# Mathematical Model of Surface to Air Intercepting Missile

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*Abstract:*
*The purpose of this design is to lay up a precised intercepting missile trajectory equations for military rapid response of counter attacks of enemies' high invading targets. The set of non linear mathematical equations that model the trajectory of an intercepting missile is presented here with considerations to characteristics such as Angle of Attack (AoA), Lift off speed, Optimized flight direction (wind direction). A numerical solution using MATLAB is also derived for rapid computation and feed back into the missile launcher command computer system. This paper include scientific data for atmospheric variables such as pressure, density, gravity and temperature to an altitude below the tropopause (11km) beyond which the data seized to be true. An iterative computational process which divides the earth's altitude into finite strata is used in MATLAB program to predict the flight parameters (AoA, lift off speed, intercept time) given input parameters (safe intercept position, speed of enemy's target, position of enemy's target)*

*Keywords: Intercepting, missile, mathematical model, target*

## I. Introduction

Surface to air missiles (SAMs) are a powerful weapon in both defensive and offensive tactics. The purpose of the system is to detect and destroy enemy aircraft or missile approaching a defended area. The system must be capable of defending strategic areas against attack from high altitude, high speed enemy target that could cause damage. Missile systems can be categorized into two classes- strategic and tactical. Strategic missiles are designed to travel long distances towards known, stationary targets. Tactical missiles track or intercept shorter range, maneuvering and non-maneuvering targets where its guidance and control technologies turn out to be more critical. Strategic missiles primarily operate in exoatmospheric conditions while tactical missiles most commonly operate in endoatmospheric conditions. The missile type of interest in this study is the tactical missiles designed to intercept a non maneuvering target. Sophisticated missile launcher systems are used to point the missile to the target.

Another classification of missiles includes: Surface-to-Surface Missiles (SSM), Surface-to-Air Missiles (SAM), Air-to-Air Missiles (AAM) and Air-to-Surface Missiles (ASM).The one of interest in this study is the Surface to Air Missiles (SAMs).This system consists of the Search radar, Missile Launch system and the Control Station.
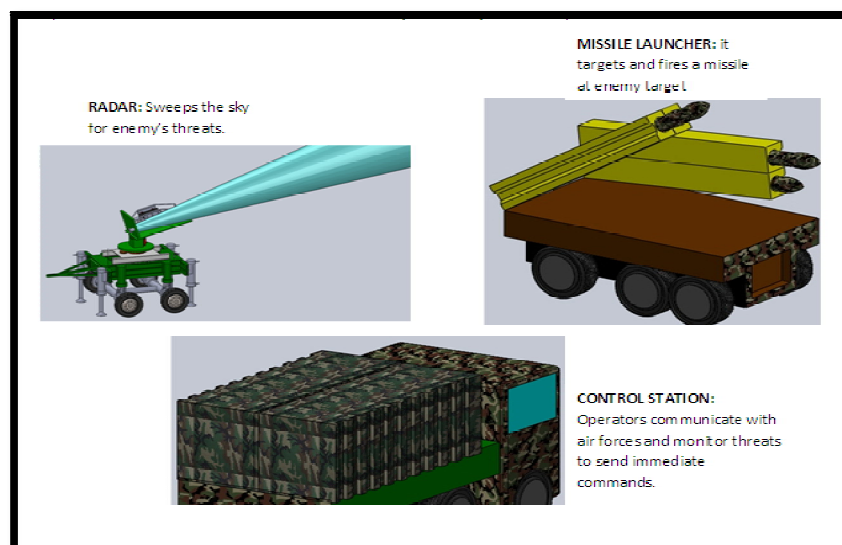


*Figure 1 Sub-Systems of Surface to Air Missile*

## 2. Statement of Problem and Methodology
The problem to be solved in this paper is as follows:
- A missile is to follow a trajectory to intersect an incoming enemy target.
- This engagement is to take place at a safe minimum altitude $H_c$ in the atmosphere
- The Intercepting missile ($M2$) has the following informations on the target ($E1$) the velocity and direction (angle to the normal).
- The Angle of Attack (AoA) $\theta$ and engagement time $t$ for the intercepting missile of given velocity ($v$) are to be determined for rapid response.
- The trajectory of the intercepting missile ($M2$) is parabolic whose acceleration is subject to aerodynamic force of Drag and atmospheric Gravity throughout its flight.
- The interaction of the intercepting missile with the atmospheric parameters is to be considered. The procedural methods followed to obtain the solution are outlined below;
- The consideration of the physics of forces acting on flying objects
- Influence parameters of flight varying in the atmosphere are considered and modeled appropriately
- The flight model of the approaching enemy's target is outlined and analyzed
- The flight model of the intercepting missile is then created and analyzed
- The general solution is obtained and presented as graphical results

