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Fuzzy Neutrosophic Equivalence Relations

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Abstract:
This paper introduces the concept of fuzzy neutrosophic equivalence relations and discuss some of their properties. Also we define fuzzy neutrosophic transitive closure and investigate their properties.

Keywords: *Fuzzy neutrosophic equivalence relation, fuzzy neutrosophic equivalence class, fuzzy neutrosophic transitive closure.*
MSC 2000: 03B99,03E99.

1. Introduction

Relations are a suitable tool for describing correspondences between objects. Crisp relations like $\in, \subseteq, =$,...... have served well in developing mathematical theories. The use of fuzzy relations originated from the observation that real life objects can be related to each other to certain degree. Fuzzy relations are able to model vagueness, but they cannot model uncertainty. Intuitionistic fuzzy sets, as defined by Atanassov [4,5], give us a way to incorporate uncertainty in an additional degree.In1995, Florentine Smarandache [13]extended the concept of intuitionistic fuzzy sets to a tri component logic set withnon-standard interval namely Neutrosophic set. Motivated by this concept I. Arockiaraniet al., [2] defined the fuzzy neutrosophic set in which the non-standard interval is taken as standard interval.

In 1996, Bustince and Burillo [7] introduced the concept of intuitionistic fuzzy relations and studied some of its properties. In 2003, Deschrijver and Kerre [9] investigated some properties of the composition of intuitionistic fuzzy relations.

In this paper, we introduce and study some properties of fuzzy neutrosophic equivalence relations and fuzzy neutrosophic transitive closures.

2.Preliminaries

- *Definition 2.1: [2]*

A Fuzzy neutrosophic set A on the universe of discourse X is defined as
A= $\langle x, T_A(x), I_A(x), F_A(x)\rangle, x \in X$ where $T, I, F: X \rightarrow [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$

- *Definition 2.2:[2]*

A Fuzzy neutrosophic set A is a subset of a Fuzzy neutrosophic set B (i.e.,) A $\subseteq$B for all x if
$T_A(x) \le T_B(x)$, $I_A(x) \le I_B(x)$, $F_A(x) \ge F_B(x)$

- *Definition 2.3: [2]*

Let X be a non-empty set, and $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle, B = \langle x, T_B(x), I_B(x), F_B(x)\rangle$ be two Fuzzy neutrosophic sets.
Then $A \cup B = \langle x, \max(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \min(F_A(x), F_B(x))\rangle$
$A \cap B = \langle x, \min(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \max(F_A(x), F_B(x))\rangle$

- *Definition 2.4: [2]*

The difference between two Fuzzy neutrosophic sets A and B is defined as A\B (x)=
$\langle x, \min(T_A(x), F_B(x)), \min(I_A(x), 1 - I_B(x)), \max(F_A(x), T_B(x))\rangle$

- *Definition 2.5: [2]*

A Fuzzy neutrosophic set A over the universe X is said to be null or empty Fuzzy neutrosophic set if $T_A(x) = 0$,$I_A(x) = 0$, $F_A(x) = 1$ for allx $\in$ X. It is denoted by 0_N

- *Definition 2.6: [2]*

A Fuzzy neutrosophic set A over the universe X is said to be absolute (universe) Fuzzy neutrosophic set if $T_A(x) = 1$ $I_A(x) = 1$, $F_A(x) = 0$for allx $\in$ X. It is denoted by 1_N

- *Definition 2.7: [2]*

The complement of a Fuzzy neutrosophic set A is denoted by A^c and is defined as

$A^c = \left\langle x, T_{A^c}(x), I_{A^c}(x), F_{A^c}(x) \right\rangle$ where $T_{A^c}(x) = F_A(x)$, $I_{A^c}(x) = 1 - I_A(x)$, $F_{A^c}(x) = T_A(x)$

The complement of a Fuzzy neutrosophic set A can also be defined as $A^c = 1_N - A$.

- *Definition 2.8: [3]*

A fuzzy neutrosophic set relation is defined as a fuzzy neutrosophic subset of$X \times Y$ having the form
$R = \{ \langle (x,y), T_R(x,y), I_R(x,y), F_R(x,y) \rangle : x \in X, y \in Y \}$ where $T_R, I_R, F_R : X \times Y \to [0,1]$
Satisfy the condition $0 \le T_R(x,y) + I_R(x,y) + F_R(x,y) \le 3 \forall\ (x,y) \in X \times Y$.
We will denote with$FNR(X \times Y)$ the set of all fuzzy neutrosophic subsets in $X \times Y$.

