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Extended Pair of Vague Filters in Residuated Lattices

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Abstract:
The focus of this paper is to develop the theory of vague filters on residuated lattices. Characterizations of vague filters in residuated lattice are established. We discuss some properties of vague filters in terms of its level subsets. Also the notion of extended pair of vague filter is introduced and characterize their properties.

Keywords: *vague filter, extended pair of vague filter, distributive lattice.*

1. Introduction

The notion of residuated lattices is initiated in order to provide a reliable logical foundation for uncertain information processing theory and establish a logical system with truth value in a relatively general lattice. The concept of fuzzy set was introduced by Zadeh (1965) [19]. Since then this idea has been applied to other algebraic structures. Since the fuzzy set is single function, it cannot express the evidence of supporting and opposing. Hence the concept of vague set [6] is introduced in 1993 by W.L.Gau and Buehrer. D.J. In a vague set A, there are two membership functions: a truth membership fucntion t_A and a false membership function f_A, where t_A and f_A are lower bound of the grade of membership respectively and t_A(x) + f_A(x) ≤ 1. Thus the grade of membership in a vague set A is a subinterval [t_A(x), 1-f_A(x)] of [0, 1]. Vague sets is an extension of fuzzy sets. The idea of vague sets is that the membership of every elements which can be divided into two aspects including supporting and opposing. With the development of vague set theory, some structure of algebras corresponding to vague set have been studied. R.Biswas [3] initiated the study of vague algebras by studying vague groups. T.Eswarlal [5] study the vague ideals and normal vague ideals in semirings. H.Hkam , etc [13] study the vague relations and its properties. Quotient algebras are basic tool for exploring the structures of algebras. There are close correlations among filters, congruences and quotient algebras. In this paper, we introduce the concept of vague filters and we discuss some properties of Vague filters in terms of its level subsets. Also by introducing the notion of extended vague filters, it is proved that the set of all vague filters forms a bounded distributive lattice.

2. Preliminaries

2.1. Definition 2.1: [17]

A residuated lattice is an algebraic structure L = (L, ∨, ∧, *, →, 0, 1) satisfying the following axioms:

1. (L, ∨, ∧, 0, 1) is a bounded lattice
2. (L, *, 1) is a commutative monoid.
3. (*, 1) is an adjoint pair, i.e., for any x, y, z, w∈ L,
   i. if x ≤ y and z ≤ w , then x * z ≤ y * w.
   ii. if x ≤ y and y → z ≤ x → z then z → x ≤ z → y.
   iii. (adjointness condition) x * y ≤ z if and only if x ≤ y → z.

In this paper, denote L as residuation lattice unless otherwise specified.

2.2. Definition 2.2: [20]

Let U ≠ φ. A mapping f : U → [0, 1] is called a fuzzy set. Let f and g be fuzzy sets on U. Then tip- extended pair of f and g [19, 20] can be defined by

$$f^g(\text{x}) = \begin{Bmatrix} f(x), & x \neq 1 \\ f(1) \vee g(1), & x = 1 \end{Bmatrix}$$

$g^f(x) = \begin{cases} g(x), & x \neq 1 \\ g(1) \vee f(1), & x = 1 \end{cases}.$

2.3. Theorem 2.3: [17, 16]

In each residuated lattice L, the following properties hold for all x, y, z ∈ L:

1. (x * y) → z = x → (y → z).
2. z ≤ x → y ⇔ z * x ≤ y.
3. x ≤ y ⇔ z * x ≤ z * y.
4. x → (y → z) = y→ (x → z).
5. x ≤ y ⇒ z → x ≤ z → y.
6. x ≤ y ⇒ y → z ≤ x → z, $y' \leq x'$.
7. y → z ≤ (x → y) → (x → z).
8. y → x ≤ (x → z) → (y → z).
9. 1 → x = x, x → x = 1.
10. $x^m \leq x^n$, m, n ∈ N, m ≥ n.
11. x ≤ y ⇔ x → y = 1.
12. $0' = 1$, $1' = 0$, $x' = x^m$, x ≤ x^n.
13. x ∨ y → z = (x → z) ∧ (y → z).
14. $x * x' = 0$.
15. x → (y ∧ z) = (x → y) ∧ (x → z).

2.4. Definition 2.4: [20]

A non-empty subset F of a residuated lattice L is called a filter of L if it satisfies

1. x, y ∈ F ⇒ x * y ∈ F.
2. x ∈ F, x ≤ y ⇒ y ∈ F.

2.5. Theorem 2.5: [20]

A non-empty subset F of a residuated lattice L is called a filter of L if it satisfies, for any x,y∈ L,

1. 1 ∈ F.
2. x ∈ F, x → y ∈ F ⇒ y ∈ F.

