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A Note on Fuzzy Ideals in Near-Rings and its Anti-homomorphism

M. Rupa Lavanya
Assistant Professor, DVR & Dr. HS MIC College of Engineering & Technology, Kanchikacherla, India
Dr. T. V. Pradeep Kumar
Professor, ANU College of Engineering and Technology, Acharya Nagarjuna University, Guntur, AP, India

Abstract:
The Theory of Fuzzy sets introduced in 1964 by Zadeh [10]. The study of Fuzzy algebraic structures has started by Rosen field [9] in 1970. In this paper we investigate anti-homomorphic images and pre-images of semi prime, strongly primary, irreducible, strongly irreducible Fuzzy ideals and f-invariant, semi prime Fuzzy ideals of a Near-Ring N.
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1. Introduction

The theory of fuzzy sets, proposed by Zadeh [10], has provided a useful mathematical device for unfolding the behavior of the systems that are too complex or nonspecific to admit precise mathematical analysis by conventional methods and tools. The study of fuzzy algebraic structure has started by Rosenfeld [9] and since then this concept has been applied to a variety ofalgebraic structures. Liu introduced the concept of a fuzzy ideal of aNear- ring in[11].

The concepts of prime fuzzy ideals, primary fuzzy ideals were introduced in [7]. T Shah and M.Saeed has introduced strongly primary fuzzy ideals and strongly irreducible fuzzy ideals of a Near- ring [13]. Sheikabdullah and Jeyaramanhas discussed anti- homomorphic images and pre images of prime fuzzy ideals and anti-homomorphic image ofprimary fuzzy ideals in a ringin [12].

In this note we investigate some anti- homomorphic images and pre - images ofsemi prime, strongly primary, irreducible and strongly irreducible fuzzy ideals. We also prove that: for a surjective anti-homomorphism $f: N \to N^1$, if every fuzzy ideal of R is f-invariant and has a fuzzy primary respectively, strongly primary) decomposition in N, then every fuzzy ideal of N^1has a fuzzy primary (respectively, strongly primary) decomposition in N^1.1

2. Preliminaries

2.1. Definition

A non-empty set N with two binary operations + and ·is called a Near-ring if:

i. (N,+) is a group
ii. (N,·) is a semi group
iii. $x.(y+z) = x.y + x.z$for all $x, y, z \in N$

we will use the word Near-Ring to mean left Near-Ring.
We denote xy insteadof $x.y$.
Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in N$

2.2. Example

Consider $Z_8 = \{0,1,2,3,4,5,6,7,\}$ with respect to addition and multiplication modulo8.This is a Near-Ring

2.3. Example

Let N={a,b,c,d} be a set with two binary operations as follows

$+$	a	bc	d			
a	a	bc	da	a	aa	a

$.$	a	bc	d			
b	b	ad	cb	a	aa	a
c	c	db	ac	a	aa	a
d	d	ca	bd	a	ab	b

Here (N, +, **.**) is a left Near- Ring

2.4. Definition

Let X be a non-empty universal set. A fuzzy subset μ of X is a function
$\mu: X \to [0,1]$.

2.5. Definition

A Fuzzy subset μof a Near-Ring N is called a Fuzzy sub near-ring of N if for all $x, y \in N$

(i) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

2.6. Definition

A Fuzzy subset μof a Near-Ring N is called a Fuzzy ideal of N if μ is a Fuzzy sub near-ring of N and

(i) $\mu(x) = \mu(y+x-y)$
(ii) $\mu(xy) \geq \mu(y)$
(iii)$\mu((x+i)y-xy) \geq \mu(i)$ for any $x, y, i \in N$

2.7. Definition

A Fuzzy ideal μ of a Near- ring *N*is called a prime fuzzy ideal if for any two fuzzy ideals σ and θ of *N* the condition $\sigma\theta \subseteq \mu$ implies that $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

2.8. Definition

For a fuzzy ideal μ of a Near-ring *N,* thefuzzy radical of μ, denoted by$\sqrt{\mu}$, is defined by $\sqrt{\mu} = \cap$ *{*σ:σ is a fuzzy prime ideal of *N,* $\sigma \subseteq \mu$, $\sigma_* \subseteq \mu_*$*}*. We denote $\mu_* = \{x \in N : \mu(x) = \mu(0)\}$

2.9. Definition

A fuzzy ideal μ of anear- ring *N*is known as a $\sigma\theta \subseteq \mu$ and $\sigma \subseteq \mu$together imply that
$\theta \subseteq \sqrt{\mu}$

2.10. Definition

A Fuzzy ideal μ of a Near-Ring N is called strongly primary Fuzzy ideal of a Near-Ring N if μ is a primary Fuzzy ideal and $(\sqrt{\mu})^n \subset \mu$ for some n∈N.

