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# On the Eigenvalues of a Norlund Infinite Matrix as an Operator on Some Sequence Spaces

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***Abstract:***
*In various papers some authors have previously investigated [1], [2], [3], [4], [5] and determined the spectrum of weighted mean matrices considered as bounded operators on various sequence spaces. In this study, we determine eigen values of a Norlund matrix as a bounded operator over the sequence space$c_0$. This will be achieved by applying Banach space theorems of functional analysis as well as summability methods of summability theory. We are also going to apply eigenvalue problem i.e. Ax= λ x. Where λ arenumbers (realorcomplex) and vector columnsx $(x \neq 0)$ ;suchthatx $\in c_0$. In which case it is shown that the set of Eigen values of $A \in B(c_0) = \emptyset$ . Also it is shown that the set of Eigenvalues of $A^* \in B(l_1)$ is*

$$\{\lambda \in \mathbb{C}: |\lambda + 1| < 2\} \cup \{1\}$$

***Keywords:*** *Spectrum, Norlund means, Sequence spaces and Boundedness*

## 1. Introduction

Functional analysis finds a lot of applications through summability theory. Broadly, summability is the theory of assignment of limits, which is fundamental in analysis. The results from this research will provide useful information to engineers to improve on areas of application of eigenvalues and eigenvectors in engineering. It will also be useful to mathematicians when solving similar problems.

### *1.1. Eigenvalues*

Given a square matrix A , let us consider the problem of finding numbers λ ( real or complex ) and vectors ( vector columns ) $x$ $(x \neq 0)$ such that$Ax = \lambda x$.This problem is called the eigenvalue problem, the number λ are called the eigenvalues of the matrix $A$ , and the non-zero vector $x$ are called the eigenvectors corresponding to the eigenvalues λ .
To find eigenvalues; we note that $\lambda x = \lambda I x$, where $I$ is the identity matrix. Then we can rewrite $Ax = \lambda x$ in the form$Ax - \lambda I x = 0$
Matrix equation $Ax - \lambda I x = 0$ (which in fact represents the linear system) has a non-trivial solution $x \neq 0$ if and only if the matrix $A - \lambda I$ of this system is singular, which is the case if and only if $det(A - \lambda I) = 0$ Thus we have the equation for finding eigenvalues λ which is called the characteristic equation.

### *1.2. Classical Summability*

The central problem in summability is to find means of assigning a limit to a divergent sequence or sum to a divergent series. In such a way that the sequence or series can be manipulated as though it converges, (Ruckel, 1981), pp. 159-161. The most common means of summing divergent series or sequences, is that of using an infinite matrix of complex numbers or by a power series.

#### 1.2.1. Definition

*Sequence to Sequence transformation*
Let $A = (a_{nk})$ , $n$ , $k = 0, 1, 2, \ldots$ be an infinite matrix of complex numbers. Given a sequence $x = (x_k)_{k=0}^{\infty}$ define $y_n = \sum a_{nk} x_k$ , $n = 0, 1, 2, \ \ldots$ . If the series, converges for all $n$, then we call the sequence $(y_n)_{n=0}^{\infty}$ , the $A - transform$ of the sequence $(x_k)_{k=0}^{\infty}$ . If further,
$y_n \to a \ \ as \ \ n \to \infty$ , we say that $(x_k)_{k=0}^{\infty}$ is summable $A \ to \ a.$ .
There are various sequences to sequence transformations, here we state Norlund means below which is the matrix of interest in this paper.

1.2.2. (Norlund means)

*The transformation given by* $y_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}x_{k}{}_{,n=0,1,2,...}$

*where* $P_n = p_0 + p_1 + \cdots + p_n \neq 0$ *, is called a Norlund means and is denoted by (N, p).*

*Its matrix is given by*

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n}, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

*In the matrix above if* $p_0 = 1, p_1 = -2, p_2 = p_3 = \cdots = 0, then\ A = a_{nk}$ *.i.e.*

$$a_{nk} = \begin{cases} 1,\ n = k = 0 \\ 2,\ n-1 \le k \le n \\ -1,\ n = k \\ 0,\ otherwise \end{cases}$$

*or*

$$A = a_{nk} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 2 & -1 & 0 & 0 & 0 & . & . & . \\ 0 & 2 & -1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & -1 & 0 & . & . & . \\ 0 & 0 & 0 & 2 & -1 & . & . & . \\ & . & . & . & & & & \end{pmatrix}$$

1.2.3. Adjoint of A ($A^*$)

*It is the transpose of the matrix A and we denote it here by* $\boldsymbol{A}^*$.