### 2.1. Force Parameterization for Intercepting Missile
Drag force, Thrust force and Inertia force are the forces acting on an intercepting missile in flight as shown in Figure 2.
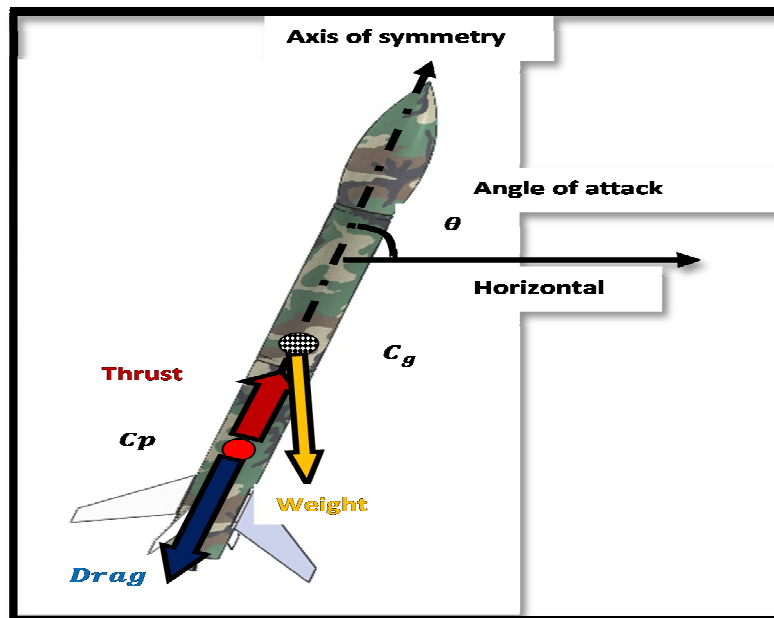


*Figure 2: Schematic of Forces Acting on a Missile In Flight*

### 2.1.1. Assumptions
- The trajectory of the intercepting missile is within the atmosphere
- The forces relevant in the flight performance are the aerodynamic drag, the thrust and the inertial weight of the missile.
- The coefficient of drag for the missile is 0.75 (coefficient of Drag for circular cross section in air)

### 2.2. Model of Atmospheric Variables Influencing Missile Flight

### 2.2.1. Variation of Atmospheric Density and Altitude
The work of Mustafa Cavcar [1];mathematically modeled the atmospheric density from the perfect gas equation given the standard pressure and temperatures as;

$$\rho_h = \rho_0 \left( 1 - 0.0065 \frac{h}{T_0} \right)^{4.2561} \qquad (1)$$

$\rho_0 \, mean sea level density$ , $T_0 \, mean sea level Temperature$

| Quantity | Notation | Value | Unit |
|----------|----------|-------|------|
| Pressure | $p_0$ | 101325 | $N/m^2$ |
| Temperature | $T_0$ | 288.15 | $°K$ |
| Density | $\rho_0$ | 1.225 | $Kg/m^3$ |
| Acceleration(gravity) | $g_0$ | 9.80665 | $m/s^2$ |
| Speed of sound | $a_0$ | 340.294 | $m/s$ |

*Table1 ISA Mean Sea Level Conditions [1]*

## 2.2.2. Model of Gravitational Acceleration and Latitude

The World Geodetic System [2] (WGS) gives the international gravity formula at given latitude as;

$$g = 9.7803267714 \left( \frac{1 + 0.00193185138639\sin^2 \zeta}{\sqrt{1 - 0.00669437999013\sin^2 \zeta}} \right) \tag{2}$$

Where, $\zeta$ is the geographic latitude of the missile launch location. For the sake of calculation in this paper, Kaduna Nigeria is located on latitude $10.51^o$ north [3] is used as the launch location.