2.6. Note 2.6: [20]

A fuzzy set A on a residuated lattice L is a mapping from L' to [0, 1]

2.7. Definition 2.7: [20]

A fuzzy set A of a residuated lattice L is called a fuzzy filter, if it satisfies, for any x, y ∈ L

1. A(1) ≥ A(x).
2. A(x * y) ≥ min{A(x), A(y)}.

2.8. Theorem 2.8: [20]

A fuzzy set A of a residuated lattice L is a fuzzy filter, if and only if it satisfies, for any x, y ∈ L,

1. A(1) ≥ A(x).
2. A(y) ≥ min{A(x → y), A(x)}

2.9. Definition 2.9: [3]

A Vague set A in the universe of discourse S is a Pair (t_A, f_A) where t_A : S → [0,1] and f_A : S → [0,1] are mappings (called truth membership function and false membership function respectively) where t_A(x) is a lower bound of the grade of membership of x derived from the evidence for x and f_A(x) is a lower bound on the negation of x derived from the evidence against x and $t_A(x) + f_A(x) \leq 1$ ∀x∈ S.

2.10. Definition 2.10: [18]

Let δ be a mapping from [0, 1] × [0, 1] to [0, 1]. δ is called a t-norm (resp. s-norm) on [0, 1], if it satisfies the following conditions: for any x, y, z ∈ [0, 1]

1. δ(x, 1) = x (resp. δ(x, o) = x),
2. δ(x, y) = δ(y, x),
3. δ(δ(x, y), z) = δ(x, δ(y, z)),
4. if x ≤ y, then δ(x, z) ≤ δ(y, z).

3. Vague Filters on Residuated Lattice

3.1. Definition 3.1:

A Vague set A of L is called a vague filter of L, if for any x, y ∈ L:

1. V_A(I) ≥ V_A(x)
2. V_A(y) ≥ min(V_A(x → y), V_A(x))

3.2. Theorem 3.2:

Let A be a vague filter of L. Then, for any x, y ∈ L : if x ≤ y, then V_A(x) ≤ V_A(y) .

- Proof:

Since x ≤ y, it follows that x → y = I. Since A is a vague filter of L, we have V_A(y) ≥ min(V_A(x → y), V_A(x)) and V_A(I) ≥ V_A(x) for any x, y ∈ L. Therefore V_A(y) ≥ min (V_A(x → y), V_A(x)) = min (V_A(I), V_A(x)) ≥ min (V_A(x), V_A(x)) = V_A(x). Therefore V_A(x) ≤ V_A(y).

Theorem 3.3:

Let A be a vague se t on L. Then A is vague filter of L, if and only if , for any x, y, z ∈ L V_A(I) ≥ V_A(x) and V_A(x → z) ≥ min(V_A(y → (x → z)) , V_A(y)) .

- Proof:

Let A be vague filter of L, obviously V_A(I) ≥ V_A(x) and V_A(I) ≥ V_A(x) and V_A(x → z) ≥ min(V_A(y → (x → z)) , V_A(y)) holds for any x, y, z ∈ L. Taking x = I in V_A(x → z) ≥ min(V_A(y → (x → z)) , V_A(y)), we have V_A(z) = V_A(I → z) ≥ min(V_A(y → (I → z)), V_A(y)) = min(V_A(y → z), V_A(y)). Since V_A(I) ≥ V_A(x) holds, and so A is a vague filter of L.

3.4. Theorem 3.4:

Let A be a vague set on L. Then A is a vague filter of L, if and only if, for any x, y, z ∈ L, A satisfies if x ≤ y, then V_A(x) ≤ V_A(y) for any x, y ∈ L and V_A(x * y) ≥ min(V_A(x), V_A(y)) .

- Proof:

Assume that A is a vague filter of L, obviously if x ≤ y, then V_A(x) ≤ V_A(y) holds for any x,y ∈ L. Since x ≤ y → (x * y), we have V_A(y → (x * y)) ≥ V_A(x). By Definition 3.1 (2), it follows that V_A(x * y) ≥ min(V_A(y), V_A(y → (x * y))) ≥ min(V_A(y), V_A(x)). Conversely, assume that if x ≤ y, then V_A(x) ≤ V_A(y) and V_A(x * y) ≥ min(V_A(x), V_A(y)) holds for any x, y ∈ L. Taking y = I, we get V_A(I) ≥ V_A(x) . As x * (x → y) ≤ y, thus V_A(y) ≥ V_A(x * (x → y)). Therefore V_A(y) ≥ min(V_A(x), V_A(x → y)). Hence A is a vague filter of L.

3.5. Remark 3.5:

A vague set on L is a vague filter of L, if and only if, for any x, y, z ∈ L : if x → (y → z) = I then V_A(z) ≥ min(V_A(x), V_A(y)).

3.6. Remark 3.6:

A vague set on L is a vague filter of L, if and only if, for any x, y, z ∈ L:
if a_n → (a_{n-1} → ….→ (a_1 → x) …..) = I, then V_A(x) ≥ min($V_A(a_n)$, …….,$V_A(a_1)$)

3.7. Theorem 3.7:

A vague set on L is a vague filter of L, if and only if, for any x, y, z ∈ L, A satisfies Remark 3.5 and V_A((x → (y → z)) → z) ≥ min(V_A(x), V_A(y)).