1.2.4. Dual space of $c_0$

*It is* $c_0^*$ *and it is the space* $l_1$ *; the space of absolutely convergent series.*

*1.3. General Results in Classical Summability*

→ Definition 1.3.1*(regular method, conservative method)*

Let $A = (a_{nk}), n = 0, 1, 2, 3, ...$ be an infinite matrix of complex numbers.

i. If the $A$ transform of any convergent sequence of complex numbers exists and converges then $A$ is called a conservative method. We then write $A \in (c, c)$
ii. If the $A$ transform of any convergent sequence of complex numbers exists and converges, then $A$ is called regular.

→ Theorem 1.3.1 $A \in (c_0, c_0)$ if and only if

i. $\lim_{n\to\infty} a_{nk} = 0$ for each fixed $k$
ii. $sup_{n\ge0}\{\sum_{k=0}^{\infty}|a_{nk}|\} < \infty$

Proof: (Hardy, 1948), pp. 42 - 60; (Maddox, 1970), pp. 165 - 167.

**2. The Eigen values of Operator A On $c_0$**

*2.1. Boundedness of operator A on sequence space* $c_0$.

In this section we show that $\in B(c_0)$ . The corollary below arises from theorem (1.3.1) above.

*Corollary 2.1.1 It is clear that* $A \in B(c_0)$ *. since* $lim_{n\to\infty}a_{nk} = 0 foreachfixedk$ *from matrix A.*

$$\|A\| = sup_{n\ge0}\sum_{k=0}^{\infty}|a_{nk}| = \sup(1, 3, 3, 3, ...) = 3$$

*Also*

$$\|A\| = \|A^*\| = 3$$

*Lemma 2.1.1 Each bounded linear operator* $T{:}X \to Y, where x = c_0, l_1,$ *and* $Y = c_0, l_p(1 \le p < \infty), l_\infty$ *determines and is determined by an infinite matrix of complex numbers.*

*Proof.* see (Taylor, 1958) pages 217-219

*Lemma 2.1.2Let* $T: c_0 \to c_0$ *be a linear map and define* $T^*: l_1 \to l_1$ *by* $T^* o\ g = g\ o\ T, g \in c_0^* = l_1$ *then T must be given by a matrix by lemma (2.1.1) and moreover* $T^*: l_1 \to l_1$ *is the transposed matrix of* $T$.

*Corollary 2.1.2 LetA*: $c_0 \to c_0$ *where A is our matrix of interest. Then* $A^* \in B(l_1)$, *moreover*

$$A^* = \begin{pmatrix} 1 & 2 & 0 & . & . & . \\ 0 & -1 & 2 & . & . & . \\ 0 & 0 & -1 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ & . & . & . & & \end{pmatrix}$$

*2.2. Eigenvalues of A on the sequence space* $c_0$
*Theorem 2.2.1.A* $\in B(c_0)$ *has no Eigenvalue.*

Proof: Suppose Ax=λ x for $x \neq 0$ *in* $c_0$ *and* $\lambda \in \mathbb{C}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 2 & -1 & 0 & 0 & 0 & . & . & . \\ 0 & 2 & -1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & -1 & 0 & . & . & . \\ 0 & 0 & 0 & 2 & -1 & . & . & . \\ & & . & . & . & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \end{pmatrix}$$

Implies

$$\begin{aligned} & x_0 = \lambda x_0 \\ & 2x_0 - x_1 = \lambda x_1 \\ & 2x_1 - x_2 = \lambda x_2 \\ & 2x_2 - x_3 = \lambda x_3 \\ & 2x_3 - x_4 = \lambda x_4 \\ & \ldots \\ & 2x_{n-1} - x_n = \lambda x_n, n \geq 1 \end{aligned} \qquad (2.2.1)$$

solving system (2.2.1) we have that if $x_0$ is the first non zero entry of x, then λ =1,but

λ =1impliesthat $x_0 = x_1 = x_2 = \ldots = x_n = \ldots$ i.e.

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ . \\ . \\ . \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \end{pmatrix}$$

which shows that x is in the span of δ. But δ =(1,1, 1,)$\notin c_0$ . That is x does not tend to zero as n tends to infinity, so λ =1 is not an eigenvalue of $A \in B(c_0)$ .

If $x_{n+1}$ ,n=0,1,2, 3, is the first non zero entry, then λ =-1. Solving the system with λ = -1 results in $x_n = 0$,n=0,1,2, 3, a contradiction. Hence λ =-1 cannot be an eigen value of $A \in B(c_0)$ .