- *Definition 2.9: [3]*

Given a binary fuzzy neutrosophic relation between $X$and Y ,we can define R^{-1} between Y and X by means of$T_{R^{-1}}(y,x) = T_R(x,y), I_{R^{-1}}(y,x) = I_R(x,y), F_{R^{-1}}(y,x) = F_R(x,y) \forall (x,y) \in X \times Y$to which we call inverse relation ofR.

- *Definition 2.10: [3]*

Let R and P be two fuzzy neutrosophic relations between X and Y, for every $(x,y) \in X \times Y$

We can define,
1)$R \le P \Leftrightarrow T_R(x,y) \le T_P(x,y), I_R(x,y) \le I_P(x,y), F_R(x,y) \ge F_P(x,y)$
2)$R \preccurlyeq P \Leftrightarrow T_R(x,y) \le T_P(x,y), I_R(x,y) \le I_P(x,y), F_R(x,y) \le F_P(x,y)$
3)$R \vee P = \{ \langle (x,y), T_R(x,y) \vee T_P(x,y), I_R(x,y) \vee I_P(x,y), F_R(x,y) \wedge F_P(x,y) \rangle$
4)$R \wedge P = \{ \langle (x,y), T_R(x,y) \wedge T_P(x,y), I_R(x,y) \wedge I_P(x,y), F_R(x,y) \vee F_P(x,y) \rangle$
5)$R^c = \{ \langle (x,y), F_R(x,y), 1 - I_R(x,y), T_R(x,y) \rangle : x \in X, y \in Y \}$

- *Definition 2.11: [3]*

Let $\alpha, \beta, \lambda, \rho$ be t-norms or t-conorms not necessarily dual two – two,$R \in FNR(X \times Y)$ and$P \in FNR(Y \times Z)$. We will call composed relation $P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R \in FNR(X \times Z)$ to the one defined by

$$P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R = \left\{ \langle (x,z), T_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z), I_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z), F_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) \rangle / x \in X, z \in Z \right\}$$

Where,

$$T_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) = \overset{\alpha}{\underset{y}{}} \{ \beta [T_R(x,y), T_P(y,z)] \},$$

$$I_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) = \overset{\alpha}{\underset{y}{}} \{ \beta [I_R(x,y), I_P(y,z)] \},$$

$$F_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) = \overset{\lambda}{\underset{y}{}} \{ \rho [F_R(x,y), F_P(y,z)] \}$$

Whenever $0 \le T_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) + I_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) + F_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) \le 3\ \forall (x,z) \in X \times Z$

The choice of the t-norms and t-conorms $\alpha, \beta, \lambda, \rho$in the previous definition, is evidently conditioned by the fulfilment of

$$0 \le T_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) + I_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) + F_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) \le 3\ \forall (x,z) \in X \times Z.$$

- *Definition2.12: [3]*

1) The relation $\Delta \in FNR(X \times X)$ is called the relation of identity if$\forall$ $(x,y) \in X \times X T_{\Delta}(x,y) = \begin{cases} 1 if x = y \\ 0 if x \neq y \end{cases}$, $I_{\Delta}(x,y) = \begin{cases} 1 if x = y \\ 0 if x \neq y \end{cases}$, $F_{\Delta}(x,y) = \begin{cases} 0 if x = y \\ 1 if x \neq y \end{cases}$

2) The complementary relation$\Delta^c = \nabla$is defined by

$T_{\nabla}(x,y) = \begin{cases} 0 if x = y \\ 1 if x \neq y \end{cases}$,$I_{\nabla}(x,y) = \begin{cases} 0 if x = y \\ 1 if x \neq y \end{cases}$, $F_{\nabla}(x,y) = \begin{cases} 1 if x = y \\ 0 if x \neq y \end{cases}$.

3.Fuzzy neutrosophic equivalence relations

- *Definition 3.1:*

Let X be a set and let $P,Q \in FNR(X)$. Then the composition $Q \circ P$ of P and Q can also bedefined as follows : for any $x,y \in X$

$T_{Q \circ P}(x,y) = \bigvee_{z \in X}[T_P(x,z) \wedge T_Q(z,y)]$, $I_{Q \circ P}(x,y) = \bigvee_{z \in X}[I_P(x,z) \wedge I_Q(z,y)]$ and $F_{Q \circ P}(x,y) = \bigwedge_{z \in X}[F_P(x,z) \vee F_Q(z,y)]$

