b_{23}b_{33})f_xf_y^2 + (b_{21}b_{11}^2 +$$
$$b_{21}b_{11}b_{12} + b_{21}b_{11}b_{13} + b_{21}^2b_{12} + b_{21}b_{12}b_{22} + b_{21}b_{12}b_{23} + b_{21}b_{13}b_{31} + b_{21}b_{13}b_{32}$$
$$+b_{21}b_{13}b_{33} + b_{22}b_{11}b_{21} + b_{22}b_{12}b_{21} + b_{22}b_{13}b_{21} + b_{22}^2b_{21} + b_{22}^3 + b_{22}^2b_{23} +$$
$$b_{22}b_{23}b_{31} + b_{22}b_{23}b_{32} + b_{22}b_{23}b_{33} + b_{23}b_{31}b_{11} + b_{23}b_{31}b_{12} + b_{23}b_{31}b_{13} +$$
$$b_{23}b_{32}b_{21} + b_{23}b_{32}b_{22} + b_{23}^2b_{32} + b_{23}b_{33}b_{31} + b_{23}b_{33}b_{32} + b_{23}b_{33}^2)ff_y^3 +$$
$$(a_1b_{11}b_{21} + a_1b_{21}b_{12} + a_1b_{21}b_{13} + a_2b_{22}b_{21} + a_2b_{22}^2 + a_2b_{22}b_{23} + a_3b_{23}b_{31} +$$
$$a_3b_{23}b_{32} + a_3b_{23}b_{33} + a_2b_{21}b_{11} + a_2b_{21}b_{12} + a_2b_{21}b_{13} + a_2b_{22}b_{21} +$$
$$a_2b_{22}^2 + a_2b_{22}b_{23} + a_2b_{23}b_{31} + a_2b_{23}b_{32} + a_2b_{23}b_{33})ff_yf_{xy} + \frac{1}{2}(b_{21}b_{11}^2 +$$
$$2b_{21}b_{11}b_{12} + 2b_{21}b_{11}b_{13} + b_{21}b_{12}^2 + 2b_{21}b_{12}b_{13} + b_{21}b_{13}^2 + b_{22}b_{21}^2 + 2b_{22}^2b_{21} +$$
$$2b_{22}b_{21}b_{23} + b_{22}^3 + 2b_{22}^2b_{23} + b_{22}b_{23}^2 + b_{23}b_{31}^2 + 2b_{23}b_{31}b_{32} + 2b_{23}b_{31}b_{33} + b_{23}b_{32}^2$$
$$+2b_{23}b_{32}b_{33} + b_{23}b_{33}^2 + 2b_{21}^2b_{11} + 2b_{21}^2b_{12} + 2b_{21}^2b_{13} + 2b_{21}^2b_{22} + 2b_{21}^2b_{22}$$
$$+2b_{21}b_{22}b_{23} + 2b_{21}b_{22}b_{11} + 2b_{21}b_{22}b_{12} + 2b_{21}b_{22}b_{13} + 2b_{21}b_{23}b_{31} + 2b_{21}b_{23}b_{32}$$
$$+2b_{21}b_{23}b_{33} + 2b_{21}b_{23}b_{11} + 2b_{21}b_{23}b_{12} + 2b_{21}b_{23}b_{13} + 2b_{22}^2b_{21} + 2b_{22}^3 +$$
$$2b_{22}^2b_{23} + 2b_{22}b_{23}b_{31} + 2b_{22}b_{23}b_{32} + 2b_{22}b_{23}b_{33} + 2b_{22}b_{23}b_{21} + 2b_{22}^2b_{23}$$
$$+2b_{22}b_{23}^2 + 2b_{23}^2b_{31} + 2b_{23}^2b_{32} + 2b_{23}^2b_{33})f^2f_yf_{yy} + (a_1a_2b_{21} + a_2^2b_{22} + a_2a_3b_{23})$$
$$f_xf_{xy} + (a_1b_{21}^2 + a_2b_{21}b_{22} + a_1b_{21}b_{22} + a_3b_{21}b_{23} + a_1b_{21}b_{23} + a_2b_{22}^2$$
$$+a_3b_{22}b_{23} + a_2b_{22}b_{23} + a_3b_{23}^2)ff_xf_{yy} + \frac{1}{2}a_2^2(b_{21} + b_{22} + b_{23})ff_{xxy} + \frac{1}{2}a_2(b_{21}^2 +$$
$$2b_{21}b_{22} + 2b_{21}b_{23} + b_{22}^2 + 2b_{22}b_{23} + b_{23}^2)f^2f_{xyy} + \frac{1}{6}(b_{21}^3 + 3b_{21}^2b_{22} +$$
$$3b_{21}^2b_{23} + 3b_{21}b_{22}^2 + 6b_{21}b_{22}b_{23} + 3b_{21}b_{23}^2 + b_{22}^3 + 3b_{22}^2b_{23} + 3b_{22}b_{23}^2 + b_{23}^3)f^3f_{yyy}$$

$$\emptyset_3 = \frac{1}{6}a_3^3f_{xxx} + \frac{1}{2}(a_1^2b_{31} + a_2^2b_{32} + a_3^2b_{33})f_{xx}f_y + (a_1b_{31}b_{11} + a_2b_{31}b_{12} +$$
$$a_3b_{31}b_{13} + a_1b_{32}b_{21} + a_2b_{32}b_{22} + a_3b_{32}b_{23} + a_1b_{31}b_{31} + a_2b_{33}b_{32} + a_3b_{33}^2)f_xf_y^2 +$$
$$(b_{31}b_{11}^2 + b_{31}b_{11}b_{12} + b_{31}b_{11}b_{13} + b_{31}b_{12}b_{21} + b_{31}b_{12}b_{22} + b_{31}b_{12}b_{23} + b_{31}^2b_{13}$$
$$+b_{31}b_{13}b_{32} + b_{31}b_{13}b_{33} + b_{32}b_{11}b_{21} + b_{32}b_{12}b_{21} + b_{32}b_{13}b_{21} + b_{32}b_{13}b_{22} +$$
$$b_{32}b_{22}^2 + b_{32}b_{22}b_{23} + b_{32}b_{23}b_{31} + b_{32}^2b_{23} + b_{32}b_{23}b_{33} + b_{33}b_{31}b_{11} + b_{33}b_{31}b_{12}$$
$$+b_{33}b_{31}b_{21} + b_{33}b_{32}b_{22} + b_{33}b_{32}b_{23} + b_{33}^2b_{31} + b_{33}^2b_{32} + b_{33}^3)ff_y^3 + (a_1b_{31}b_{11}$$