2.11. Definition

A Fuzzy ideal μ of a Near-Ring N is called semi prime if $\mu^2(x) = \mu(x) \quad \forall\, x \in N$

2.12. Definition

Let X and Y be two non-empty sets,$f: X \to Y$,μ and σ be Fuzzy subsets of X and Y respectively then $f(\mu)$, the image of μ under f is a Fuzzy subset of Y denoted by

$$f(\mu)(y) = \begin{cases} sup\mu(x): f(x) = y & if\ f^{-1}(y) \neq \emptyset \\ 0 & if\ f^{-1}(y) = \emptyset \end{cases}$$

And $f^{-1}(\sigma)$,the pre-image of σ under f is a Fuzzy subset of X defined by

$$f^{-1}(\sigma)(x) = \sigma\big(f(x)\big) \quad \forall x \in X$$

2.13. Definition

Let N and N^1 be two Near-Rings, a mapping $f: N \to N^1$ is called a Fuzzy homomorphism if $f(\mu+\sigma) = f(\mu) + f(\sigma)$ and $f(\mu\sigma) = f(\mu)f(\sigma)$
where μ and σ are Fuzzy ideals of N.

2.14. Definition

Let N and N[1] be two Near-Rings, a mapping $f: N \rightarrow N^1$ is called a Fuzzy anti- homomorphism if $f(\mu+\sigma) = f(\mu) + f(\sigma)$ and $f(\mu\sigma) = f(\sigma)f(\mu)$ where μ and σ are Fuzzy ideals of N.

2.15. Definition

A function $f: N \rightarrow N^1$,a Fuzzy subset μ of a Near-Ring is called f-invariant if
$f(x) = f(y)$ implies $\mu(x) = \mu(y)$, $x, y \in N$.

2.16. Proposition

The anti-homomorphic image of a Fuzzy ideal of N is a Fuzzy ideal of N'

Proof: Let $f: N \rightarrow N'$ be an anti-homomorphism.
Let μ be a Fuzzy ideal of a Near- Ring N.

First we have to prove that f (μ) is a Fuzzy sub Near- Ring of N'

Let $x', y' \in N'$ $\exists\, x, y \in N$ $\ni f(x) = x', f(y) = y'$

Now consider $f(\mu)(x'-y') = f(\mu)(f(x)-f(y))$
$= f(\mu)(f(x-y))$
$= sup\mu(x-y): f(x-y) = x'-y'$
$\geq$ sup (min$\{\mu(x), \mu(y)\}$)
= min $\{sup\mu(x): f(x) = x', sup\mu(y): f(y) = y'\}$
= min $\{f(\mu)(x'), f(\mu)(y')\}$

And $f(\mu)(x'y') = f(\mu)(f(x)f(y))$
$= f(\mu)(f(yx))$
$= sup\mu(yx): f(yx) = y'x'$
$\geq$ sup (min $\{\mu(y), \mu(x)\}$)
= sup (min$\{\mu(x), \mu(y)\}$)
= min $\{sup\mu(x): f(x) = x', sup\mu(y): f(y) = y'\}$
= min $\{f(\mu)(x'), f(\mu)(y')\}$

Therefore $f(\mu)$ is a Fuzzy sub Near- Ring.

Now we have to prove that f (μ) is a Fuzzy ideal of a Near- Ring N'

Let $x', y' \in N'$ then there exists $x, y \in N$ $\ni f(x) = x', f(y) = y'$

(i) $f(y'+x'-y') = sup\mu(y+x-y): f(z) = y'+x'-y'$
$= sup\mu(x): f(x) = x'$
$= f(\mu)(x')$

(ii) $f(\mu)(x'y') = sup\mu(xy): f(xy) = x'$
$\geq sup\mu(y): f(y) = y'$
$= f(\mu)(y')$

(iii) $f(\mu)((x'+i')y'-x'y') = sup\mu((x+i)y-xy)$
$: f\big((x+i)y-xy\big) = (x'+i')y'-x'y'$
$\geq sup\mu(i): f(i) = i'$
$= f(\mu)(i')$

There fore $f(\mu)$ is a Fuzzy ideal of N'

2.17. Proposition

The Homomorphic pre-image of a Fuzzy ideal of N' is a Fuzzy ideal of N

Proof: Let $f: N \rightarrow N'$ be a near-ring homomorphism.
Let μ' be a Fuzzy ideal of a near-ring N'.