- *Definition 3.2:*

Let X be a set and let $R_1, R_2, R_3, Q_1, Q_2 \in FNR(X)$ .Then (1) $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$
(2) If $R_1 \subset R_2$ and $Q_1 \subset Q_2$,then $R_1 \circ Q_1 \subset R_2 \circ Q_2$ .In particular, if $Q_1 \subset Q_2$,then $R_1 \circ Q_1 \subset R_2 \circ Q_2$.
(3)$R_1 \circ (R_2 \cup R_3) = (R_1 \circ R_2) \cup (R_1 \circ R_3)$ (4)$R_1 \circ (R_2 \cap R_3) = (R_1 \circ R_2) \cap (R_1 \circ R_3)$
(5) If $R_1 \subset R_2$,then $R_1^{-1} \subset R_2^{-1}$ (6)$\left(R^{-1}\right)^{-1} = R$ and $\left(R_1 \circ R_2\right)^{-1} = R_2^{-1} \circ R_1^{-1}$
(7)$\left(R_1 \cup R_2\right)^{-1} = R_1^{-1} \cup R_2^{-1}$ (8)$\left(R_1 \cap R_2\right)^{-1} = R_1^{-1} \cap R_2^{-1}$

- *Proposition 3.1:*

Let P and Q be any fuzzy neutrosophic relations on a set X .If $Q \circ P = P \circ Q$,then $(Q \circ P) \circ (Q \circ P) = (Q \circ Q) \circ (P \circ P)$

- *Proof:*

Proof follows from definition 3.2(1)

- *Definition 3.3:*

A fuzzy neutrosophic relation R on a set X is called a fuzzy neutrosophic equivalence relation (in short FNER) on X if it satisfies the following conditions:

(i) It is fuzzy neutrosophic reflexive, (i.e.,) $R(x,x) = (1,1,0)$ for each $x \in X$
(ii) It is fuzzy neutrosophic symmetric (i.e.,) $R^{-1} = R$
(iii) It is fuzzy neutrosophic transitive (i.e.,) $R \circ R \subset R$

We will denote the set of all FNERs on X as $FNE(X)$.
The following proposition is the immediate result of definition 3.1.

- *Proposition 3.2:*

Let X be a set and let $R, Q \in FNR(X)$

(i) If R is fuzzy neutrosophic reflexive (respectively symmetric, transitive) then R^{-1} is fuzzy neutrosophic reflexive (respectively symmetric, transitive)
(ii) If R is fuzzy neutrosophic reflexive (respectively symmetric, transitive), then $R \circ R$ is fuzzy neutrosophic reflexive (respectively symmetric, transitive)
(iii) If R is fuzzy neutrosophic reflexive, then $R \subset R \circ R$
(iv) If R is fuzzy neutrosophic symmetric then $R \cup R^{-1}$ and $R \cap R^{-1}$ are symmetric and $R \circ R^{-1} = R^{-1} \circ R$
(v) If R and Q are fuzzy neutrosophic reflexive (respectively symmetric, transitive). Then $R \cap Q$ is fuzzy neutrosophic reflexive (respectively symmetric, transitive)

(vi) If R and Q are fuzzy neutrosophicsymmetric, then $R \cup Q$ is fuzzy neutrosophic symmetric.

- *Proof:*

It is the immediate result of definition 3.3.
The following two results are easily seen.

- *Proposition 3.4:*
  Let X be a set. If $R \in FNE(X)$ then $R \circ R = R$.

- *Proposition 3.5:*
  Let $\{R_\alpha\}_{\alpha\in\Gamma}$ be a non-empty family of FNERs on a set X .Then $\bigcap_{\alpha\in\Gamma} R_\alpha \in FNE(X)$ .However , in general, $\bigcup_{\alpha\in\Gamma} R_\alpha$ need not be a FNER on X .

  - Example 3.1:

Let $X = \{a,b,c\}$ .Let P and Q be the FNRs on X represented by matrices are given below

$$\begin{pmatrix} P & a & b & c \\ a & (1,1,0) & (0.7,0.4,0.2) & (0.6,0.3,0.1) \\ b & (0.7,0.4,0.2) & (1,1,0) & (0.6,0.3,0.1) \\ c & (0.6,0.3,0.1) & (0.6,0.3,0.1) & (1,1,0) \end{pmatrix}$$

$$\begin{pmatrix} Q & a & b & c \\ a & (1,1,0) & (0.7,0.4,0.2) & (1,1,0) \\ b & (0.7,0.4,0.2) & (1,1,0) & (0.6,0.3,0.1) \\ c & (1,1,0) & (0.6,0.3,0.1) & (1,1,0) \end{pmatrix}$$