## 2.2.3. Variation of Atmospheric Gravity and Altitude

From mathematical ratios of Newton's gravitational formula

$g = \dfrac{GM}{R^2}$ It can be shown that;

$$g_h = g_0 \left[ 1 - \frac{2h}{R} \right] \tag{3}$$

Where, $g_h$ = gravity at an altitude, $h$ = Altitude, $R_e = 6378137m$ Earth's equatorial radius

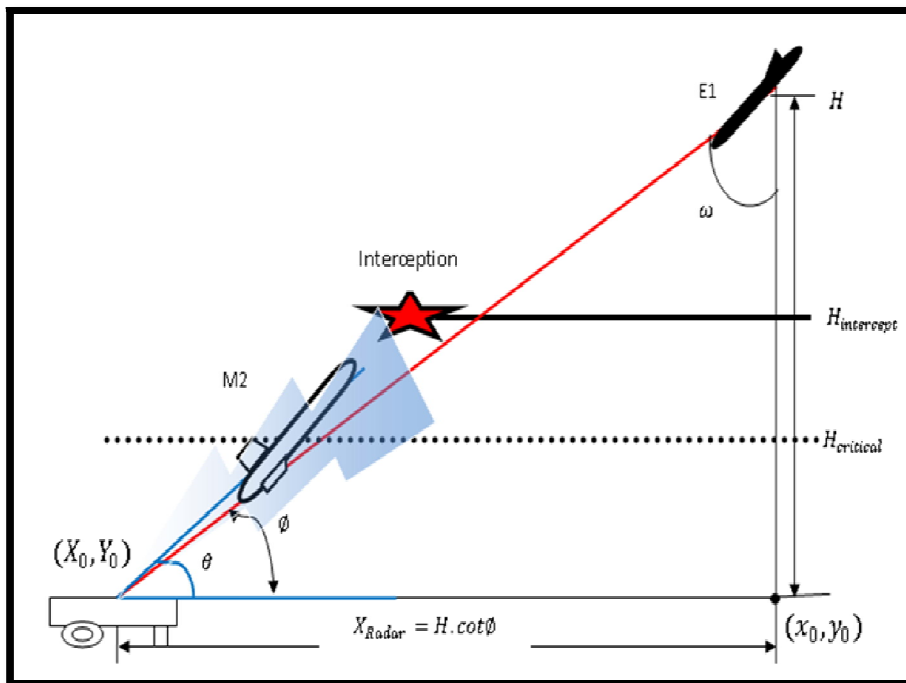## 2.3. Conceptualization of Missile Intercepting Target



*Figure 3: Conceptual View of Missile Intercepting Target*

## 2.3.1. Equations of Motion (EOM)

In the derivation of the equations of motion, the missile motion and the target motion are both divided into two coordinate axes, the vertical ($y$) and the horizontal ($x$). The vector balance of the forces is shown below;
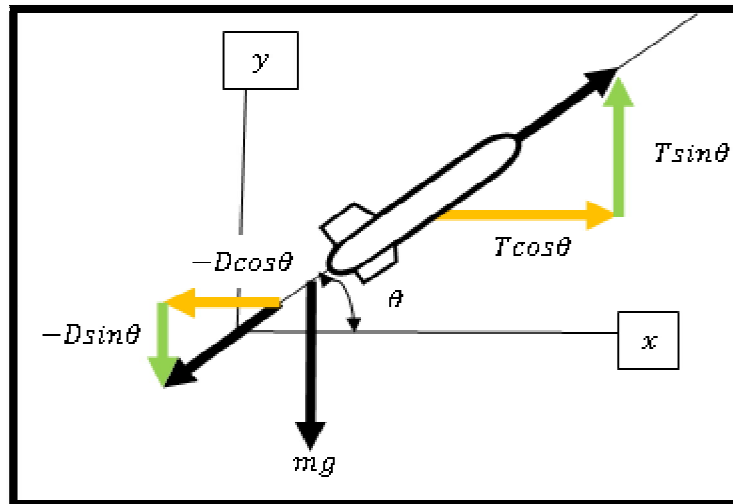
## 2.3.1.1. Missile Motion



*Figure 4 Free Body Diagrams of the Forces on the Missile*

### 2.3.2. Resolution of the Forces

#### 2.3.2.1. Vertical forces
The vertical forces can be resolved into;

$$\therefore a_y = \frac{(T-D)\sin\theta}{m} - g \qquad (4)$$

The quadratic model of the drag (Air Resistance) $D = kv^2$ (5)