To prove $f^{-1}(\mu')$ is a Fuzzy ideal of a near-ringN

Let $x, y \in N$

(i) $f^{-1}(\mu')(x-y) = \mu'\big(f(x-y)\big)$
$= \mu'(f(x)-f(y))$
$\geq \min\{\mu'\big(f(x)\big), \mu'\big(f(y)\big)$
$= \min\{f^{-1}(\mu')(x), f^{-1}(\mu')(y)\}$

(ii) $f^{-1}(\mu')(xy) = \mu'\big(f(xy)\big)$
$= \mu'(f(x)f(y))$
$\geq \min\{\mu'\big(f(x)\big), \mu'\big(f(y)\big)$
$= \min\{f^{-1}(\mu')(x), f^{-1}(\mu')(y)\}$

(iii) $f^{-1}(\mu')(y+x-y) = \mu'\big(f(y+x-y)\big)$
$= \mu'(f(y)+f(x)-f(y))$
$\geq \mu'\big(f(x)\big)$
$= f^{-1}(\mu')(x)$

(iv)$f^{-1}(\mu')(xy) = \mu'\big(f(xy)\big)$
$= \mu'(f(x)f(y))$
$\geq \mu'\big(f(y)\big)$
$= f^{-1}(\mu')(y)$

(v) $f^{-1}(\mu')((x+i)y-xy) \quad = \mu'\Big(f\big((x+i)y-xy\big)\Big)$
$= \mu'(f(x+i)y - f(xy))$
$= \mu'(f(x+i)f(y)-f(xy))$
$= \mu'(f(x)+f(i))f(y)-f(x)f(y))$
$\geq \mu'(f(i))$
$= f^{-1}(\mu')(i)$

There fore $f^{-1}(\mu')$ is a Fuzzy ideal of N

3. Main Results

In this section we proved some main results related to fuzzy prime ideals.

3.1. Proposition

Let$f: N \to N'$ be a subjective near ring anti-homomorphism, letμ' is a Fuzzy prime ideal of N' then $f^{-1}(\mu')$ is a Fuzzy prime ideal of N.

- Proof: Let μ and σ be two Fuzzy ideals of N such that

$\mu\sigma \subseteq f^{-1}(\mu')$
$\Rightarrow f(\mu\sigma) \subseteq \mu'$
$\Rightarrow f(\sigma)f(\mu) \subseteq \mu'$

Since μ' is a Fuzzy prime ideal of N'

$\Rightarrow f(\sigma) \subseteq \mu' or f(\mu) \subseteq \mu'$
$\Rightarrow \sigma \subseteq f^{-1}(\mu')$ Or $\mu \subseteq f^{-1}(\mu')$

There fore $f^{-1}(\mu')$ is a Fuzzy prime ideal of N

3.2. Proposition

Let $f: N \to N'$ be a subjective near ring anti-homomorphism, Let μ' is a primaryFuzzy ideal of N' then $f^{-1}(\mu')$ is a primary Fuzzy ideal of N.

- Proof: Let μ and σ be two Fuzzy ideals of N

Such that $\mu\sigma \subseteq f^{-1}(\mu')$ and $\sigma \nsubseteq f^{-1}(\mu')$
$\Rightarrow f(\mu\sigma) \subseteq \mu'andf(\sigma) \nsubseteq \mu'$
$\Rightarrow f(\sigma)f(\mu) \subseteq \mu'andf(\sigma) \nsubseteq \mu'$
$\Rightarrow f(\mu) \subseteq \sqrt{\mu'}$ (Since μ' is a primary Fuzzy ideal)

$\Rightarrow \mu \subseteq f^{-1}\sqrt{(\mu')}$
$\Rightarrow \mu \subseteq \sqrt{f^{-1}(\mu')}$

There fore $f^{-1}(\mu')$ is a primary Fuzzy ideal of N.