$$+a_1b_{31}b_{12} + a_1b_{31}b_{13} + a_2b_{32}b_{21} + a_2b_{32}b_{22} + a_2b_{32}b_{23} + a_3b_{33}b_{31} + a_3b_{33}b_{32} + a_3b_{33}^2 + a_3b_{31}b_{11} + a_3b_{31}b_{12} + a_3b_{31}b_{13} + a_3b_{32}b_{21} + a_3b_{32}b_{22} + a_3b_{32}b_{23} + a_3b_{33}b_{31} + a_3b_{33}b_{32} + a_3b_{33}^2)ff_yf_{xy} + \frac{1}{2}(b_{31}b_{11}^2 + 2b_{31}b_{11}b_{12} + 2b_{31}b_{11}b_{13} + b_{31}b_{21}^2 + 2b_{31}b_{12}b_{13} + b_{31}b_{13}^2 + b_{32}b_{21}^2 + 2b_{32}b_{21}b_{22} + 2b_{32}b_{21}b_{23} + b_{32}b_{22}^2 + 2b_{32}b_{22}b_{23} + b_{32}b_{23}^2 + b_{33}b_{31}^2 + 2b_{33}b_{31}b_{32} + 2b_{33}^2b_{31} + b_{33}b_{32}^2 + 2b_{33}^2b_{32} + b_{33}^3 + 2b_{31}^2b_{11} + 2b_{31}^2b_{12} + 2b_{31}^2b_{13} + 2b_{31}b_{32}b_{21} + 2b_{31}b_{32}b_{22} + 2b_{31}b_{32}b_{11} + 2b_{31}b_{32}b_{12} + 2b_{31}b_{32}b_{13} + 2b_{31}^2b_{33} + 2b_{31}b_{32}b_{33} + 2b_{31}b_{33}^2 + 2b_{31}b_{33}b_{11} + 2b_{31}b_{33}b_{12} + 2b_{31}b_{33}b_{13} + 2b_{32}^2b_{21} + 2b_{32}^2b_{22} + 2b_{32}^2b_{23} + 2b_{32}b_{33}b_{31} + 2b_{32}^2b_{33} + 2b_{32}b_{33}^2 + 2b_{32}b_{33}b_{21} + 2b_{32}b_{33}b_{22} + 2b_{32}b_{33}b_{23} + 2b_{33}^2b_{31} + 2b_{33}^2b_{32} + 2b_{33}^3)f^2f_yf_{yy} + (a_1a_3b_{31} + a_2a_3b_{32} + a_3^2b_{33})f_xf_{xy} + (a_1b_{31}^2 + a_2b_{31}b_{32} + a_1b_{31}b_{32} + a_3b_{31}b_{33} + a_1b_{31}b_{33} + a_3b_{32}b_{33} + a_2b_{32}^2 + a_2b_{32}b_{33} + a_3b_{33}^2)ff_xf_{yy} + \frac{1}{2}a_3^2(b_{31} + b_{32} + b_{33})ff_{xxy} + \frac{1}{2}a_3(b_{31}^2 + 2b_{31}b_{32} + 2b_{31}b_{33} + b_{32}^2 + 2b_{32}b_{33} + b_{33}^2) f^2f_{xyy} + \frac{1}{6}(b_{31}^3 + 3b_{31}^2b_{32} + 3b_{31}^2b_{33} + 3b_{31}b_{32}^2 + 6b_{31}b_{32}b_{33} + 3b_{31}b_{33}^2 + b_{32}^3 + 3b_{32}^2b_{33} + 3b_{32}b_{33}^2 + b_{33}^3)f^3f_{yyy} \quad (10)$$

From (4), we know that:

$$k_1 = \alpha_1 + h\beta_1 + h^2\theta_1 + h^3\emptyset_1$$
$$k_2 = \alpha_2 + h\beta_2 + h^2\theta_2 + h^3\emptyset_2$$
$$k_3 = \alpha_3 + h\beta_3 + h^2\theta_3 + h^3\emptyset_3 \quad (11)$$

$Also\ from$ (1), $\emptyset(x,y,h) = \sum_{r=1}^{3} c_rk_r = c_1k_1 + c_2k_2 + c_3k_3$ (12)

Putting (11) into (12), we have:

$$\emptyset(x,y,h) = c_1(\alpha_1 + h\beta_1 + h^2\theta_1 + h^3\emptyset_1) + c_2(\alpha_2 + h\beta_2 + h^2\theta_2 + h^3\emptyset_2) + c_3(\alpha_3 + h\beta_3 + h^2\theta_3 + h^3\emptyset_3) \quad (13)$$

Arranging (13) in power of h, we have:

$$\emptyset(x,y,h) = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + h(c_1\beta_1 + c_2\beta_2 + c_3\beta_3) + h^2(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) + h^3(c_1\emptyset_1 + c_2\emptyset_2 + c_3\emptyset_3) + O(h^4) \quad (14)$$

The necessary Taylor's series expansion is:

$$\emptyset_T(x,y,h) = f + \frac{h}{2!}(f_x + ff_y) + \frac{h^2}{3!}(f_{xx} + 2ff_{xy} + f_xf_y + ff_y^2 + f^2f_{yy}) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + 4f^2f_yf_{yy} + f_xf_y^2 + ff_y^3 + f^3f_{yyy}) \quad (15)$$