- Proof:

If A is a vague filter of L then Remark 3.5 holds. Since V_A((x → (y → z)) → z) → z) ≥ min (V_A((x → (y → z)) → (y → z)), V_A(y)). As (x → (y → z)) → (y → z) = x ∨ (y → z) ≥ x , by Theorem 3.2 we have V_A((x → (y → z)) → (y → z)) ≥ V_A(x). Therefore, V_A((x → (y → z)) → z) ≥ min (V_A(x), V_A(y)). Conversely, suppose V_A((x → (y → z)) → z) ≥ min(V_A(x), V_A(y)) is valid. Since V_A(y) = V_A(I → y) = V_A(((x → y) → (x → y)) → y) ≥ min(V_A(x → y), V_A(x)). Hence by Definition 3.1, A is s vague filter of L.

3.8. Theorem 3.8:

Let A be a vague set on L. Then A is a vague filter of L, for any x, y, z ∈ L, A satisfies Definition 3.1(1) and V_A(x → z) ≥ min(V_A(x → y), V_A(y → z)).

- Proof:

Assume that A is vague filter of L. Since (x → y) ≤ (y → z) → (x → z), it follows from Theorem 3.2 that V_A((y → z) → (x → z)) ≥ V_A(x → y). As A is a vague filter , so V_A(x → z) ≥ min(V_A(y → z), V_A((y → z) → (x → z))). We have V_A(x → z) ≥ min(V_A(y → z), V_A(x → z)). Conversely, if V_A(x → z) ≥ min(V_A(x → y), V_A(y → z)) for any x, y, z ∈ L, then V_A(I → z) ≥ min(V_A(I → y), V_A(y → z)) that is V_A(z) ≥ min(V_A(y), V_A(y → z)). Hence by definition 3.1 A is a vague filter of L.

3.9. Theorem 3.9:

Let A be a vague set on L. Then A is a vague filter of L, if and only if, for any α, β ∈ [0, 1] and α + β ≤ 1, the sets U(t_A, α) (≠ φ) and L(1-f_A, β) (≠ φ) are filters of L, where U(t_A, α) = {x ∈ L / t_A(x) ≥ α} , L(1- f_A(x), β) = {x ∈ L/ 1- f_A(x) ≥ β}.

- Proof:

Assume A is a vague filter of L, then V_A(I) ≥ V_A(x) . By the condition U(t_A, α) ≠ φ, it follows that there exist a ∈ L such that t_A(a) ≥ α , and so t_A(I) ≥ α, hence I ∈ U(t_A, α). Let x, x → y ∈ U(t_A, α), then t_A(x) ≥ α, t_A(x → y) ≥ α. Since A is a filter of L, then t_A(y) ≥ min(t_A(x), t_A(x → y)) ≥ min (α, α) = α. Hence y ∈ U(t_A, α). Therefore U(t_A, α) is a filter of L. We will show that L(1- f_A(x), β) is a filter of L. Since A is a vague filter of L, then 1- f_A(I) ≥ 1- f_A(x). By the condition L(1- f_A(x), β) ≠ φ, it follows that there exist a ∈ L such that 1- f_A(a) ≥ β. Therefore we have 1- f_A(I) ≥ 1- f_A(a) ≥ β. Hence I∈ L(1- f_A(x), β). Let x, x → y ∈ L(1- f_A(x), β), then 1- f_A(x) ≥ β , 1- f_A(x → y) ≥ β. Since A is a vague filter of L, then 1- f_A(y) ≥ min (1- f_A(x), 1- f_A(x → y)) ≥ min (β, β) = β. It follows that 1- f_A(y) ≥ β, hence y ∈ L(1- f_A(x), β). Therefore L(1- f_A(x), β) is a filter of L. Conversely, suppose that U(t_A, α) ≠ φ and L(1- f_A(x), β) ≠ φ are filters of L, then, for any x ∈ L, x ∈ U(t_A, t_A(x)) and x ∈ L(1- f_A, 1- f_A(x)). By U(t_A, t_A(x)) ≠ φ and L(1- f_A, 1- f_A(x)) ≠ φ are filters of L, it follows that I∈ U(t_A, t_A(x)) and I ∈ L(1- f_A, 1- f_A(x)), and so V_A(I) ≥ V_A(x). For any x, y ∈ L, let α = min(t_A(x), t_A(x → y)) and β = min (1- f_A(x), 1- f_A(x → y)), then x, x → y ∈ U(t_A, α) and x, x → y ∈ L(1- f_A, β). And so y ∈ U(t_A, α) and y ∈ L(1- f_A(x), β). Therefore t_A(y) ≥ α = min(t_A(x), t_A(x → y)) and 1- f_A(y) ≥ β = min(1- f_A(x), 1- f_A(x → y)). From theorem 3.2, we have A is a vague filter of L.