Then clearly $P,Q \in FNE(X)$ and $P \cup Q$ is the fuzzy neutrosophic relation on X represented by the following matrix

$$\begin{pmatrix} P \cup Q & a & b & c \\ a & (1,1,0) & (0.7,0.4,0.2) & (1,1,0) \\ b & (0.7,0.4,0.2) & (1,1,0) & (0.6,0.3,0.1) \\ c & (1,1,0) & (0.6,0.3,0.1) & (1,1,0) \end{pmatrix}$$

On the other hand, $T_{(P\cup Q)\circ(P\cup Q)}(b,c) = 0.7 > 0.6 = T_{P\cup Q}(b,c)$

$I_{(P\cup Q)\circ(P\cup Q)}(b,c) = 0.4 > 0.3 = I_{P\cup Q}(b,c)$ and $F_{(P\cup Q)\circ(P\cup Q)}(b,c) = 0 < 0.1 = F_{P\cup Q}(b,c)$

Thus $(P \cup Q) \circ (P \cup Q) \not\subset P \cup Q$ .So $P \cup Q$ is not fuzzy neutrosophic transitive. Hence $P \cup Q \notin FNE(X)$.

- *Proposition 3.6:*

Let P and Q be fuzzy neutrosophic reflexive relations on a set X .Then $Q \circ P$ is also a fuzzy neutrosophic reflexive relation on X .

- *Proof:*
  Let $x \in X$ .Then
  $T_{Q\circ P}(x,x) = \vee_{t\in X}[T_P(x,t) \wedge T_Q(t,x)]$
  $\geq T_P(x,x) \wedge T_Q(x,x)$ (Since P and Q are fuzzy neutrosophic reflexive)
  $=1$
  Similarly, $I_{Q\circ P}(x,x) = 1$
  $F_{Q\circ P}(x,x) = \wedge_{t\in X}[F_P(x,t) \vee F_Q(t,x)]$

$\leq F_P(x,x) \vee F_Q(x,x) = 0$

Thus $Q \circ P(x,x) = (1,1,0)$ for each $x \in X$ .Hence $Q \circ P$ is a fuzzy neutrosophic relation on X .

- *Proposition 3.7:*

Let X be a set and let $P, Q \in FNE(X)$.If $Q \circ P = P \circ Q$ then $P \circ Q \in FNE(X)$.

- *Proof:*

Let $x \in X$ .Since P and Q are fuzzy neutrosophic reflexive.

$T_{P \circ Q}(x,x) = \vee_{y \in X} [T_Q(x,y) \wedge T_P(y,x)] \geq T_Q(x,x) \wedge T_P(x,x) = 1$

Similarly, $I_{P \circ Q}(x,x) = 1$ and

$F_{P \circ Q}(x,x) = \wedge_{y \in X} [F_Q(x,y) \wedge F_P(y,x)] \leq F_Q(x,x) \vee F_P(x,x) = 0$

Thus $P \circ Q(x,x) = (1,1,0)$ .So $P \circ Q$ is fuzzy neutrosophic reflexive. Let $x, y \in X$ .Then

$T_{P \circ Q}(x,z) = \vee_{y \in X} [T_Q(x,y) \wedge T_P(y,z)] = \vee_{y \in X} [T_P(z,y) \wedge T_Q(y,x)]$

(Since P and Q are fuzzy neutrosophic symmetric)

$= T_{Q \circ P}(z,x) = T_{P \circ Q}(z,x)$ [$\because P \circ Q = Q \circ P$]

Similarly

$I_{P \circ Q}(x,z) = I_{P \circ Q}(z,x)$

$F_{P \circ Q}(x,z) = \wedge_{y \in X} [F_Q(x,y) \vee F_P(y,z)] = \wedge_{y \in X} [F_P(z,y) \vee F_Q(y,x)] = F_{Q \circ P}(z,x) = F_{P \circ Q}(z,x)$

So $P \circ Q$ *is fuzzy neutrosophic symmetric*

On the other hand

$(P \circ Q) \circ (P \circ Q) = (P \circ P) \circ (Q \circ Q)$

(By proposition3.1)

$\subset P \circ Q$

(Since P and Q are fuzzy neutrosophic transitive)

$P \circ Q \in FNE(X)$

Hence

- *Definition 3.4:*

Let R be a fuzzy neutrosophic equivalence relation on a set X and let $a \in X$ .We define a complex mapping $Ra : X \to I \times I$ as follows : for each $x \in X$, $Ra(x) = R(a,x)$ .Then clearly $Ra \in FNS(X)$ .The fuzzy neutrosophic set Ra in X is called a fuzzy neutrosophic equivalence class of R containing $a \in X$ .The set $\{Ra : a \in X\}$ is called the fuzzy neutrosophic quotient set of X by R and denoted by X / R .