Thus $A \in B(c_0)$ has no eigen values i.e. the set of eigen values is empty.

→ Corollary 2.2.1 *The set of Eigenvalues of A* $\in B(bv_0)$ *&A* $\in B(l_1)$ *is empty*

Proof: This follows from the fact that $bv_0 \subset c_0$ also $l_1 \subset c_0$

→ Theorem 2.2.2. *The Eigenvalues of* $A^* \in B(l_1)$ *is the set*

$$\{\lambda \in \mathbb{C} : |\lambda + 1| < 2\} \cup \{1\}$$

Proof: Suppose $A^* x = \lambda x \ \ for \ \ x \neq 0 \ \ and \ \ \lambda \in \mathbb{C}$

Then: $$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & . & . & . \\ 0 & -1 & 2 & 0 & 0 & . & . & . \\ 0 & 0 & -1 & 2 & 0 & . & . & . \\ 0 & 0 & 0 & -1 & 2 & . & . & . \\ 0 & 0 & 0 & 0 & -1 & . & . & . \\ & & . & . & . & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \end{pmatrix}$$

That is

$$\begin{aligned} & x_0 + 2x_1 = \lambda x_0 \\ & -x_1 - 2x_2 = \lambda x_1 \\ & -x_2 - x_3 = \lambda x_2 \\ & -x_3 - 2x_4 = \lambda x_3 \\ & -x_4 - 2x_5 = \lambda x_4 \\ & \ldots \\ & -x_n + 2x_{n+1} = \lambda x_n, n \geq 1 \end{aligned} \qquad (2.2.2)$$

solving system (2.2.2) for $x_1, x_2, x_3, \ldots, x_n$ in terms of $x_0$ gives

$$x_1 = 2^{-1}\lambda\left(1-\frac{1}{\lambda}\right)x_0$$
$$x_2 = 2^{-2}\lambda^{-2}\left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda}\right)x_0$$
$$x_3 = 2^{-3}\lambda^{-3}\left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda}\right)^2 x_0$$
$$x_4 = 2^{-4}\lambda^{-4}\left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda}\right)^3 x_0$$
$$\ldots$$
$$in\ \ general$$
$$x_n = 2^{-n}\lambda^{n}\left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda}\right)^{n-1} x_0$$

By ratio test

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\left|\frac{2^{-n}2^{-1}\lambda^n\lambda\left(1+\frac{1}{\lambda}\right)^n\left(1-\frac{1}{\lambda}\right)x_0}{2^{-n}\lambda^n\left(1+\frac{1}{\lambda}\right)^n\left(1+\frac{1}{\lambda}\right)^{-1}\left(1-\frac{1}{\lambda}\right)x_0}\right|$$
$$\lim_{n\to\infty}\left|\frac{2^{-1}\lambda}{\left(1+\frac{1}{\lambda}\right)^{-1}}\right|$$
$$\left|\frac{1}{2}\lambda\left(1+\frac{1}{\lambda}\right)\right| = l\ \ for\ some\ real\ number\ l \geq 0$$

By ratio test $x_n \in l_1 \quad iff\ l < 1$

That is if $\left|\frac{1}{2}\lambda + \frac{1}{2}\right| < 1$

or

$| \lambda +1|<2$

That is the series $\sum_{n=0}^{\infty}|x_n|$converges for all λ in the circular disc centred at the point ( -1,0) of radius 2.

It is clear that λ =1 is an eigenvalue corresponding to the eigenvector $( x_0 ,0,0,0,\ldots)^t$ . Where$x_0$ is any real or complex number. This is the case since $( x_0 ,0,0,0,\ldots)^t \subset l_1$ forany$x_0 \in$ □

Hence the Eigenvalues of$A^* \in B(l_1)$ is the set {λ ∈ :| λ +1|<2} ∪ {1}

## 3. Conclusions

In this paper the following results were obtained

i. $A\in B(c_0)$ has no Eigen values
ii. Also $A \in B( bv_0 )$&$A \in B( l_1 )$has no Eigenvalues
iii. The set of Eigenvalues for $A^* \in B(l_1)$ is { λ ∈ :| λ +1|<2 } ∪ {1}

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## 5. Notations

$\|.\|$ norm of

In general, {...} will denote the set of, (...) the set sequence of and $(\ldots)^t$ the transpose of the sequence of; unless otherwise specified.

$c_0$ the set of sequences which converge to zero (null sequences),$bv_0$ the space of null bounded variation, $l_1$ the space of absolutely convergent series.

$A^*$ adjoint of $A$

$c_0^*$dual space of $c_0$

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