The dynamic pressure of air $k = \frac{1}{2}\rho C_d A$ (6)

But $v$ component along y is: $v_y = v\sin\theta$

$$\therefore a_y = \frac{\left[T - k(v\sin\theta)^2\right]\sin\theta}{m} - g \qquad (7)$$

#### 2.3.2.2. Horizontal Forces
The horizontal forces can be resolved into;

$$\therefore a_x = \frac{(T-D)\cos\theta}{m} \qquad (8)$$

But $v$ component along x is: $v_x = v\cos\theta$

$$\therefore a_x = \frac{\left(T - k(v\cos\theta)^2\right)\cos\theta}{m} \qquad (9)$$

#### 2.3.2.3. Missile Position Vector
The position vector of the missile can be derived from Newton's second equation of motion

$$r(t) = \left(X_0 + v\cos\theta.t + \left\{\frac{\left(T - k(v\cos\theta)^2\right)\cos\theta}{2m}\right\}t^2\right)i + \left(Y_0 + v\sin\theta.t + \left\{\frac{\left(T - k(v\sin\theta)^2\right)\sin\theta}{2m} - \frac{g}{2}\right\}t^2\right)j \quad (10)$$
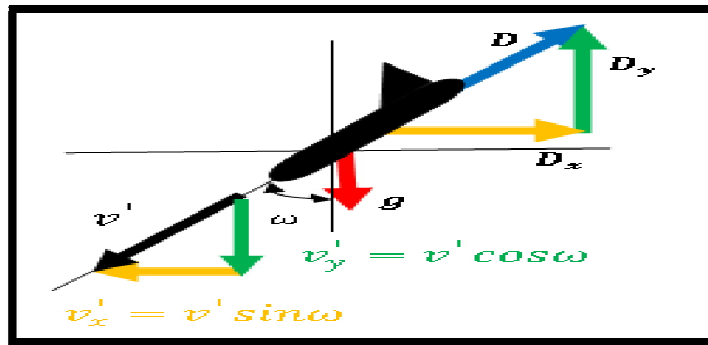
### 2.3.3. Invading Target Motion



*Figure 5 Free Body Diagram of Target*

Target Position Vector: The target position vector is given by;

$$r^{'}(t) = \left[ x_0 - \left( v_x^{'}.t + \frac{1}{2}a_x^{'}t^2 \right) \right]i + \left[ y_0 - \left( v_y^{'}.t + \frac{1}{2}a_y^{'}.t^2 \right) \right]j \qquad (11)$$

$x_0$, horizontal distance of the invader from the missile system detection base

$y_0$, vertical distance of the invader from the missile system detection base

$x_0 = \frac{H_d}{\tan\theta} = H_d \cot\theta$, $y_0 = H_d$

$H_d$ = Height of detection from the ground

$a_x^{'} = 0$, since velocity is constant for free falling target

$a_y^{'} = g\left(\frac{m}{s^2}\right)$ Is the decent acceleration

$$r^{'}(t) = \left[ x_0 - \left( v^{'}\sin\omega \right)t \right]i + \left[ y_0 - \left( v^{'}\cos\omega \right)t - \frac{g}{2}.t^2 \right]j \qquad (12)$$

$v^{'}$ Target speed, $\omega$ Target direction

*2.4. Calculation of Critical Response Time*

In order to obtain an estimate of the response time required before the invading target reaches the critical height$H_c$ substitute the $j$ component in equation $(12)$and equate to the critical height.

$$H_c = H_d - \left( v^{'}\cos\omega \right).t_c - \frac{g}{2}.t_c^{\,2}$$

Which can be solved to obtain the critical time of response before the enemy flight reaches the safe intercept altitude. Once the critical time is found, it can be compared with the time delays of the different missiles that could be used to intercept the incoming target. Only those with a time delay smaller than the critical time $t_c$will be considered by the missile detection base system as possible interceptors.

