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Use of Variational Calculus to Evolve Third Order Functionals for Continuum Analysis

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Abstract:
This paper presents use of variational calculus to evolve third order functionals for continuum analysis. The governing equilibrium equation of forces of a line continuum was integrated in the open domain with respect to deflection to obtain three different valid forms of total energy functional for the continuum. They are second order (Ritz energy functional), fourth order (work error functional) and any functional hereinafter called the third order energy functional. Third order energy functional for a rectangular plate was also formulated. These third order energy functionals were subjected to direct variation (differentiating with respect to the coefficient of deflection) to obtain the weak form equilibrium of forces of continuums. Line continuum of four different boundary conditions and a plate with one edge clamped and the other three edges simply supported were used to test this new third order energy functional. In this numerical study, pure bending, buckling and free vibration analysis were performed. The results obtained indicated that the values obtained using this new method are exactly the same as the values obtained using either Ritz or work error energy functionals. Thus, one can comfortably and confidently use the third order energy functionals in continuum analysis

Keywords: *variational calculus, continuum, deflection, energy functional, third order energy functional, equilibrium of forces, energy functional*

1. Introduction
Flexural continuum (beams, columns rectangular plates) energy approach analysis is dominated by the use of second order functional (functional with highest derivative of deflection being two) and fourth order functional (functional with highest derivative of deflection being four). Typical examples of second order functional are Ritz and Rayleigh-Ritz energy functionals (El-Naschie, 1990; Ugural, 1999; Long et al., 2009; Chakraverty, 2009; Ibearugbulem and Ezeh, 2013; Ibearugbulem et al., 2013;). Galerkin and work-error energy functionals are examples of fourth order functionals (Ezeh et l., 2013; Njoku et al., 2013; Ibearugbulem et al.,2014a; Ibearugbulem et al.,2014b). These energy functionals are very sufficient in continuum analyses. However, curiosity demands for a third order function for continuum analyses. The probing question is whether a reliable third order energy functional will be adequate for continuum analyses. It is in the bid to answer this probing question that gave rise to the present subject matter " use of variational calculus to evolve third order functionals for continuum analysis"

2. Energy Functionals from Governing Differential Equation
The governing differential equation of a line continuum subject to uniform lateral load, axial load and vibration is given by Ibearugbulem et al. (2014) as:

$$\frac{\mathrm{EI}d^4w}{dx^4} = \mathrm{q} + \mathrm{Nx}\frac{d^2w}{dx^2} + \rho a\lambda^2 w \qquad (1)$$

Where w, q, Nx, ρ, a, λ and EI respectively symbolize lateral deflection, lateral uniform load, axial load, density, cross section area, vibration frequency and flexural stiffness of the line continuum. This equation is the equation of equilibrium of forces acting at any arbitrary point along the continuum. The term on the left hand of the equation is the internal resistance force of the continuum, where as the terms on the right hand sum up to external forces trying to deform the continuum. Summation of all these forces (internal and external) at any point along the continuum must be zero. By employing principles variational calculus, equation (1) can be integrated in open domain with respect to deflection, w to obtain:

$$\frac{\mathrm{EI}d^4(w)^2}{2dx^4} = \mathrm{qw} + \mathrm{Nx}\,\frac{d^2(w)^2}{2dx^2} + \frac{\rho a\lambda^2(w)^2}{2} + e_i \quad (2)$$

'e_i' is the arbitrary constant of integration and i denote arbitrary position along the continuum. Equation (2) can be rewritten in three different mathematically valid ways as:

$$ei = \frac{\mathrm{EI}}{2}\left(\frac{d^2w}{dx^2}\right)^2 - \mathrm{qw} - \frac{\mathrm{Nx}}{2}\left(\frac{dw}{dx}\right)^2 - \frac{\rho a\lambda^2(w)^2}{2} \quad (3)$$

$$ei = \frac{\mathrm{EI}}{2}\frac{d^4w}{dx^4}.\mathrm{w} - \mathrm{qw} - \frac{\mathrm{Nx}}{2}\frac{d^2w}{dx^2}.w - \frac{\rho a\lambda^2(w)^2}{2} \quad (4)$$

$$ei = \frac{\mathrm{EI}}{2}\frac{d^3w}{dx^3}.\frac{dw}{dx} - \mathrm{qw} - \frac{\mathrm{Nx}}{2}\frac{d^2w}{dx^2}.w - \frac{\rho a\lambda^2(w)^2}{2} \quad (5)$$