Putting (7),(8),(9), and (10) into (14) and equating with (15), we have the equations below:

$$(c_1 + c_2 + c_3)f = f \quad (16)$$

$$h(c_1a_1 + c_2a_2 + c_3a_3)f_x = \frac{h}{2!}f_x \quad (17)$$

$$h[c_1(b_{11} + b_{12} + b_{13}) + c_2(b_{21} + b_{22} + b_{23}) + c_3(b_{31} + b_{32} + b_{33})]ff_y = \frac{h}{2!}ff_y \quad (18)$$

$$\frac{h^2}{2}(c_1a_1^2 + c_2a_2^2 + c_3a_3^2)f_{xx} = \frac{h^2}{3!}f_{xx} \quad (19)$$

$$h[c_1(b_{11}a_1 + b_{12}a_2 + b_{13}a_3) + c_2(b_{21}a_1 + b_{22}a_2 + b_{23}a_3) + c_3(b_{31}a_1 + b_{32}a_2 + b_{33}a_3)]f_xf_y = \frac{h^2}{3!}f_xf_y \quad (20)$$

$$h^2[c_1(b_{11}^2 + b_{11}b_{12} + b_{11}b_{13} + b_{12}b_{21} + b_{12}b_{22} + b_{12}b_{23} + b_{13}b_{31} + b_{13}b_{32} + b_{13}b_{33}) + c_2(b_{11}b_{21} + b_{12}b_{21} + b_{13}b_{21} + b_{21}b_{22} + b_{22}^2 + b_{22}b_{23} + b_{23}b_{31} + b_{23}b_{32} + b_{23}b_{33}) + c_3(b_{31}b_{11} + b_{31}b_{12} + b_{31}b_{13} + b_{32}b_{21} + b_{32}b_{22} + b_{32}b_{23} + b_{33}b_{31} + b_{33}b_{32} + b_{33}^3)] ff_y^2 = \frac{h^2}{3!}ff_y^2 \quad (21)$$

$$h^2[c_1(a_1b_{11} + a_1b_{12} + a_1b_{13}) + c_2(a_2b_{21} + a_2b_{22} + a_2b_{23}) +$$
$$c_3(a_3b_{31} + a_3b_{32} + a_3b_{33})]ff_{xy} = \frac{2h^2}{3!}ff_{xy} \qquad (22)$$

$$\frac{h^2}{2}[c_1(b_{11}^2 + 2b_{11}b_{12} + 2b_{11}b_{13} + b_{12}^2 + 2b_{12}b_{13} + b_{13}^2) + c_2(b_{21}^2 + 2b_{21}b_{22} +$$
$$2b_{21}b_{23} + b_{22}^2 + 2b_{22}b_{23} + b_{23}^2) + c_3(b_{31}^2 + 2b_{31}b_{32} + 2b_{31}b_{33} + b_{32}^2 + 2b_{32}b_{33} + b_{33}^2)$$
$$]f^2f_{yy} = \frac{h^2}{3!}f^2f_{yy} \qquad (23)$$

$$\frac{h^3}{6}(c_1a_1^3 + c_2a_2^3 + c_3a_3^3)f_{xxx} = \frac{h^3}{4}f_{xxx} \qquad (24)$$

$$\frac{h^3}{2}[c_1(a_1^2b_{11} + a_2^2b_{12} + a_3^2b_{13}) + c_2(a_1^2b_{21} + a_2^2b_{22} + a_3^2b_{23}) + c_3(a_1^2b_{31} +$$
$$a_2^2b_{32} + a_3^2b_{33})]f_{xx}f_y = \frac{h^3}{4!}f_{xx}f_y \qquad (25)$$

$$h^3[c_1(a_1b_{11}^2 + a_2b_{11}b_{12} + a_3b_{11}b_{13} + a_1b_{12}b_{21} + a_2b_{12}b_{22} + a_3b_{12}b_{23} +$$
$$a_1b_{13}b_{31} + a_2b_{13}b_{32} + a_3b_{13}b_{33}) + c_2(a_1b_{11}b_{21} + a_2b_{21}b_{12} + a_3b_{21}b_{13} +$$
$$a_1b_{21}b_{22} + a_2b_{22}^2 + a_3b_{22}b_{23} + a_1b_{23}b_{31} + a_2b_{23}b_{32} + a_3b_{23}b_{33}) + c_3(a_1b_{31}b_{11} +$$
$$a_2b_{31}b_{12} + a_3b_{31}b_{13} + a_1b_{32}b_{21} + a_2b_{32}b_{22} + a_3b_{32}b_{23} + a_1b_{33}b_{31} + a_2b_{33}b_{32} +$$
$$a_3b_{33}^2)f_xf_y^2 = \frac{h^3}{4!}f_xf_y^2 \qquad (26)$$

$$h^3[c_1(b_{11}^3 + b_{11}^2b_{12} + b_{11}^2b_{13} + b_{11}b_{12}b_{21} + b_{11}b_{12}b_{22} + b_{11}b_{12}b_{23} + b_{11}b_{13}b_{31}$$
$$+b_{11}b_{13}b_{32} + b_{11}b_{13}b_{33} + b_{11}b_{12}b_{21} + b_{12}^2b_{21} + b_{12}b_{13}b_{21} + b_{12}b_{21}b_{22} +$$
$$b_{12}b_{22}^2 + b_{12}b_{22}b_{23} + b_{12}b_{23}b_{31} + b_{12}b_{23}b_{32} + b_{12}b_{23}b_{33} + b_{13}b_{31}b_{11} + b_{13}b_{31}b_{12}$$
$$+b_{13}^2b_{31} + b_{13}b_{32}b_{21} + b_{13}b_{32}b_{22} + b_{13}b_{32}b_{23} + b_{13}b_{33}b_{31} + b_{13}b_{33}b_{32} + b_{13}b_{33}^2)$$
$$+c_2(b_{21}b_{11}^2 + b_{21}b_{11}b_{12} + b_{21}b_{11}b_{13} + b_{21}^2b_{12} + b_{21}b_{12}b_{22} + b_{21}b_{12}b_{23} +$$
$$b_{21}b_{13}b_{31} + b_{21}b_{13}b_{32} + b_{21}b_{13}b_{33} + b_{22}b_{11}b_{21} + b_{22}b_{12}b_{21} + b_{22}b_{13}b_{21} + b_{22}^2b_{21}$$
$$+b_{22}^3 + b_{22}^2b_{23} + b_{22}b_{23}b_{31} + b_{22}b_{23}b_{32} + b_{22}b_{23}b_{33} + b_{23}b_{31}b_{11} + b_{23}b_{31}b_{12} +$$
$$+b_{23}b_{31}b_{13} + b_{23}b_{32}b_{21} + b_{23}b_{32}b_{22} + b_{23}^2b_{32} + b_{23}b_{33}b_{31} + b_{23}b_{33}b_{32} + b_{23}b_{33}^2)$$
$$+c_3(b_{31}b_{11}^2 + b_{31}b_{11}b_{12} + b_{31}b_{11}b_{13} + b_{31}b_{12}b_{21} + b_{31}b_{12}b_{22} + b_{31}b_{12}b_{23} +$$
$$b_{31}^2b_{13} + b_{31}b_{13}b_{32} + b_{31}b_{13}b_{33} + b_{32}b_{11}b_{21} + b_{32}b_{12}b_{21} + b_{32}b_{13}b_{21} + b_{32}b_{21}b_{22} +$$
$$b_{32}b_{22}^2b_{32}b_{22}b_{23} + b_{32}b_{23}b_{31} + b_{32}^2b_{23} + b_{32}b_{23}b_{33} + b_{33}b_{31}b_{11} + b_{33}b_{31}b_{12} +$$
$$b_{33}b_{31}b_{13} + b_{33}b_{32}b_{21} + b_{33}b_{32}b_{22} + b_{33}b_{32}b_{23} + b_{33}^2b_{31} + b_{33}^2b_{32} + b_{33}^3)]ff_y^3 =$$
$$\frac{h^3}{4!}ff_y^3 \qquad (27)$$