- *Theorem 3.1:*

Let X be a set and let $R \in FNE(X)$ .Then the following hold:

(i) $Ra = Rb$ if and only if $R(a,b) = (1,1,0)$ for any $a, b \in X$

(ii) $R(a,b) = (0,0,1)$ if and only if $Ra \cap Rb = 0_N$ for any $a, b \in X$

(iii) $\bigcup_{a \in X} Ra = 1_N$

(iv) There exists a surjection $p : X \to X / R$ (called the natural mapping) defined by $p(x) = Rx$ for each $x \in X$.

- *Proof:*

(i) $\Rightarrow$ Suppose $Ra = Rb$ .Since R is a fuzzy neutrosophic equivalence relation,

$R(a,b) = Ra(b) = R(b,b) = (1,1,0)$ .Hence $R(a,b) = (1,1,0)$.

Conversely, suppose $R(a,b)=(1,1,0)$ .Then $T_R(a,b)=1, I_R(a,b)=1, F_R(a,b)=0$ .Let $x\in X$ .Then

$T_{Ra}(x)=T_R(a,x)\geq \underset{z\in X}{\vee}[T_R(a,z)\wedge T_R(z,x)]$ (Since R is fuzzy neutrosophic transitive)

$\geq T_R(a,b)\wedge T_R(b,x)=1\wedge T_R(b,x)=T_R(b,x)=T_{Rb}(x)$

Similarly $I_{Ra}(x)\geq I_{Rb}(x)$

$F_{Ra}(x)=F_R(a,x)\leq \underset{z\in X}{\wedge}[F_R(a,z)\vee F_R(z,x)]\leq F_R(a,b)\vee F_R(b,x)=0\vee F_R(b,x)=F_R(b,x)=F_{Rb}(x)$ Thus

$Ra\supset Rb$ .By the similar arguments, we have $Ra\subset Rb.$ Hence $Ra=Rb.$

The proofs of (ii), (iii) and (iv) are easy.

This completes the proof.

- *Definition 3.5:*

  Let X be a set, let $R\in FNR(X)$ and let $\{R_\alpha\}_{\alpha\in\Gamma}$ be the family of all the FNERs on X containing R .Then $\bigcap_{\alpha\in\Gamma} R_\alpha$ is called the FNER generated by R and denoted by R^e .

  It is easily seen that R^e is the smallest fuzzy neutrosophic equivalence relation containing R .

- *Definition 3.6:*

  Let X be a set and let $R\in FNR(X)$ .Then the fuzzy neutrosophic transitive closure of R ,denoted by R^∞ ,is defined as follows: $R^\infty=\bigcup_{n\in N} R^n$,where $R^n=R\circ R\circ R....\circ R$ in which R occurs n times.

- *Proposition 3.7:*

  Let X be a set and let $R\in FNR(X)$ .Then (i) R^∞ is the smallest fuzzy neutrosophic transitive relation on X containing R .

(ii)If there exists $n\in N$ such that $R^{n+1}=R^n$, then $R^\infty=R\cup R^2\cup.....\cup R^n$.

- *Example 3.2:*

  Let $X=\{a,b,c\}$ and let $R=\langle T_R, I_R, F_R\rangle$ be the FNR on X defined as follows:

$$\begin{pmatrix} R & a & b & c \\ a & (0.8,0.4,0.1) & (1,1,0) & (0.1,0.2,0.9) \\ b & (0,0,1) & 0.3,0.3,0.6 & (0,0,1) \\ c & (0.2,0.3,0.8) & (0,0,1) & (0.3,0.2,0.7) \end{pmatrix}.$$

Then $\begin{pmatrix} R^2 & a & b & c \\ a & (0.8,0.4,0.1) & (0.8,0.4,0.1) & (0.1,0.2,0.9) \\ b & (0,0,1) & (0.3,0.3,0.6) & (0,0,1) \\ c & (0.2,0.3,0.8) & (0.2,0.3,0.8) & (0.3,0.2,0.7) \end{pmatrix}\begin{pmatrix} R^3 & a & b & c \\ a & (0.8,0.4,0.1) & (0.8,0.4,0.1) & (0.1,0.2,0.9) \\ b & (0,0,1) & (0.3,0.3,0.6) & (0,0,1) \\ c & (0.2,0.3,0.8) & (0.2,0.3,0.8) & (0.3,0.2,0.7) \end{pmatrix}$ .Thus