Equations (3), (4) and (5) means that summation of all the works (internal and external) performed on the continuum at any arbitrary point along the continuum is not equal to zero but equal to a constant, ei. Let us sum these works at all the points along the continuum. This is an indefinite summation (integration):

$$\Pi = \int_0^L ei\,dx = \frac{\mathrm{EI}}{2}\int_0^L\left(\frac{d^2w}{dx^2}\right)^2 dx - \mathrm{q}\int_0^{\mathrm{L}}\mathrm{w\,dx} - \frac{\mathrm{Nx}}{2}\int_0^L\left(\frac{dw}{dx}\right)^2 dx - \frac{\rho a\lambda^2}{2}\int_0^L (w)^2\,dx \quad (6)$$

$$\Pi = \int_0^L ei\,dx = \frac{\mathrm{EI}}{2}\int_0^L\frac{d^4w}{dx^4}.\mathrm{w\,dx} - \mathrm{q}\int_0^{\mathrm{L}}\mathrm{w\,dx} - \frac{\mathrm{Nx}}{2}\int_0^{\mathrm{L}}\frac{d^2w}{dx^2}.w\,dx - \frac{\rho a\lambda^2}{2}\int_0^L (w)^2\,dx \quad (7)$$

$$\Pi = \int_0^L ei\,dx = \frac{\mathrm{EI}}{2}\int_0^L\frac{d^3w}{dx^3}.\frac{dw}{dx}\mathrm{dx} - \mathrm{q}\int_0^{\mathrm{L}}\mathrm{w\,dx} - \frac{\mathrm{Nx}}{2}\int_0^L\frac{d^2w}{dx^2}.w\,dx - \frac{\rho a\lambda^2}{2}\int_0^L (w)^2\,dx \quad (8)$$

Equation (6) is a typical Raleigh-Ritz equation of total potential energy functional of flexural line continuum under external flexural disturbance. Equation (7) looks like the equation of total work error functional of flexural line continuum under external flexural disturbance (Ibearugbulem et al., 2014). The first term (internal work or internal energy of the continuum) of equation (8) is new and has not been used for continuum analysis. However, it is supposed to be mathematically equal to the internal work terms of equations (6) and (7). At this point, it shall be proper to ascertain the actual sign (positive or negative) of the various derivatives of the deflection function. In doing this, common trigonometric ($H = \sin \pi R$) and polynomial ($H = R - 2R^3 + R^4$) deflection functions for simply supported (SS) line continuum shall be used. Polynomial deflection functions for clamped, C-C, ($H = R^2 - 2R^3 + R^4$) and propped cantilever, CS, ($H = 1.5R^2 - 2.5R^3 + R^4$) line continua were also used. The signs of these deflection functions and their derivative are presented on figures 1 to 4. Note: w = A H; A is the coefficient and H is deflection curve (profile, shape); R = x/L; $0 \le R \le 1$; G is as defined in the figures.