$$h^3[c_1(a_1b_{11}^2 + a_1b_{11}b_{12} + a_1b_{11}b_{13} + a_2b_{12}b_{21} + a_2b_{12}b_{22} + a_2b_{12}b_{23} +$$
$$a_3b_{13}b_{31} + a_3b_{13}b_{32} + a_3b_{13}b_{33} + a_1b_{11}^2 + a_1b_{11}b_{12} + a_1b_{11}b_{13} + a_1b_{12}b_{21} +$$
$$a_1b_{12}b_{22} + a_1b_{12}b_{23} + a_1b_{13}b_{31} + a_1b_{13}b_{32} + a_1b_{13}b_{33}) + c_2(a_1b_{11}b_{21} + a_1b_{21}b_{12}$$
$$+a_1b_{21}b_{13} + a_2b_{22}b_{21} + a_2b_{22}^2 + a_2b_{22}b_{23} + a_3b_{23}b_{31} + a_3b_{23}b_{32} + a_3b_{23}b_{33}$$
$$+a_2b_{21}b_{11} + a_2b_{21}b_{12} + a_2b_{21}b_{13} + a_2b_{22}b_{21} + a_2b_{22}^2 + a_2b_{22}b_{23} + a_2b_{23}b_{31} + a_2b_{23}b_{32} + a_2b_{23}b_{33}) + c_3(a_1b_{31}b_{11} + a_1b_{31}b_{12}$$
$$+ a_2b_{31}b_{13} + a_2b_{32}b_{21}$$
$$+a_2b_{32}b_{22} + a_2b_{32}b_{23} + a_3b_{33}b_{31} + a_3b_{33}b_{32} + a_3b_{33}^2 + a_3b_{31}b_{11} + a_3b_{31}b_{12} +$$
$$a_3b_{31}b_{13} + a_3b_{32}b_{21} + a_3b_{32}b_{22} + a_3b_{32}b_{23} + a_3b_{33}b_{31} + a_3b_{33}b_{32} + a_3b_{33}^2)]$$
$$ff_yf_{xy} = \frac{5h^3}{4!}ff_yf_{xy} \qquad (28)$$

$$\frac{h^3}{2}[c_1(b_{11}^3 + 2b_{11}^2b_{12} + 2b_{11}^2b_{13} + b_{11}b_{12}^2 + 2b_{11}b_{13}b_{12} + b_{11}b_{13}^2 + b_{12}b_{12}^2 +$$
$$2b_{12}b_{21}b_{22} + 2b_{12}b_{21}b_{23} + b_{12}b_{22}^2 + 2b_{12}b_{22}b_{23} + b_{12}b_{23}^2 + b_{13}b_{31}^2 + 2b_{13}b_{31}b_{32} +$$
$$2b_{13}b_{31}b_{33} + b_{13}b_{32}^2 + 2b_{13}b_{32}b_{33} + b_{13}b_{33}^2 + 2b_{11}^3 + 2b_{11}^2b_{12} + 2b_{11}^2b_{13} +$$
$$2b_{11}b_{12}b_{21} + 2b_{11}b_{12}b_{22} + 2b_{11}b_{12}b_{23} + 2b_{11}^2b_{12} + 2b_{11}b_{12}^2 + 2b_{11}b_{12}b_{13} +$$
$$2b_{11}b_{13}b_{31} + 2b_{11}b_{13}b_{32} + 2b_{11}b_{13}b_{33} + 2b_{11}^2b_{13} + 2b_{11}b_{12}b_{13} + 2b_{11}b_{13}^2 + 2b_{12}^2b_{21} +$$