$R^2=R^3$ .So $R^\infty=R\cup R^2$

Moreover $R^\infty\circ R^\infty\subset R^\infty$. $\begin{pmatrix} R^\infty & a & b & c \\ a & (0.8,0.4,0.1) & (0.8,0.4,0.1) & (0.1,0.2,0.9) \\ b & (0,0,1) & (0.3,0.3,0.6) & (0,0,1) \\ c & (0.2,0.3,0.8) & (0.2,0.3,0.8) & (0.3,0.2,0.7) \end{pmatrix}$.

$$\begin{pmatrix} R^{\infty} \circ R^{\infty} & a & b & c \\ a & (0.8,0.4,0.1) & (0.8,0.4,0.1) & (0.1,0.2,0.9) \\ b & (0,0,1) & (0.3,0.3,0.6) & (0,0,1) \\ c & (0.2,0.3,0.8) & (0.2,0.3,0.8) & (0.3,0.2,0.7) \end{pmatrix}$$

.Hence $R^{\infty} = R \cup R^2$ is fuzzy neutrosophic transitive.

- *Proposition 3.9:*
  If R is fuzzy neutrosophic symmetric,then so is R^{∞}.

- *Proof:*
  For $n \geq 1$ and $x, y \in X$

$T_{R^n}(x,y) = \underset{z_1,z_2,\dots z_{n-1}}{\vee}[T_R(x,z_1) \wedge T_R(z_1,z_2) \wedge \dots\dots \wedge T_R(z_{n-1},y)] = \underset{z_{n-1},\dots z_1}{\vee}[T_R(y,z_{n-1}) \wedge \dots\dots \wedge T_R(z_1,x)] = T_{R^n}(y,x)$

Similarly, $I_{R^n}(x,y) = I_{R^n}(y,x)$.

$F_{R^n}(x,y) = \underset{z_1,z_2,\dots z_{n-1}}{\wedge}[F_R(x,z_1) \vee F_R(z_1,z_2) \vee \dots\dots \vee F_R(z_{n-1},y)] = \underset{z_{n-1},\dots z_1}{\wedge}[F_R(y,z_{n-1}) \vee \dots\dots \vee F_R(z_1,x)] = F_{R^n}(y,x)$. Thus R^n is fuzzy neutrosophic symmetric for any $n \geq 1$.Hence R^{∞} is fuzzy neutrosophic symmetric.

- *Another proof:*

It is clear that $R^1 = R$ is fuzzy neutrosophic symmetric. Suppose R^k is fuzzy neutrosophic symmetric for $k > 1$.We show that R^{k+1} is fuzzy neutrosophic symmetric.Let $x, y \in X$ .Then

$T_{R^{k+1}}(x,y) = T_{R \circ R^k}(x,y) = \underset{z \in X}{\vee}[T_{R^k}(x,z) \wedge T_R(z,y)] = \underset{z \in X}{\vee}[T_{R^k}(z,x) \wedge T_R(y,z)]$

$= \underset{z \in X}{\vee}[T_{R^k}(y,z) \wedge T_R(z.x)] = T_{R^k \circ R}(y,x) = T_{R^{k+1}}(y,x)$

Similarly, $I_{R^{k+1}}(x,y) = I_{R^{k+1}}(y,x)$ and $F_{R^{k+1}}(x,y) = F_{R \circ R^k}(x,y) = \underset{z \in X}{\wedge}[F_{R^k}(x,z) \vee F_R(z,y)] = \underset{z \in X}{\wedge}[F_{R^k}(z,x) \vee F_R(y,z)]$

$= \underset{z \in X}{\wedge}[F_{R^k}(y,z) \vee F_R(z,x)] = F_{R^k \circ R}(y,x) = F_{R^{k+1}}(y,x)$

So R^n is fuzzy neutrosophic symmetric for any $n \geq 1$.Hence R^{∞} is fuzzy neutrosophic symmetric.