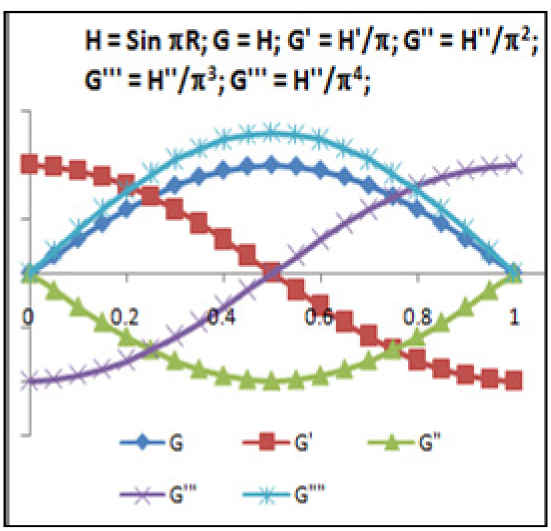

Figure 1: Trigonometric funtion for SS

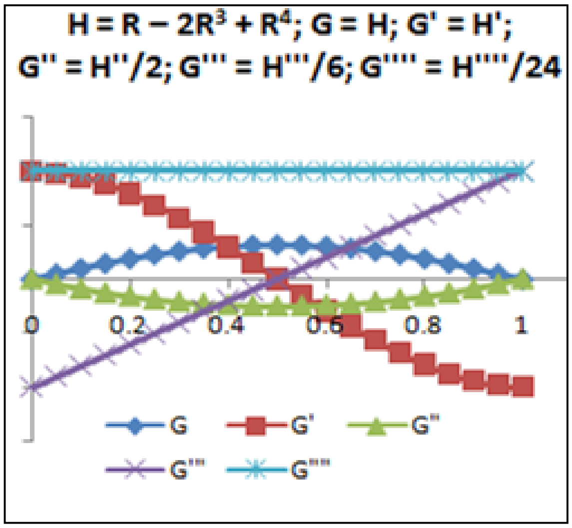

Figure 2: Polynomial funtion for SS

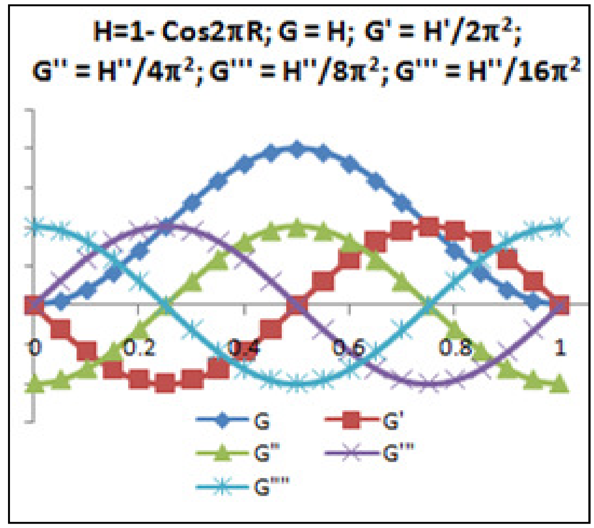

Figure 3: Trigonometric funtion for CC

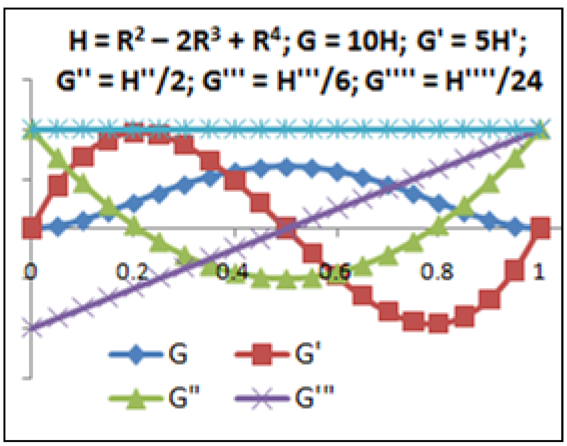

Figure 4: Polynomial funtion for CC

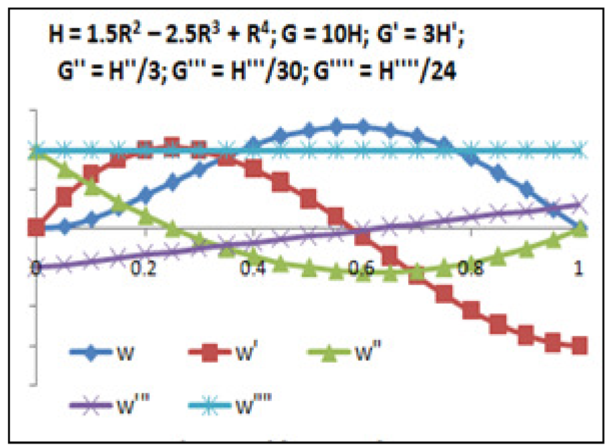

Figure 5: Polynomial funtion for CS

From figures 1 to 5, only the deflection function, w and its fourth derivatives, w'''' are positive at all the points along the continuum. Other derivatives – w'; w''; w''' – are positive at some points and negative at some other points along the continuum.