3.10. Theorem 3.10:

Let A, B be two vague filters of L, then A∩B is also a vague filter of L.

- Proof:

Let x , y, z ∈ L such that z ≤ x → y, then z → (x → y) = I. Since A, B be two vague filters of L, we have V_A(y) ≥ min (V_A(z), V_A(x)) and V_B(y) ≥ min (V_B(z), V_B(x)). Since $V_{A\cap B}$(y) = min(V_A(y), V_B(y)) ≥ min(min(V_A(z), V_A(x)), min (V_B(z), V_B(x))) = min(min(V_A(z), V_B(z)), min(V_A(x), V_B(x)) = min($V_{A\cap B}$(z), $V_{A\cap B}$(x)). Since A, B be two vague filters of L, we have V_A(I) ≥ V_A(x) and V_B(I) ≥ V_B(x). Hence $V_{A\cap B}$(I) = min(V_A(I), V_B (I)) ≥ min(V_A(x), V_B(x)) = $V_{A\cap B}$(x) . Then A ∩ B is a vague filters of L.

3.11. Remark 3.11:

Let A_i be a family of vague sets on L, where i is an index set. Denoting C by the intersection of A_i, i.e. $\bigcap_{i\in I} A_i$, where V_C(x) = min(V_{A_1}(x), V_{A_2}(x), ……) for any x ∈ L.

3.12. Note 3.12:

Let A_i be a family of vague filters of L, where i ∈ I, I is an index set, then $\bigcap_{i\in I} A_i$ is also a vague filters of L.

3.13. Theorem 3.13:

Let A be a vague set on L. Then

a. For any α, β ∈ [0, 1], if $A_{(\alpha,\beta)}$ is a filter of L. Then, for any x, y, z ∈ L, V_A(z) ≤ min(V_A(x → y), V_A(x)) imply V_A(z) ≤ V_A(y).
b. If A satisfy Definition 3.1(1) and condition (a), then, for any α, β ∈ [0, 1], $A_{(\alpha,\beta)}$ is a filter of L.

Proof:

a. Assume that $A_{(\alpha,\beta)}$ is a filter of L for any α, β ∈ [0, 1].
Since V_A(z) ≤ min(V_A(x → y), V_A(x)), it follows that V_A(z) ≤ V_A(x → y), V_A(z) ≤ V_A(x). Therefore, x → y ∈ $A_{(t_A(z),1-f_A(z))}$, x ∈ $A_{(t_A(z),1-f_A(z))}$. As V_A(z) ∈ [0, 1], and $A_{(t_A(z),1-f_A(z))}$ is a filter of L, so y ∈ $A_{(t_A(z),1-f_A(z))}$. Thus V_A(z) ≤ V_A(y).
b. Assume A satisfy (a) and (b). For any x, y ∈ L, α, β ∈ [0, 1], we have x → y ∈ $A_{(\alpha,\beta)}$, x ∈ $A_{(\alpha,\beta)}$,therefore t_A(x → y) ≥ α, 1- f_A(x → y) ≥ β and t_A(x) ≥ α , 1- f_A(x) ≥ β , and so min(t_A(x → y) , t_A(x)) ≥ min(α, α) = α. By (a), we have t_A(y) ≥ α and 1- f_A(y) ≥ β, that is, y ∈ $A_{(\alpha,\beta)}$. Since V_A(I) ≥ V_A(x) for any x ∈ L, it follows that t_A(I) ≥ α and 1- f_A(I) ≥ β, that is, I ∈ $A_{(\alpha,\beta)}$. Then for any α, β ∈ [0, 1], $A_{(\alpha,\beta)}$ is a filter of L.

3.14. Theorem 3.14:

Let A be a vague filter of L, then for any α, β ∈ [0, 1], $A_{(\alpha,\beta)}$ (`≠ φ) is a filter of L.

- Proof:

Since $A_{(\alpha,\beta)} \neq \varphi$, there exist α, β ∈ [0, 1] such that t_A(x) ≥ α , 1- f_A(x) ≥ β . And A is a vague filter of L, we have t_A(I) ≥ t_A(x) ≥ α, 1- f_A(I) ≥ 1- f_A(x) ≥ β, therefore I ∈ $A_{(\alpha,\beta)}$. Let x, y ∈ L and x ∈ $A_{(\alpha,\beta)}$, x → y ∈ $A_{(\alpha,\beta)}$, therefore t_A(x) ≥ α, 1- f_A(x) ≥ β , t_A(x → y) ≥ α, 1- f_A(x → y) ≥ β. Since A is a vague filter of L, thus t_A(y) ≥ min(t_A(x → y), t_A(x)) ≥ α and 1- f_A(y) ≥ min(1- f_A(x → y), 1- f_A(x)) ≥ β, it follows that y ∈ $A_{(\alpha,\beta)}$. Therefore, $A_{(\alpha,\beta)}$ is a filter of L.