- *Proposition 3.10:*
  Let X be a set and let $P, Q \in FNR(X)$ .Then (1) If $P \subset Q$ then $P^{\infty} \subset Q^{\infty}$

(2)If $P \circ Q = Q \circ P$ and $P, Q \in FNE(X)$,then $(P \circ Q)^{\infty} = P \circ Q$

- *Proof:*

(1) It is clear that $P^2 \subset Q^2$,by definition 3.2(2). Suppose $P^k \subset Q^k$ for any $k > 2$.Then by definition 3.2(2) $P^{k+1} \subset Q^{k+1}$ .Hence $P^{\infty} \subset Q^{\infty}$

(2) Suppose $P \circ Q = Q \circ P$ and $P, Q \in FNE(X)$ .Then it is clear that $(P \circ Q)^1 = P \circ Q$

Suppose $(P \circ Q)^k = P \circ Q$ for any $k \geq 2$ .Then

$(P \circ Q)^{k+1} = (P \circ Q)^k \circ (P \circ Q) = (P \circ Q) \circ (P \circ Q) = (P \circ P) \circ (Q \circ Q) = P \circ Q$ .So $(P \circ Q)^n = P \circ Q$ for any $n \geq 1$.Hence $(P \circ Q)^{\infty} = P \circ Q$.

- *Theorem 3.2:*
  If R is a fuzzy neutrosophic relation on a set X,then $R^e = [R \cup R^{-1} \cup \Delta]^{\infty}$.

- *Proof:*

Let $Q=\left[R\cup R^{-1}\cup\Delta\right]^{\infty}$ .Then clearly $R\subset Q$ .By proposition 3.8(1) Q is fuzzy neutrosophic transitive.Let $x\in X$ .Since $\Delta\subset Q, 1=T_{\Delta}(x,x)\leq T_{Q}(x,x), 1=I_{\Delta}(x,x)\leq I_{Q}(x,x)$ *and* $0=F_{\Delta}(x,x)\geq F_{Q}(x,x)$.
Thus $T_{Q}(x,x)=1, I_{Q}(x,x)=1, F_{Q}(x,x)=0$ .So Q is fuzzy neutrosophic reflexive. It is clear that $\left[R\cup R^{-1}\cup\Delta\right]$ is fuzzy neutrosophic symmetric. By proposition 3.8 Q is fuzzy neutrosophic symmetric. Hence $Q\in FNE(X)$ .Now $K\in FNE(X)$ such that $R\subset K$ .Then $\Delta\subset K$ and $R^{-1}\subset K^{-1}=K$ .Thus $R\cup R^{-1}\cup\Delta\subset K$ .By definition3.4(2), $\left[R\cup R^{-1}\cup\Delta\right]^{n}\subset K^{n}=K$ for any $n\geq 1$ .So $Q\subset K$ .Hence $R^{e}=Q=\left[R\cup R^{-1}\cup\Delta\right]^{\infty}$ .This completes the proof.

- *Proposition 3.11:*

Let X be a set and let $P,Q\in FNE(X)$ .We define $P\vee Q$ as follows: $P\vee Q=(P\cup Q)^{\infty}$,(i.e.,) $P\vee Q=\underset{n\in N}{\cup}(P\cup Q)^{n}$ .Then $P\vee Q\in FNE(X)$.

- *Proof:*

By proposition 3.8, $P\vee Q$ is fuzzy neutrosophic transitive. Let $x\in X$ .Since P and Q are fuzzy neutrosophic reflexive.

$$(P\vee Q)(x,x)=\left(\underset{n\in N}{\vee}\left[T_{P}(x,x)\vee T_{Q}(x,x)\right]^{n},\underset{n\in N}{\vee}\left[I_{P}(x,x)\vee I_{Q}(x,x)\right]^{n},\underset{n\in N}{\wedge}\left[F_{P}(x,x)\wedge F_{Q}(x,x)\right]^{n}\right)$$
$$=\left(\underset{n\in N}{\vee}(1\vee 1)^{n},\underset{n\in N}{\vee}(1\vee 1)^{n},\underset{n\in N}{\wedge}(0\wedge 0)^{n}=(1,1,0)\right)$$

Thus $P\vee Q$ is fuzzy neutrosophic reflexive. Now let $x,y\in X$. Since P and Q are fuzzy neutrosophic symmetric,

$$(P\vee Q)(x,y)=\left(\underset{n\in N}{\vee}\left[T_{P}(x,y)\vee T_{Q}(x,y)\right]^{n},\underset{n\in N}{\vee}\left[I_{P}(x,y)\vee I_{Q}(x,y)\right]^{n},\underset{n\in N}{\wedge}\left[F_{P}(x,y)\wedge F_{Q}(x,y)\right]^{n}\right)$$
$$=\left(\underset{n\in N}{\vee}\left[T_{P}(y,x)\vee T_{Q}(y,x)\right]^{n},\underset{n\in N}{\vee}\left[I_{P}(y,x)\vee I_{Q}(y,x)\right]^{n},\underset{n\in N}{\wedge}\left[F_{P}(y,x)\wedge F_{Q}(y,x)\right]^{n}\right)=(P\vee Q)(y,x)$$

Thus $P\vee Q$ is fuzzy neutrosophic symmetric. Hence $P\vee Q\in FNE(X)$.
The following result gives another description for $P\vee Q$ of two FNERs P and Q.