$$2b_{12}^2b_{23} + 2b_{12}^2b_{23} + 2b_{12}b_{13}b_{31} + 2b_{12}b_{13}b_{32} + 2b_{12}b_{13}b_{33} + 2b_{12}b_{13}b_{21}$$
$$+2b_{12}b_{13}b_{22} + 2b_{12}b_{13}b_{23} + 2b_{13}^2b_{31} + 2b_{13}^2b_{32} + 2b_{13}^2b_{33}) + c_2(b_{21}b_{11}^2 + 2b_{21}b_{11}b_{12}$$
$$+2b_{21}b_{11}b_{13} + b_{21}b_{12}^2 + 2b_{21}b_{12}b_{13} + b_{21}b_{13}^2 + b_{22}b_{21}^2 + 2b_{22}^2b_{21} + 2b_{22}b_{21}b_{23}$$
$$+b_{22}^3 + 2b_{22}^2b_{23} + b_{22}b_{23}^2 + b_{23}b_{31}^2 + 2b_{23}b_{31}b_{32} + 2b_{23}b_{31}b_{33} + b_{23}b_{32}^2 +$$
$$2b_{23}b_{32}b_{33} + b_{23}b_{33}^2 + 2b_{21}^2b_{11} + 2b_{22}^2b_{12} + 2b_{21}^2b_{13} + 2b_{21}^2b_{22} + 2b_{21}b_{22}^2 +$$
$$2b_{21}b_{22}b_{23} + 2b_{21}b_{22}b_{11} + 2b_{21}b_{22}b_{12} + 2b_{21}b_{22}b_{13} + 2b_{21}b_{23}b_{31} +$$
$$2b_{21}b_{23}b_{32} + 2b_{21}b_{23}b_{33} + 2b_{21}b_{23}b_{11} + 2b_{21}b_{23}b_{12} + 2b_{21}b_{23}b_{13} +$$
$$2b_{22}^2b_{21} + 2b_{22}^3 + 2b_{22}^2b_{23} + 2b_{22}b_{23}b_{31} + 2b_{22}b_{23}b_{32} + 2b_{22}b_{23}b_{33} +$$
$$2b_{22}b_{23}b_{21} + 2b_{22}^2b_{23} + 2b_{22}b_{23}^2 + 2b_{23}^2b_{31} + 2b_{23}^2b_{32} + 2b_{23}^2b_{33}) +$$
$$c_3(b_{31}b_{11}^2 + 2b_{31}b_{11}b_{12} + 2b_{31}b_{11}b_{13} + b_{31}b_{21}^2 + 2b_{31}b_{12}b_{13} + b_{31}b_{13}^2 +$$
$$b_{32}b_{21}^2 + 2b_{32}b_{21}b_{22} + 2b_{32}b_{21}b_{23} + b_{21}b_{22}^2 + 2b_{32}b_{22}b_{23} + b_{32}b_{23}^2 +$$
$$b_{33}b_{31}^2 + 2b_{33}b_{31}b_{32} + 2b_{33}^2b_{31} + b_{33}b_{32}^2 + 2b_{33}^2b_{32} + b_{33}^3 + 2b_{31}^2b_{11} +$$
$$2b_{31}^2b_{12} + 2b_{31}^2b_{13} + 2b_{31}b_{32}b_{21} + 2b_{31}b_{32}b_{22} + 2b_{31}b_{32}b_{23} + 2b_{31}b_{32}b_{11} +$$
$$2b_{31}b_{32}b_{12} + 2b_{31}b_{32}b_{13} + 2b_{31}^2b_{33} + 2b_{31}b_{33}b_{32} + 2b_{31}b_{33}^2 + 2b_{31}b_{33}b_{11} +$$
$$2b_{31}b_{33}b_{12} + 2b_{31}b_{32}b_{13} + 2b_{32}^2b_{21} + 2b_{32}^2b_{22} + 2b_{32}^2b_{22} + 2b_{32}b_{33}b_{31} +$$
$$2b_{32}^2b_{33} + 2b_{32}b_{33}^2 + 2b_{32}b_{33}^2 + 2b_{32}b_{33}b_{21} + 2b_{32}b_{33}b_{22} + 2b_{32}b_{33}b_{23} +$$
$$2b_{33}^2b_{31} + 2b_{33}^2b_{32} + 2b_{33}^3)f^2f_yf_{yy} = \frac{4h^3}{4!}f^2f_yf_{yy} \quad (29)$$

$$h^3[c_1(a_1^2b_{11} + a_1a_2b_{12} + a_1a_3b_{13}) + c_2(a_1a_2b_{21} + a_2^2b_{22} + a_2a_3b_{23}) +$$
$$c_3(a_1a_3b_{31} + a_2a_3b_{32} + a_3^2b_{33})]f_xf_{xy} = \frac{3h^3}{4!}f_xf_{xy} \quad (30)$$

$$h^3[c_1(a_1b_{11}^2 + a_2b_{11}b_{12} + a_1b_{11}b_{12} + a_3b_{11}b_{13} + a_1b_{11}b_{13} + a_2b_{12}^2 +$$
$$a_3b_{12}b_{13} + +a_2b_{12}b_{13} + a_3b_{13}^2) + c_2(a_1b_{21}^2 + a_2b_{21}b_{22} + a_1b_{21}b_{22} +$$
$$a_3b_{21}b_{23} + a_1b_{21}b_{23} + a_2b_{22}^2 + a_3b_{22}b_{23} + +a_2b_{22}b_{23} + a_3b_{23}^2) + c_3(a_1b_{31}^2 +$$
$$a_2b_{31}b_{32} + a_1b_{31}b_{32} + a_3b_{31}b_{33} + a_1b_{31}b_{33} + a_2b_{32}^2 + +a_3b_{32}b_{33} + +a_2b_{32}b_{33}$$
$$+a_3b_{33}^2)ff_xf_{yy} = \frac{3h^3}{4!}ff_xf_{yy} \quad (31)$$

$$\frac{h^3}{2}[c_1a_1^2(b_{11} + b_{12} + b_{13}) + c_2a_2^2(b_{21} + b_{22} + b_{23}) + c_3a_3^2(b_{31} + b_{32} + b_{33})]$$
$$ff_{xxy} = \frac{3h^3}{4!}ff_{xxy} \quad (32)$$

$$\frac{h^3}{2}[c_1a_1(b_{11} + b_{12} + b_{13})^2 + c_2a_2(b_{21} + b_{22} + b_{23})^2 + c_3a_3(b_{31} + b_{32} + b_{33})^2]$$
$$f^2f_{xyy} = \frac{3h^3}{4!}f^2f_{xyy} \quad (33)$$

$$\frac{h^3}{6}[c_1(b_{11} + b_{12} + b_{13})^3 + c_2(b_{21} + b_{22} + b_{23})^3 + c_3(b_{31} + b_{32} + b_{33})^3]$$
$$f^3f_{yyy} = \frac{h^3}{4!}f^3f_{yyy} \quad (34)$$

$Note$: $a_1 = b_{11} + b_{12} + b_{13}$, $a_2 = b_{21} + b_{22} + b_{23}$, $a_3 = b_{31} + b_{32} + b_{33}$,

Separating the $f(y)$ $functional\ derivatives$ and their individual equations from the $f(x,y)$ $functional\ derivatives$, we have tables 1 and 2 below:

Equations	Derivatives
$c_1 + c_2 + c_3 = 1$	f
$c_1a_1 + c_2a_2 + c_3a_3 = \frac{1}{2}$	ff_y
$c_1a_1^2 + c_2a_2^2 + c_3a_3^2 = \frac{1}{3}$	f^2f_{yy}
$c_1a_1^3 + c_2a_2^3 + c_3a_3^3 = \frac{1}{4}$	f^3f_{yyy}
$c_1(b_{11}a_1 + b_{12}a_2 + b_{13}a_3) + c_2(b_{21}a_1 + b_{22}a_2 + b_{23}a_3) + c_3(b_{31}a_1 + b_{32}a_2 + b_{33}a_3) = \frac{1}{6}$	ff_y^2
$c_1(b_{11}a_1^2 + b_{12}a_1a_2 + b_{13}a_1a_3) + c_2(b_{21}a_1a_2 + b_{22}a_2^2 + b_{23}a_2a_3) + c_3(b_{31}a_1a_3 + b_{32}a_2a_3 + b_{33}a_3^2) = \frac{1}{8}$	$f^2f_yf_{yy}$
$c_1(b_{11}a_1^2 + b_{12}a_2^2 + b_{13}a_3^2) + c_2(b_{21}a_1^2 + b_{22}a_2^2 + b_{23}a_3^2) + c_3(b_{31}a_1^2 + b_{32}a_2^2 + b_{33}a_3^2) = \frac{1}{12}$	$f^2f_yf_{yy}$
$c_1[a_1(b_{11}^2 + b_{12}b_{21} + b_{13}b_{31}) + a_2(b_{11}b_{12} + b_{12}b_{22} + b_{13}b_{32}) + a_3(b_{11}b_{13} + b_{12}b_{23} + b_{13}b_{33})] + c_2[a_1(b_{11}b_{21} + b_{21}b_{22} + b_{23}b_{31}) + a_2(b_{12}b_{21} + b_{22}^2 + b_{23}b_{32}) + a_3(b_{21}b_{13} + b_{22}b_{23} + b_{23}b_{33})] + c_3[a_1(b_{11}b_{31} + b_{21}b_{32} + b_{31}b_{33}) + a_2(b_{12}b_{31} + b_{22}b_{32} + b_{32}b_{33}) + a_3(b_{13}b_{31} + b_{23}b_{32} + b_{33}^2)] = \frac{1}{24}$	ff_y^3

Table 1:The equations from the $f(y)$ functional derivatives only are:

Equations	Derivatives
$c_1 + c_2 + c_3 = 1$	f
$c_1a_1 + c_2a_2 + c_3a_3 = \frac{1}{2}$	f_x
$c_1a_1^2 + c_2a_2^2 + c_3a_3^2 = \frac{1}{3}$	f_{xx}
$c_1a_1^3 + c_2a_2^3 + c_3a_3^3 = \frac{1}{3}$	ff_{xy}
$c_1a_1^3 + c_2a_2^3 + c_3a_3^3 = \frac{1}{4}$	f_{xxx}
$c_1a_1^3 + c_2a_2^3 + c_3a_3^3 = \frac{1}{4}$	ff_{xxy}
$c_1a_1^3 + c_2a_2^3 + c_3a_3^3 = \frac{1}{4}$	f^2f_{xyy}
$c_1(b_{11}a_1 + b_{12}a_2 + b_{13}a_3) + c_2(b_{21}a_1 + b_{22}a_2 + b_{23}a_3) + c_3(b_{31}a_1 + b_{32}a_2 + b_{33}a_3) = \frac{1}{6}$	f_xf_y
$c_1(b_{11}a_1^2 + b_{12}a_1a_2 + b_{13}a_1a_3) + c_2(b_{21}a_1a_2 + b_{22}a_2^2 + b_{23}a_2a_3) + c_3(b_{31}a_1a_3 + b_{32}a_2a_3 + b_{33}a_3^2) = \frac{1}{8}$	ff_yf_{xy}
$c_1(b_{11}a_1^2 + b_{12}a_1a_2 + b_{13}a_1a_3) + c_2(b_{21}a_1a_2 + b_{22}a_2^2 + b_{23}a_2a_3) + c_3(b_{31}a_1a_3 + b_{32}a_2a_3 + b_{33}a_3^2) = \frac{1}{8}$	f_xf_{xy}
$c_1(b_{11}a_1^2 + b_{12}a_1a_2 + b_{13}a_1a_3) + c_2(b_{21}a_1a_2 + b_{22}a_2^2 + b_{23}a_2a_3) + c_3(b_{31}a_1a_3 + b_{32}a_2a_3 + b_{33}a_3^2) = \frac{1}{8}$	ff_xf_{yy}
$c_1(b_{11}a_1^2 + b_{12}a_2^2 + b_{13}a_3^2) + c_2(b_{21}a_1^2 + b_{22}a_2^2 + b_{23}a_3^2) + c_3(b_{31}a_1^2 + b_{32}a_2^2 + b_{33}a_3^2) = \frac{1}{12}$	ff_yf_{xy}
$c_1(b_{11}a_1^2 + b_{12}a_2^2 + b_{13}a_3^2) + c_2(b_{21}a_1^2 + b_{22}a_2^2 + b_{23}a_3^2) + c_3(b_{31}a_1^2 + b_{32}a_2^2 + b_{33}a_3^2) = \frac{1}{12}$	$f_{xx}f_y$
$c_1[a_1(b_{11}^2 + b_{12}b_{21} + b_{13}b_{31}) + a_2(b_{11}b_{12} + b_{12}b_{22} + b_{13}b_{32}) + a_3(b_{11}b_{13} + b_{12}b_{23} + b_{13}b_{33})] + c_2[a_1(b_{11}b_{21} + b_{21}b_{22} + b_{23}b_{31}) + a_2(b_{12}b_{21} + b_{22}^2 + b_{23}b_{32}) + a_3(b_{21}b_{13} + b_{22}b_{23} + b_{23}b_{33})] + c_3[a_1(b_{11}b_{31} + b_{21}b_{32} + b_{31}b_{33}) + a_2(b_{12}b_{31} + b_{22}b_{32} + b_{32}b_{33}) + a_3(b_{13}b_{31} + b_{23}b_{32} + b_{33}^2)] = \frac{1}{24}$	$f_xf_y^2$

Table 2: The equations from the $f(x, y)$ functional derivatives only are:

$From\ tables\ 1\ and\ 2, we\ can\ see\ that\ the\ two\ sets\ of\ equations\ are\ the\ same\ and\ are\ eight (8)$. Therefore, we can solve as follows: from the first equation in table1, $set\ c_1 = \frac{1}{4}, \quad c_2 = \frac{1}{2}, \ c_3 \frac{1}{4}, \ the\ second, third\ and\ sixth\ equations\ become$:

$$a_1 + 2a_2 + a_3 = 2 \quad (35)$$

$$a_1^2 + 2a_2^2 + a_3^2 = \frac{4}{3} \quad (36)$$

$$a_1^3 + 2a_2^3 + a_3^3 = 1 \quad (37)$$

Also, set$a_2 = \frac{1}{2}, from\ (35), a_1 = 1 - a_3, (36)\&(37) become$:

$$(1 - a_3)^2 + a_3^2 = \frac{5}{6} \rightarrow 12a_3^2 - 12a_3 + 1 = 0 \quad (38)$$

$$(1 - a_3)^3 + a_3^3 = \frac{3}{4} \rightarrow 12a_3^2 - 12a_3 + 1 = 0 \quad (39)$$

$Solving\ any\ of\ (38) or\ (39), we\ have$:

$$a_3 = \frac{1}{2} + \frac{\sqrt{6}}{6}\ or\ \frac{1}{2} - \frac{\sqrt{6}}{6} \quad (40)$$

$But\ a_1 = 1 - a_3$

$$\therefore\ a_1 = \frac{1}{2} - \frac{\sqrt{6}}{6}, a_2 = \frac{1}{2}, a_3 = \frac{1}{2} + \frac{\sqrt{6}}{6} \quad (41)$$

Or

$$a_1 = \frac{1}{2} + \frac{\sqrt{6}}{6}, a_2 = \frac{1}{2}, a_3 = \frac{1}{2} - \frac{\sqrt{6}}{6}, \ c_1 = \frac{1}{4}, \ c_2 = \frac{1}{2}, \ c_3 = \frac{1}{4}, \quad (42)$$

Putting (42) into the fourth, fifth, seventh and eighth equations in table1, we have:

$$b_{11} = \frac{1}{8}, b_{12} = \frac{6-\sqrt{6}}{24}, b_{13} = \frac{1-\sqrt{6}}{8}, b_{21} = \frac{6+\sqrt{6}}{48}, b_{22} = \frac{1}{4},$$
$$b_{23} = \frac{6-\sqrt{6}}{48}, b_{31} = \frac{1+\sqrt{6}}{8}, b_{32} = \frac{6+\sqrt{6}}{24}, b_{33} = \frac{1}{8} \quad (43)$$

$The\ 3rd - stage\ 4th - order\ implicit\ Runge - kutta\ method\ is$:

$$y_{n+1} - y_n = h(c_1k_1 + c_2k_2 + c_3k_3)$$
$$k_1 = f(x_n + ha_1, y_n + h(b_{11}k_1 + b_{12}k_2 + b_{13}k_3))$$
$$k_2 = f(x_n + ha_2, y_n + h(b_{21}k_1 + b_{22}k_2 + b_{23}k_3))$$
$$k_3 = f(x_n + ha_3, y_n + h(b_{31}k_1 + b_{32}k_2 + b_{33}k_3)) \quad (44)$$

Putting (42) and (43) into (44), we have the formula below:

$$y_{n+1} - y_n = \frac{h}{4}(k_1 + 2k_2 + k_3)$$
$$k_1 = f\left(x_n + h\left(\frac{3-\sqrt{6}}{6}\right), \quad y_n + h\left(\frac{k_1}{8} + \left(\frac{6-\sqrt{6}}{24}\right)k_2 + \left(\frac{1-\sqrt{6}}{8}\right)k_3\right)\right)$$
$$k_2 = f\left(x_n + \frac{h}{2}, \quad y_n + h\left(\left(\frac{6+\sqrt{6}}{48}\right)k_1 + \frac{k_2}{4} + \left(\frac{6-\sqrt{6}}{48}\right)k_3\right)\right)$$
$$k_3 = f\left(x_n + h\left(\frac{3+\sqrt{6}}{6}\right), y_n + h\left(\left(\frac{1+\sqrt{6}}{8}\right)k_1 + \left(\frac{6+\sqrt{6}}{24}\right)k_2 + \frac{k_3}{8}\right)\right) \quad (45)$$

3.1. Properties for stability Analysis for Implicit Runge-Kutta Method

According to Butcher (1975, 2010), an implicit Runge-Kutta method is A – Stable and B -Stable if the following conditions are satisfied by that method:

(i) B is a non – singular matrix,
(ii) $c_1, c_2, \ldots c_s$ are non – negative,

(iii) $a_1, a_{2,} \; . \; . \; . \; a_5$ are distinct,

(iv) $0 \leq \sum_{i,j=1}^{s} c_i b_{ij}^{(-1)} \leq 2,$

(v) The order is at least 2s – 2,

(vi) $\sum_{j=1}^{s} b_{ij} a_j^{m-1} = \frac{a_j^m}{m}, for\; m = 1, 2, . . ., s-1, i = 1, 2, . . ., s,$

(vii) $\sum_{j=1}^{s} c_i a_j^{l-1} b_{ij} = \frac{c_j(1-a_j^l)}{l}, for\; l = 1, 2, . . ., s-1, j = 1, 2, . . ., s$

3.2. Test for Stability of Our Method

The Butcher's tableau for our method is:

$$\begin{array}{c|ccc} \frac{3-\sqrt{6}}{6} & \frac{1}{8} & \frac{6-\sqrt{6}}{24} & \frac{1-\sqrt{6}}{8} \\ \frac{1}{2} & \frac{6+\sqrt{6}}{48} & \frac{1}{4} & \frac{6-\sqrt{6}}{48} \\ \frac{3+\sqrt{6}}{6} & \frac{1+\sqrt{6}}{8} & \frac{6+\sqrt{6}}{24} & \frac{1}{8} \\ \hline & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}$$

From our third – stage fourth – order method above, s = 3, $i = 1,2,3$.

$a_1 = \frac{3-\sqrt{6}}{6}, \; a_2 = \frac{1}{2}, \; a_3 = \frac{3+\sqrt{6}}{6}$ Which are distinct and satisfies condition (iii).

Also $c_1 = \frac{1}{4}, \; c_2 = \frac{1}{2}, \; c_3 = \frac{1}{4}$which are non – negative and satisfies condition (ii).