### 2.4.1. Position Vector for Interception of Missile and Target

At point of interception in the air; $r(t) = r^{'}(t)$hence; from equations $(10)$ and $(12)$we can equate their $i$ and $\mathbf{j}$ components as follows:

$$X_0 + v\cos\theta.t + \left\{ \frac{\left( T - k(v\cos\theta)^2 \right)\cos\theta}{2m} \right\}t^2 = x_0 - \left( v^{'}\sin\omega \right)t \qquad (13a)$$

$$Y_0 + v\sin\theta.t + \left\{ \frac{\left( T - k(v\sin\theta)^2 \right)\sin\theta}{2m} - \frac{g}{2} \right\}t^2 = y_0 - \left( v^{'}\cos\omega \right)t - \frac{g}{2}.t^2 \qquad (13b)$$

The Solution to the 'EOM' of missile intercepting the invading target can be obtained on MATLAB platform by solving the equations (13a) and (13b) simultaneously by considering the following parameters:

| 1 | The missile detection system base will thus have its coordinate vector thus; $(X_0, Y_0) = (0,0)$ |
|---|---|
| 2 | $H_d = 10000$ height of target (m) |
| 3 | The coordinate position of the target will be; $(x_0, y_0) = (H_d \cot\phi, H_d)$ |
| 4 | $v' = 2000$ velocity of target |
| 5 | $\omega = 80^o$ angle of target to the normal(degrees) |
| 6 | $v$ velocity of missile (m/s) |
| 7 | $T = 500$ Thrust of missile (N) |
| 8 | $\rho = 1.29$ Air density ($kg/m^3$) |
| 9 | $Cd = 0.75$ Drag coefficient of circular cross section in air |
| 10 | $Dr = 0.3$ Diameter of missile (m) |
| 11 | $A$ Cross sectional area of missile |
| 12 | $m = 30$ Mass of missile (kg) |

*Table 2: Parameters for Solving the EOM*

This solution results in the angle of missile launch $\theta$ and time of interception $t_{int}$.

### 2.4.2. Determination of the Intercept Position

To obtain the altitude of interception, substitute the intercept time $t_{int}$ into the $i$ and $j$ components of either equation (10) or (12) and solve for the position of interception $P_{int}(x, y)$.

| S/No | Missile | Missile velocity $v$ ($\frac{m}{s}$) | Operation Delay (s) |
|---|---|---|---|
| 1 | jaguar | 1320 | 5 |
| 2 | tiger | 980 | 1.0 |
| 3 | leopard | 750 | 0.5 |

*Table 3: Input Parameters for Available Missile Options*

## 3. Result

### 3.1. Possible Intercepting Missile options

The following are graphical results of varying the missile velocity and the resulting intercept position, missile launch angle and intercept time are tabulated for considerations.
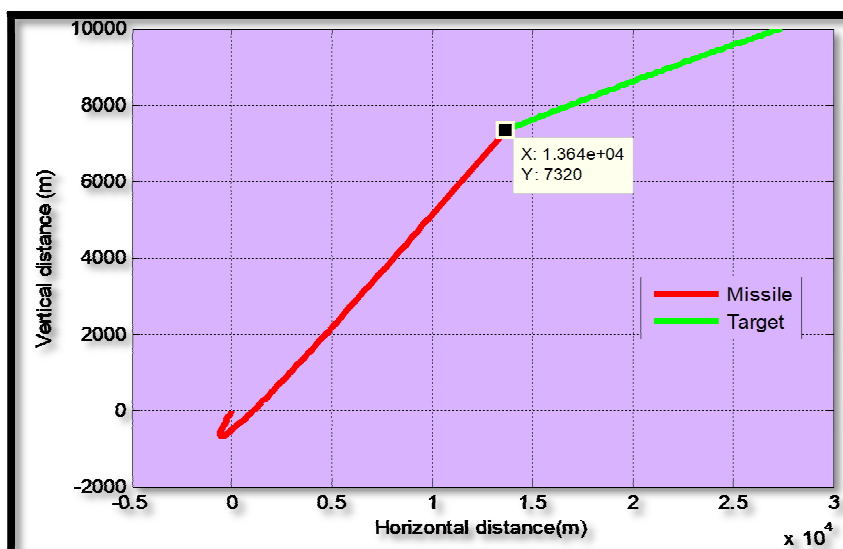