3.15. Remark 3.15:

From Theorem 3.14, the filter $A_{(\alpha,\beta)}$ is also called a vague – cut filter of L.

3.16. Theorem 3.16:

Any filter F of L is a vague –cut filter of some vague filter of L.

- Proof:

Consider the vague set A of L: A = {(x, t_A(x) / x ∈ L}, where If x ∈ F, V_A(x) = α. If x ∉ F, V_A(x) = 0. where α ∈ [0, 1]. Since F is a filter of L, we have 1 ∈ F. Therefore V_A(I) = α ≥ V_A(x). For any x, y ∈L, if y ∈ F, then V_A(y) = α = min(α, α) ≥ min (V_A(x → y), V_A(x)). If y ∉ F, then x ∉ F or x → y ∉ F. And so V_A(y) = 0 = min(0, 0) = min(V_A(x → y), V_A(x)). Therefore A is a vague filter of L.

3.17. Theorem 3.17:

Let A be a vague filter of L. Then F = {x ∈ L / t_A(x) = t_A(I), 1- f_A(x) = 1- f_A(I)} is a filter of L.

- Proof:

Since F = {x ∈ L / t_A(x) = t_A(I), 1- f_A(x) = 1- f_A(I)}, obviously I ∈ F. Let x → y ∈ F, x ∈ F, so V_A(x → y) = V_A(x) = V_A(I). Therefore V_A(y) ≥ min(V_A(x → y), V_A(x)) = V_A(I) and V_A(I) ≥ V_A(y), then V_A(y) = V_A(I). Thus y ∈ F. It follows that F is a filter of L.

4. Extended Pair of Vague Filters

4.1. Remark 4.1:

Let A, B be two vague filters of L, then A∩B is also a vague filter of L.

4.2. Remark 4.2:

For A ∈ VS (L), the intersection of all vague filters containing A is called the generated fuzzy filter by A, denoted as < A >.

4.3. Remark 4.3:

Let A ∈ VS (L). Then A is a vague filter if and only if x * y ≤ z implies V_A(x) ∧ V_A(y) ≤ V_A(z) for all x, y, z ∈ L.

4.4. Theorem 4.4:

Let A ∈ VS(L). Define a new vague set B by B = $[t_B, 1 - f_B]$ where V_B(x) = $\vee_{a_1* ....a_n \leq x}\{V_A(a_1) \wedge \ldots\ldots V_A(a_n)\}$, for all x ∈ L. where a_i∈ L, n ∈ N. Then B = < A >.

- Proof:

We complete the proof by two steps. Firstly, we verify that B is a vague filter. For all x, y ∈ L, such that x ≤ y, the Definition of B yields that V_B(x) ≤ V_B(y). For all x, y ∈ L, we have V_B(x) ∧ V_B(y) = $\vee_{a_1* ....a_n \leq x}\{V_A(a_1) \wedge \ldots\ldots V_A(a_n)\} \wedge \vee_{b_1* ....b_m \leq y}\{V_A(b_1) \wedge \ldots\ldots V_A(b_m)\}$ = $\vee_{a_1* ....a_n \leq x} \vee_{b_1* ....b_m \leq y}\{\{V_A(a_1) \wedge \ldots\ldots V_A(a_n) \wedge V_A(b_1) \wedge \ldots\ldots V_A(b_m)\}$, where a_i, b_i ∈ L, n,m ∈N, for all x, y ∈ L. ≤ $\vee_{c_1* ....c_k \leq x*y}\{V_A(c_1) \wedge \ldots\ldots V_A(c_k)\}$, c_i ∈ L, k ∈ N. = V_B(x * y), by Remark 4.3. Thus B is a vague filter. Secondly, let C be a vague filter such that C ⊇ A. By the Definition of vague filter, it holds that V_B(x) = $\vee_{a_1* ....a_n \leq x}\{V_A(a_1) \wedge \ldots\ldots V_A(a_n)\}$ ≤ $\vee_{a_1* ....a_n \leq x}\{V_C(a_1) \wedge \ldots\ldots V_C(a_n)\}$ ≤ $\vee_{a_1* ....a_n \leq x}\{V_C(a_1 * \ldots\ldots * a_n)\}$ ≤ V_C(x). Hence B ⊆ C. Thus B = < A >.

4.5. Lemma 4.5:

Let a, b, u, v ∈ [0, 1] such that 0 ≤ a + b ≤ 1 and 0 ≤ u + v ≤ 1. Then 0 ≤ a ∨ u + b ∧ v ≤ 1.

- Proof:

Without losing the generality, we assume that a ≤ u. Then a ∨ u + b ∧ v ≤ u + v ≤ 1. It obvious that 0 ≤ a ∨ u + b ∧ v. Thus it holds that 0 ≤ a ∨ u + b ∧ v ≤ 1.