From our formula

$$B = \begin{bmatrix} \frac{1}{8} & \frac{6-\sqrt{6}}{24} & \frac{1-\sqrt{6}}{8} \\ \frac{6+\sqrt{6}}{48} & \frac{1}{4} & \frac{6-\sqrt{6}}{48} \\ \frac{1+\sqrt{6}}{8} & \frac{6+\sqrt{6}}{24} & \frac{1}{8} \end{bmatrix}$$

$$B = \begin{bmatrix} 0.125 & 0.1479379 & -0.1811862 \\ 0.176031 & 0.25 & 0.073969 \\ 0.4311862 & 0.3520621 & 0.125 \end{bmatrix}$$

Det(B) = 0.010416667, this shows that B is a non – singular matrix and satisfies condition (i).

$$B^{-1} = \begin{bmatrix} 0.499998555 & -7.898978661 & 5.398979126 \\ 0.94949151 & 8.99999853 & -3.949489061 \\ -4.39897962 & 1.898977681 & 0.5000000955 \end{bmatrix}$$

Also, the order is at least 2s – 2 which is of order four, which also satisfies condition (v).

Hence, from condition (iv), $\sum_{i,j=1}^{s} c_i b_{ij}$ = 1.9999997 which is ≤ 2.

From condition (vi), m = 1, 2, $i = 1, 2, 3$

$$b_{11}a_1^0 + b_{12}a_2^0 + b_{13}a_3^0 = a_1 = 0.0917518$$

$$b_{21}a_1^0 + b_{22}a_2^0 + b_{23}a_3^0 = a_2 = 0.5$$

$$b_{31}a_1^0 + b_{32}a_2^0 + b_{33}a_3^0 = a_3 = 0.9082483$$

From condition (vii)

$$c_1a_1^0b_{11} + c_2a_2^0b_{21} + c_3a_3^0b_{31} = c_1(1 - a_1) = 0.2270621$$

$$c_1a_1^0b_{12} + c_2a_2^0b_{22} + c_3a_3^0b_{32} = c_2(1 - a_2) = 0.25$$

$$c_1a_1^0b_{13} + c_2a_2^0b_{23} + c_3a_3^0b_{33} = c_3(1 - a_3) = 0.0229379$$

Hence, from the above explanation, it shows that our implicit formula satisfies the seven (7) conditions above, we can say with proof that our implicit method is A – stable and B – stable.

It is also seen that the condition for consistency is also satisfied by our implicit method. Thus, since $\sum_{i=}^{3} c_i = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, hence our method is consistent.

4. Implementation of the Formula

The formula is implemented on the initial – value problems below with the aid of MAPLE software:

(i) $y' = y - y^2, y(0) = 0.5$, $y(x_n) = \frac{1}{1+e^{-x}}$, $h = 0.1$

(ii) $y' = -y$, $y(0) = 1$, $y(x_n) = \frac{1}{e^{xn}}$, $h = 0.1$

(iii) $y' = y$, $y(0) = 1$, $y(x_n) = e^{xn}$, $h = 0.1$

(iv) $y' = y^2$, $y(0) = 1$, $y(x_n) = \frac{1}{1-x_n}$, $h = 0.1$

5. Results

Below are Tables of Results for the above Initial-Value Problems:

PROBLEM 1

XN	YN	TSOL	ERROR
.1D+00	0.52497918942147326736	0.52497918747893998609	$-1.94253328127000000000\ 10^{-9}$
.2D+00	0.54983400113135754686	0.54983399731247790855	$-3.81887963831000000000\ 10^{-9}$
.3D+00	0.57444252237896429063	0.57444251681165898714	$-5.56730530349000000000\ 10^{-9}$
.4D+00	0.59868766724696236276	0.59868766011245200038	$-7.13451036238000000000\ 10^{-9}$
.5D+00	0.62245933968059032036	0.62245933528481665970	$-4.39577366066000000000\ 10^{-9}$

PROBLEM 2

XN	YN	TSOL	ERROR
.1D+00	0.90483741489182465982	0.90483741803595957316	$3.14413491334000000000\ 10^{-9}$
.2D+00	0.81873074738812003466	0.81873075307798185867	$5.68986182401000000000\ 10^{-9}$
.3D+00	0.74081821295911805676	0.74081822068171786607	$7.72259980931000000000\ 10^{-9}$
.4D+00	0.67032003671870962096	0.67032004603563930074	$9.31692967978000000000\ 10^{-9}$
.5D+00	0.60653064917475019756	0.60653065971263342360	$1.05378832260400000000\ 10^{-8}$

PROBLEM 3

XN	YN	TSOL	ERROR
.1D+00	1.10517092191590269340	1.10517091807564762480	$-3.84025506860000000000\ 10^{-9}$
.2D+00	1.22140276664844628820	1.22140275816016983390	$-8.48827645430000000000\ 10^{-9}$
.3D+00	1.34985882164749755120	1.34985880757600310400	$-1.40714944720000000000\ 10^{-8}$
.4D+00	1.49182471837647893640	1.49182469764127031780	$-2.07352086186000000000\ 10^{-8}$
.5D+00	1.64872129934506512840	1.64872127070012814680	$-2.86449369816000000000\ 10^{-8}$

PROBLEM 4

XN	YN	TSOL	ERROR
.1D+00	1.11111174562506629350	1.11111111111111111110000	$-6.34513955182400000000\ 10^{-7}$
.2D+00	1.25000205757237386540	1.25000000000000000000	-0.00000205757237386540
.3D+00	1.42857682385185049920	1.42857142857142857140	-0.00000539528042192780
.4D+00	1.66668055978174744090	1.66666666666666666670	-0.00001389311508077420
.5D+00	2.00003847967047852970	2.00000000000000000000	-0.00003847967047852970

Table 3

6. Conclusion

After our implementation, it shows from the tables of numerical results that the methodis highly efficient. It also revealed the fact that implicit formulas are always more efficient when properly derived. The f(x,y) functional derivatives and f(y) functional derivatives are also seen to have generated the same set of equations. This means that to avoid stress, either the f(y) or f(x,y) functional derivatives can be considered separately and still get a highly efficient formula. Since $\sum_{i=1}^{3} c_i = 1$, this shows that this formula is consistent.

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