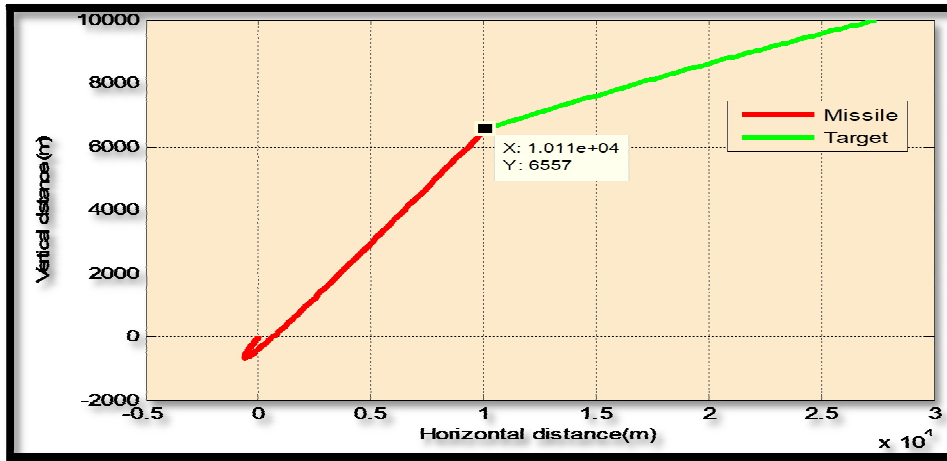*Figure 6: Trajectory of Missile-Jaguar Intercepting Target*

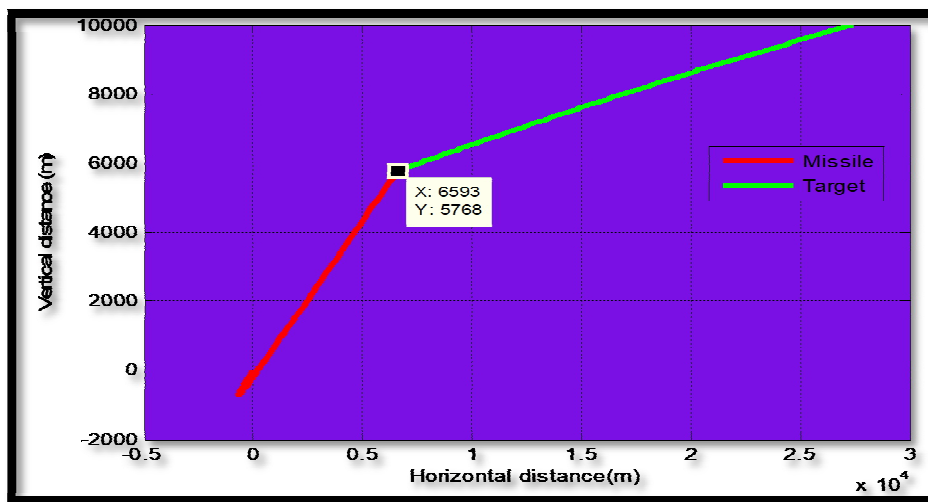*Figure 7: Trajectory of Missile-Tiger Intercepting Target*



*Figure 8: Trajectory of Missile-Leopard Intercepting Target*

| S/No | Missile | Intercept Position $P(x, y)$ | Intercept Time $t_{int}$ $(s)$ | Launch Angle $\theta$ ($rads$) |
|------|---------|------------------------------|-------------------------------|--------------------------------|
| 1 | jaguar | 13640,7320 | 7.0224 | 2.4239 |
| 2 | tiger | 10110,6557 | 8.8181 | 2.3973 |
| 3 | leopard | 6593,5768 | 10.6021 | 2.3605 |

*Table 4: Output Parameters for Appropriate Intercepting Missile Selection*

## 4. Discussion

The missile 'jaguar' of speed $1320\,m/s$ will intercept the target at an altitude of $7320m$ in $7.0224s$ after an operations delay before lift-off of $5s$, meaning, the total time taken by jaguar is $5 + 7.0224 = 12.0224s$ which is greater than $t_c = 10s$ hence it is not a good option because the enemy target will be in the critical zone $2.0224s$ before it could intercept it. The missile 'tiger' will intercept the target at an altitude of $6557m$ and in $8.8181s$ with an operation delay of $1.0s$ taking a total of $8.8181 + 1.0 = 9.8181s$ which is less than $t_c = 10s$ before the target reaches the critical height, hence, 'tiger' is a good intercept option. The missile 'leopard' of speed $750\,\frac{m}{s}$ will intercept the target at an altitude of

$5768m$ in $10.6021s$ after an operations delay before lift-off of $0.5s$, meaning, the total time taken by leopard is $0.5 + 10.6021 = 11.1021s$ which is greater than $t_c = 10s$ hence it is not a good option because the enemy target will be in the critical zone $1.1021s$ before it could intercept it.