4.6. Theorem 4.6:

Let A be a vague filter of L and for all u, v ∈ [0, 1] such that 0 ≤ u + v ≤ 1. Then $A^{u,v}$ = [$t_A{}^u$, 1-$f_A{}^v$] is a vague filter, where $t_A{}^u$ (x) = $\begin{Bmatrix} t_A(x), & x \neq 1 \\ t_A(1) \vee u, & x = 1 \end{Bmatrix}$,

$$1\text{-}f_A{}^v(x) = \begin{Bmatrix} 1-f_A(x), & x \neq 0 \\ 1-f_A(0) \vee v, & x = 0 \end{Bmatrix}.$$

- Proof:

It follows form Lemma 3.6 that $A^{u,v}$ ∈ VS(L). Now we prove that $A^{u,v}$ is a vague filter. If x ≤ y, we consider the following two cases.

Case 1: (y = 1). It is obvious that $t_A{}^u$(x) ≤ $t_A{}^u(1)$ = $t_A{}^u$(y), 1-$f_A{}^v$(x) ≤ 1-$f_A{}^v$(0) = 1-$f_A{}^v$(y).
Case 2: (y ≠ 1). The Definition of $A^{u,v}$ leads that $t_A{}^u$(x) = $t_A(x) \leq t_A(y)$ = $t_A{}^u$(y), 1-$f_A{}^v$ (x) = $1-f_A(x) \leq 1-f_A(y)$ = 1-$f_A{}^v$(y). Thus $t_A{}^u$(x) ≤ $t_A{}^u$(y), 1-$f_A{}^v$(x) ≤ 1-$f_A{}^v$(y).
Let x, y ∈ L. We consider the following two cases.
Case 1: (x * y = 1). If x = y = 1, it is obvious that $t_A{}^u$(x) ∧ $t_A{}^u$(y) ≤ $t_A{}^u$(x * y), 1-$f_A{}^v$(x) ∧ 1-$f_A{}^v$(y) ≤ 1-$f_A{}^v$(x * y). If x = 1, y ≠ 1 or x ≠ 1, y = 1, it is a contradiction. If x ≠ 1, y ≠ 1, it holds that $t_A{}^u$(x) ∧ $t_A{}^u$(y) = $t_A(x) \wedge t_A(y)$ ≤ t_A(x * y) = $t_A{}^u$(x * y), 1-$f_A{}^v$(x) ∧ 1-$f_A{}^v$(y) = $1-f_A(x) \wedge 1-f_A(y) \leq 1-f_A(x*y)$ = 1-$f_A{}^v$(x * y).
Case 2: (x * y ≠ 1), it is a contradiction. If x = y = 1, it is a contradiction. If x = 1, y ≠ 1or x ≠ 1, y = 1, it is obvious that $t_A{}^u$(x) ∧ $t_A{}^u$(y) ≤ $t_A{}^u$(x * y), 1-$f_A{}^v$(x) ∧ 1-$f_A{}^v$(y) ≤ 1-$f_A{}^v$(x * y). If x ≠ 1, y ≠ 1, we have $t_A{}^u$(x) ∧ $t_A{}^u$(y) = $t_A(x) \wedge t_A(y)$ ≤ t_A(x * y) = $t_A{}^u$(x * y) 1-$f_A{}^v$(x) ∧ 1-$f_A{}^v$(y) = $1-f_A(x) \wedge 1-f_A(y) \leq 1-f_A$(x * y) =1- $f_A{}^u$(x * y). And in all, it yields that $t_A{}^u$(x) ∧ $t_A{}^u$(y) = $t_A(x) \wedge t_A(y)$ ≤ t_A(x * y) = $t_A{}^u$(x * y), 1-$f_A{}^v$(x) ∧ 1-$f_A{}^v$(y) = $1-f_A(x) \wedge 1-f_A(y) \leq 1-f_A$(x * y) =1- $f_A{}^u$(x * y). Thus $A^{u,v}$ is a vague filter.

4.7. Definition 4.7:

For given A, B ∈ VS(L), the operation $\tilde{*}$ is defined by A $\tilde{*}$ B is defined by
A $\tilde{*}$ B = [$t_A \tilde{*} t_B, 1-f_A \tilde{*} 1-f_B$], where $V_A \tilde{*} V_B$(x) = $\bigvee_{y*z \leq x}\{V_A(y) \wedge V_B(z)\}$.

4.8. Definition 4.8:

The extended pair for vague sets A and B defined by $A^B = [t_A{}^{t_B}, 1-f_A{}^{1-f_B}]$,
$t_A{}^{t_B}$(x) = $\begin{Bmatrix} t_A(x), & x \neq 1 \\ t_A(1) \vee t_B(1), & x = 1 \end{Bmatrix}$, $1-f_A{}^{1-f_B}$(x) = $\begin{Bmatrix} 1-f_A(x), & x \neq 0 \\ 1-f_A(0) \vee 1-f_B(0), & x = 0 \end{Bmatrix}$
$B^A = [t_B{}^{t_A}, 1-f_B{}^{1-f_A}]$ where $t_B{}^{t_A}$(x) = $\begin{Bmatrix} t_B(x), & x \neq 1 \\ t_B(1) \vee t_A(1), & x = 1 \end{Bmatrix}$, $1-f_B{}^{1-f_A}$(x) = $\begin{Bmatrix} 1-f_B(x), & x \neq 0 \\ 1-f_B(0) \vee 1-f_A(0), & x = 0 \end{Bmatrix}$.