- *Theorem 3.3:*

Let X be a set and let $P,Q\in FNE(X)$. If $P\circ Q\in FNE(X)$,then $P\circ Q=P\vee Q$,where $P\vee Q$ denotes the least upper bound for $\{P,Q\}$ with respect to the inclusion.

- *Proof:*

Let $x,y\in X$ .Then $x,y\in X$ .Then $T_{P\circ Q}(x,y)=\underset{z\in X}{\vee}\left[T_{Q}(x,z)\wedge T_{P}(z,y)\right]\geq T_{Q}(x,y)\wedge T_{P}(y,y)=T_{Q}(x,y)\wedge 1$
(Since R is fuzzy neutrosophic reflexive)
$=T_{Q}(x,y)$
Similarly, $I_{P\circ Q}(x,y)=I_{Q}(x,y)$

$$F_{P\circ Q}(x,y)=\underset{z\in X}{\wedge}\left[F_{Q}(x,z)\vee F_{P}(z,y)\right]\leq F_{Q}(x,y)\vee F_{P}(y,y)=F_{Q}(x,y)\vee 0=F_{Q}(x,y)$$

Thus $P\circ Q\supset Q$.
Similarly, we have $P\circ Q\supset R$ .So $P\circ Q$ is an upper bound for $\{P,Q\}$ with respect to "$\subset$".
Now let R be any fuzzy neutrosophic equivalence relation on X such that $R\supset P$ and $R\supset Q$ .Let $x,y\in X$ .Then

$$T_{P\circ Q}(x,y)=\underset{z\in X}{\vee}\left[T_{Q}(x,z)\wedge T_{P}(z,y)\right]\leq\underset{z\in X}{\vee}\left[T_{R}(x,z)\wedge T_{R(z,y)}\right]=T_{R\circ R}(x,y)\leq T_{R}(x,y)$$

(since R is fuzzy neutrosophic transitive)
Similarly, $I_{P\circ Q}(x,y)\leq I_{R}(x,y)$ and

$$F_{P\circ Q}(x,y)=\underset{z\in X}{\wedge}\left[F_{Q}(x,z)\vee F_{P}(z,y)\right]\geq\underset{z\in X}{\wedge}\left[F_{R}(x,z)\vee F_{R}(z,y)\right]=F_{R\circ R}(x,y)\geq F_{R}(x,y)$$

Thus $P \circ Q \subset R$.So $P \circ Q$ is the least upper bound for $\{P,Q\}$ with respect to "$\subset$". Hence $P \circ Q = P \vee Q$.

- *Proposition 3.11:*
  Let X be a set. If $P,Q \in FNE(X)$,then $P \vee Q = (P \circ Q)^{\infty}$.

- *Proof:*
  Suppose $P,Q \in FNE(X)$.Then by theorem 3.2, $P \vee Q = (P \cup Q)^{e} = \left[(P \cup Q) \cup (P \cup Q)^{-1} \cup \Delta\right]^{\infty}$.Since $P,Q \in FNE(X)$, $(P \cup Q) \cup (P \cup Q)^{-1} \cup \Delta = P \cup Q$.Since $P \subset P \cup Q$ and $Q \subset P \cup Q$,by definition 3.2(2) and (1), $P \circ Q \subset (P \cup Q) \circ (P \cup Q) = P \cup Q$.Thus by proposition 3.9(1) $(P \circ Q)^{\infty} \subset (P \cup Q)^{\infty}$.On the other hand ,since $P,Q \in FNE(X), P \subset P \circ Q$ and $Q \subset P \circ Q$.Thus $P \cup Q \subset P \circ Q$.By proposition 3.9(1), $(P \cup Q)^{\infty} \subset (P \circ Q)^{\infty}$.So $(P \circ Q)^{\infty} = (P \cup Q)^{\infty}$.Hence $P \vee Q = (P \cup Q)^{\infty} = (P \circ Q)^{\infty}$.

The following is the immediate result of Proposition 3.11 and Proposition 3.10.

- *Corollary 3.1:*
  Let X be a set. If $P,Q \in FNE(X)$ such that $P \circ Q = Q \circ P$,then $P \vee Q = P \circ Q$.

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