3.3. Lemma

If μ is a primary Fuzzy ideal of a near-ring N then $\sqrt{\mu}$ is a prime Fuzzy ideal of N

- Proof: Let σ and θ be two Fuzzy ideals of N

Such that $\sigma\theta \subseteq \sqrt{\mu}$ and $\sigma \nsubseteq \sqrt{\mu}$
$\Rightarrow \sigma\theta \subseteq \mu$and$\sigma \nsubseteq \mu$
Since μ is a primary Fuzzy ideal,$\theta \subseteq \sqrt{\mu}$
There fore $\sqrt{\mu}$ is a prime Fuzzy ideal of N.

3.4. Proposition

Let $f: N \to N'$ be a surjective near ring anti-homomorphism, If μ is
An f-invariant ideal of $N$and μ ,a Fuzzy primary ideal of N then $f(\mu)$ is a Fuzzy primary ideal of N'.

- Proof: Let σ' and θ' be two Fuzzy ideals of N'
  Such that $\sigma'\theta' \subseteq f(\mu)$ and $\sigma' \nsubseteq f(\mu)$

$\Rightarrow f^{-1}(\sigma'\theta') \subseteq f^{-1}\, f(\mu)$

$\Rightarrow f^{-1}(\sigma'\theta') \subseteq \mu$ and $f^{-1}(\sigma') \nsubseteq \mu$

$\Rightarrow f^{-1}(\sigma')f^{-1}(\theta') \subseteq \mu$ and $f^{-1}(\sigma') \nsubseteq \mu$

$\Rightarrow f^{-1}(\theta') \subseteq \sqrt{\mu}$(Since μ is a primary Fuzzy ideal) $\Rightarrow \theta' \subseteq \sqrt{f(\mu)}$

Therefore $f(\mu)$ is a Fuzzy primary ideal of N'

3.5. Proposition

For a surjective near ring anti-homomorphism $f: N \rightarrow N'$, if μ is an
f-invariant strongly primary Fuzzy ideal of N then $f(\mu)$ is a strongly primary Fuzzy ideal of N'.

Proof: Let μ be an f-invariant strongly primary Fuzzy ideal of N.

$\Rightarrow \mu$ is a primary Fuzzy ideal and $(\sqrt{\mu})^n \subset \mu$ for some $n \epsilon N$

$\Rightarrow f(\mu)$ is a primary Fuzzy ideal of N'

Now we have to prove $(\sqrt{f(\mu)})^n \subset f(\mu)$ for some $n \epsilon N$

Since $f(\mu)$ is a primary Fuzzy ideal of N', $\sqrt{f(\mu)}$ is a prime Fuzzy ideal of N'

Since $\sqrt{f(\mu)} = \cap \{f(\sigma)\,, f(\sigma) \textit{is a fuzzy prime ideal of } N', f(\sigma) \subseteq f(\mu)$

There fore $(\sqrt{f(\mu)})^n \subset f(\mu)$ for some $n \epsilon N$

Hence $f(\mu)$ is a strongly primary Fuzzy ideal of N'

3.6. Proposition

For a surjective near ring homomorphism $f: N \rightarrow N'$, if μ' is a semi prime Fuzzy ideal of N', then $f^{-1}(\mu')$ is a semi prime Fuzzy ideal of N.

- Proof: Given μ' is a semi prime Fuzzy ideal of N'

$\Rightarrow \mu'$ is a Fuzzy ideal of N' and $\mu^2(x) = \mu(x)$ for all $x \epsilon N$

$\Rightarrow f^{-1}(\mu')$ is a Fuzzy ideal of N.

Let $f^{-1}(\mu') = \mu \Rightarrow \mu' = f(\mu)$

Now $\mu' = \mu'\mu' = f(\mu)f(\mu) = f(\mu\mu) = f(\mu^2)$

$\Rightarrow \mu^2 = f^{-1}(\mu') = \mu \Rightarrow [f^{-1}(\mu')]^2(x) = f^{-1}(\mu')(x)$ for all $x \epsilon N$

3.7. Proposition

Let $f: N \rightarrow N'$ be a near ring anti-homomorphism. If μ is any f-invariant semi prime fuzzy ideal of N, then f (μ) is a semi prime fuzzy ideal of N'.

- Proof: If μ is any f-invariant semi prime fuzzy ideal of N

$\Rightarrow \mu$ is a Fuzzy ideal of N and $\mu^2 = \mu$

$\Rightarrow f(\mu)$ is a primary Fuzzy ideal of N'

Now $\left(f(\mu)\right)^2 = f(\mu)f(\mu) = f(\mu\mu) = f(\mu^2) = f(\mu)$

Therefore, f (μ) is a semi prime fuzzy ideal of N'.

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