4.9. Theorem 4.9:

Let A, B ∈ VF(L). Then $A^B \tilde{*} B^A$ ∈ VF(L).

- Proof:

It is obvious that $t_A{}^{t_B} \tilde{*} t_B{}^{t_A}$ is order preserving, and $1-f_A{}^{1-f_B} \tilde{*} 1-f_B{}^{1-f_A}$ is order preserving. For all x, y ∈ L, we have $V_A{}^{V_B} \tilde{*} V_B{}^{V_A}$ (x * y) = $\bigvee_{p*q \leq x*y}\{V_A{}^{V_B}(p) \wedge V_B{}^{V_A}(q)\} \geq \bigvee_{\substack{a*b \leq x \\ c*d \leq y}}\{V_A{}^{V_B}(a*c) \wedge V_B{}^{V_A}(b*d)\} \geq \bigvee_{\substack{a*b \leq x \\ c*d \leq y}}\{V_A{}^{V_B}(a) \wedge V_A{}^{V_B}(c) \wedge V_B{}^{V_A}(b) \wedge V_B{}^{V_B}(d)\}$ = $\bigvee_{a*b \leq x}\{V_A{}^{V_B}(a) \wedge V_B{}^{V_A}(b)\} \wedge \bigvee_{c*d \leq y}\{V_A{}^{V_B}(c) \wedge V_B{}^{V_A}(d)\}$ = $V_A{}^{V_B} \tilde{*} V_B{}^{V_A}$(x) ∧ $V_A{}^{V_B} \tilde{*} V_B{}^{V_A}$(y) and hence $V_A{}^{V_B} \tilde{*} V_B{}^{V_A}$ (x * y) ≥ $V_A{}^{V_B} \tilde{*} V_B{}^{V_A}$(x) ∧ $V_A{}^{V_B} \tilde{*} V_B{}^{V_A}$(y). Thus $A^B \tilde{*} B^A$ ∈ VF(L).

4.10. Theorem 4.10:

Let A, B ∈ VF(L). Then $A^B \tilde{*} B^A$ = < A ∪ B >.

- Proof:

It is easy to prove that A, B ⊆ $A^B \tilde{*} B^A$, and hence A ∪ B ⊆ $A^B \tilde{*} B^A$. Thus < A ∪ B > ⊆ $A^B \tilde{*} B^A$. Assume that C ∈ VF(L) such that A ∪ B ⊆ C. If x = 1, we have $t_A{}^{t_B} \tilde{*} t_B{}^{t_A}$ (1) = $t_A(1) \vee t_B(1) \leq t_C(1)$, $1-f_A{}^{1-f_B} \tilde{*} 1-f_B{}^{1-f_A}$ (0) = $1-f_A(0) \wedge 1-f_B(0)$ ≤ $1-f_C(0)$. It holds that $V_A{}^{V_B} \tilde{*} V_B{}^{V_A}$ (x) = $\bigvee_{p*q \leq x}\{V_A{}^{V_B}(p) \wedge V_B{}^{V_A}(q)\}$ = $\bigvee_{\substack{p*q \leq x \\ p \neq 1, q \neq 1}}\{V_A{}^{V_B}(p) \wedge V_B{}^{V_A}(q)\} \vee \bigvee_{p \leq x}\{V_A(p)\} \vee \bigvee_{q \leq x}\{V_B(q)\}$ = $\bigvee_{\substack{p*q \leq x \\ p \neq 1, q \neq 1}} V_A(p) \wedge V_B(q)\} \vee \bigvee_{p \leq x}\{V_A(p)\} \vee \bigvee_{q \leq x}\{V_B(q)\}$ ≤ $\bigvee_{\substack{p*q \leq x \\ p \neq 1, q \neq 1}} V_C(p) \wedge V_C(q)\} \vee \bigvee_{p \leq x}\{V_C(p)\} \vee \bigvee_{q \leq x}\{V_C(q)\}$ = $\bigvee_{p \tilde{*} q \leq x}\{V_A(p) \wedge V_B(q)\} \leq V_C(x)$. It follows from Theorem 3.10 that $A^B \tilde{*} B^A$ = < A ∪ B >.

4.11. Remark 4.11:

For A,B∈ VF(L), then the operations ⊓ and ⊔ on VF(L) are defined by A⊓B=A∩B, A⊔B=$A^B \tilde{*} B^A$.