## 5. Conclusion

The missile 'tiger' should therefore be launch at $\theta = 2.3973\ rads$ and will take the time $t_{int} = 9.8181s$ to hit the target before it reaches the critical zone.

## 6. References

i. Mustafa Cavcar, The International Standard Atmosphere (ISA), Anadolu University, 26470 Eskisehir, Turkey. Pp

ii. Department of Defense World Geodetic System 1984 — Its Definition and Relationships with Local Geodetic Systems, NIMA TR8350.2, 3rd ed., Tbl. 3.4, Eq. 4-1

iii. https://www.distancesto.com/coordinates/ng/kaduna-latitude-longitude.16,June, 2017

## Appendix

*Matlab Program That Gives the Solution of the Equation of Motion of the Missile Intercepting the Target*

```
%This program computes for an appropriate angle of missile launch and the time
%of interception of an air target
%'parameters'
H=10000;%height of target(m)v=1000; %velocity of missile (m/s)v_t=2000; %velocity of target(m/s)w=80*pi/180;%angle of target to the normal(degrees)T=500; %Thrust of missile (N)rho=1.29; %air density (Kg/m^3)C_d=0.75; %Drag coefficient of circular cross sectionDr=0.3;%diameter of missile(m)A=pi*Dr^2/4 ;%cross sectional area of missile(m^2)K=rho*C_d*A/2;%dynamic air pressure m=30; %mass of missile(Kg)eta=10.5104642 ; %latitude of Kaduna(degrees) zta=eta*pi/180;
g =9.7803267714*((1+0.00193185138639*(sin(zta))^2)/sqrt(1-0.00669437999013*(sin(zta))^2 )) %Acceleration due to gravity
phi=20; %Angle at which the radar sighted the target (degrees)
z=phi*pi/180; %conversion of angle to radians
y_0=H; %Initial height of target(m)x_0=y_0*cot(z) %Initial horizontal position (m)X_0=0 %initial horizontal position of launch or observation base(m)Y_0=0 %initial vertical position of launch or observation base(m)%solving the simultaneous equation
 syms D t %positive
[Sx, Sv] =solve(X_0+v*cos(D)*t+((T-K*v.^2*(cos(D)).^2)*cos(D)*t.^2/(2*m))==(x_0-(v_t*sin(w)*t)),...
   Y_0+v*sin(D)*t+(((T-K*v.^2*(sin(D)).^2)*sin(D)/(2*m))-(g/2))*t.^2==y_0-(v_t*cos(w)*t+(g/2)*t^2),D,t)
True_tita=Sx*180/3.142
B=Sx
Time_of_intercept=Sv
%position of missile
f=0:Time_of_intercept/100:Time_of_intercept
[X_missile]=X_0+v*cos(B)*f+(((T-K*v.^2*(cos(B)).^2)/(2*m))*cos(B)*f.^2);
[Y_missile]=Y_0+v*sin(B)*f+(((T-K*v.^2*(sin(B)).^2)*sin(B)/(2*m))-(g/2))*f.^2;
plot(X_missile,Y_missile,'r')
hold on
X_target=x_0-(v_t*sin(w)*f);
Y_target=y_0-(v_t*cos(w)*f+((g/2)*f.^2));
plot(X_target,Y_target,'g')
```

*Nomenclatures*
$H_c$ Safe minimum altitude
$H_d$ = Height of target detection from the ground
$M2$ Intercepting missile
$E1$ Target
$\theta$ Angle of Attack (AoA)
$H_{intercept}$ Intercept altitude
    Angle of Radar spot on the target
$\omega$ Angle of target to the normal
$\zeta$ geographic latitude of missile launch location
$C_g$ Center of Gravity
$C_p$ Center of pressure
$g_h gravity at altitude (m/_{s^2})$
$g_0$ Mean sea level gravity $(m/_{s^2})$
$R_e = 6378137 \, m$ Earth's equatorial radius
$\varphi$ Geographic latitude
$T$ Missile Thrust force (N)
$D$ Drag force on the missile (N)
$k$ Dynamic pressure of air
$a_x$ Horizontal Acceleration
$a_y$ Vertical Acceleration
$r_{(t)}$ Missile position vector at time t
$r'_{(t)}$ Target position vector at time t
$t_c$ Critical time of response

$t_{int}$ Intercept time (**s**)