4.12. Theorem 4.12:
(VF(L), ⊓, ⊔, ∅, L) is a bounded distributive lattice.

- Proof:

We only verify the distributivity. Obviously, it holds that C ⊓ (A ∪ B) ⊇ (C ⊓A) ⊔ (C ⊓ B), so we only prove C ⊓ (A ∪ B) ⊆ (C ⊓A) ⊔ (C ⊓ B). Assume that x ∈ L for $V_C \wedge {V_A}^{V_B} \tilde{*} {V_B}^{V_A}$ (x) ≤ $(V_C \wedge V_A)^{V_C \wedge V_B} \tilde{*} (V_C \wedge V_B)^{V_C \wedge V_A}$(x), we consider the following two cases.

Case 1: (x = 1). We have $V_C \wedge V_{A \cup B}(1) = V_C(1) \wedge {V_A}^{V_B} \tilde{*} {V_B}^{V_A}$ (1) = $V_C(1) \wedge (V_A(1) \vee V_B(1))$ = $(V_C(1) \wedge (V_A(1)) \vee (V_C(1) \wedge (V_B(1))$ = $(V_C \wedge V_A)^{V_C \wedge V_B} \tilde{*} ((V_C \wedge V_B)^{V_C \wedge V_A}(1)$.

Case 2: (x ≠ 1). It holds that $V_C(x) \wedge {V_A}^{V_B} \tilde{*} {V_B}^{V_A}(x) = V_C(x) \wedge \bigvee_{p \tilde{*} q \le x}\{{V_A}^{V_B}(p) \wedge {V_B}^{V_A}(q)\}$ = $\bigvee_{p \tilde{*} q \le x}\{V_C(x) \wedge {V_A}^{V_B}(p) \wedge {V_B}^{V_A}(q)\}$ = $\bigvee_{\substack{p*q \le x \\ p \ne 1, q \ne 1}} \{V_C(x) \wedge V_A(p) \wedge V_C(x) \wedge V_B(q)\} \vee \{V_C(x) \wedge {V_A}^{V_B}(1) \wedge V_B(x)\} \vee \{V_C(x) \wedge {V_B}^{V_A}(1) \wedge V_A(x)\}$ = $\bigvee_{\substack{p*q \le x \\ p \ne 1, q \ne 1}} \{V_C(x) \wedge V_A(p) \wedge V_C(x) \wedge V_B(q)\} \vee \{V_C(x) \wedge V_C(1) \wedge {V_A}^{V_B}(1) \wedge V_B(x)\} \vee \{V_C(x) \wedge V_C(1) \wedge {V_B}^{V_A}(1) \wedge V_A(x)\}\}$ = $\bigvee_{\substack{p*q \le x \\ p \ne 1, q \ne 1}} \{V_C(x) \wedge V_A(p) \wedge V_C(x) \wedge V_B(q)\} \vee \{V_C(1) \wedge (V_A(1) \vee V_B(1))\} \wedge [(V_C(x) \wedge V_B(x)) \vee (V_C(x) \wedge V_A(x))]\}$ = $\bigvee_{\substack{p*q \le x \\ p \ne 1, q \ne 1}} \{V_C(x) \wedge V_A(p) \wedge V_C(x)\} \wedge V_B(q)\} \vee \{[(V_C(1) \wedge (V_A(1)) \vee (V_C(1) \wedge V_B(1))] \wedge [(V_C(x) \wedge V_B(x)) \vee (V_C(x) \wedge V_A(x))]\}$ ≤ $\bigvee_{\substack{p*q \le x \\ p \ne 1, q \ne 1}} \{(V_C \wedge V_A)^{V_C \wedge V_B}(p \vee x) \wedge (V_C \wedge V_B)^{V_C \wedge V_A}(q \vee x)\} \vee$ $[(V_C \wedge V_A)^{V_C \wedge V_B}(1) \wedge (V_C \wedge V_A)(p \vee x)$ = $[(V_C \wedge V_B)^{V_C \wedge V_A}(1) \wedge (V_C \wedge V_B)(q \vee x)]$ = $\bigvee_{p*q \le x}\{(V_C \wedge V_A)^{V_C \wedge V_B}(p \vee x) \wedge (V_C \wedge V_B)^{V_C \wedge V_A}(q \vee x)\}$. Let p∨x = p' and q∨x = q'. It is easy to verify that $p' * q'$ ≤ x, and then the above can be written as $\bigvee_{p*q \le x}\{(V_C \wedge V_A)^{V_C \wedge V_B}(p') \wedge (V_C \wedge V_B)^{V_C \wedge V_A}(q')\}$ = $(V_C \wedge V_A)^{V_C \wedge V_B} \tilde{*} (V_C \wedge V_B)^{V_C \wedge V_A}$(x). Thus $V_C \wedge {V_A}^{V_B} \tilde{*} {V_B}^{V_A} \le (V_C \wedge V_A)^{V_C \wedge V_B} \tilde{*} (V_C \wedge V_B)^{V_C \wedge V_A}$, that is, the distributivity holds.

5. References

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