By thorough calculations of functions and observation of figures 1 to 5, we shall note that at all times the the following are obtainable:

$w = positve; \left(\frac{d^4w}{dx^4}.w\right) = positve;$

$\left(\frac{d^3w}{dx^3}.\frac{dw}{dx}\right) = negative; \left(\frac{d^2w}{dx^2}\right)^2 = positve;$

$\left(\frac{d^2w}{dx^2}.w\right) = negative$

With these values (positive or negative), equations (7) and (8) shall be rewritten as:

$$\Pi = \frac{\mathrm{EI}}{2}\int_0^L \frac{d^4w}{dx^4}.\mathrm{w\,dx} - \mathrm{q}\int_0^{\mathrm{L}} \mathrm{w\,dx} + \frac{\mathrm{Nx}}{2}\int_0^L \frac{d^2w}{dx^2}.w\,dx$$
$$-\frac{\rho a\lambda^2}{2}\int_0^L (w)^2\,dx \qquad (9)$$

$$\Pi = -\frac{\mathrm{EI}}{2}\int_0^L \frac{d^3w}{dx^3}.\frac{dw}{dx}\mathrm{dx} - \mathrm{q}\int_0^{\mathrm{L}} \mathrm{w\,dx} + \frac{\mathrm{Nx}}{2}\int_0^L \frac{d^2w}{dx^2}.w\,dx$$
$$-\frac{\rho Aa}{2}\int_0^L (w)^2\,dx \qquad (10)$$

Equation (9) is the typical equation of total work error functional of flexural line continuum under external flexural disturbance (Ibearugbulem et al., 2014). However, equation (10) is a new equation of total energy functional (third order work function) of flexural line continuum under external flexural disturbance.

3. Verifying the Mathematical Equivalence of the Functionals

Let the three functionals represented by equations (6), (9) and (10) be designated functional 1, functional 2 and functional 3. The integrals to be tested for mathematical equivalence include:

$\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx$, $\int_0^L \frac{d^4w}{dx^4}.\text{w dx}$ and $\int_0^L \frac{d^3w}{dx^3}.\frac{dw}{dx}\text{dx}$. Also,
$\int_0^L \left(\frac{dw}{dx}\right)^2 dx$ *and* $\int_0^L \frac{d^2w}{dx^2}.w\,dx$

We shall use the five deflection equation used in plotting the graphs of figures 1 to 5 in this verification. Product of functions are presented on table 1 and their integrals are presented on table 2.

4. Applying Direct Variation Calculus on Energy Equations

If variational calculus is applied on equations (6), (9) and (10), we obtain (for each case) exactly equation (1), which is the governing differential equation. This equation is the strong form of expressing continuum flexural behavior under external disturbance because it states the equilibrium of forces at any arbitrary point along the continuum. This variation is achieved by differentiating the energy equation with respect to displacement equation, w. However, if we differentiate the energy equation with respect to the coefficient, A of the displacement equation, we obtain equilibrium of forces summed at all the points along the continuum. Note, the equilibrium of forces here is not at any arbitrary point, but summation of forces at all the points. The resulting equation is a weak form of expressing continuum flexural behavior under external disturbance because it did not state the equilibrium of forces at any arbitrary point along the continuum. This type of variation is called direct variational calculus. Thus, applying direct variation on equations (6), (9) and (10), we obtain:

$$0 = \text{EIA}\int_0^L \left(\frac{d^2H}{dx^2}\right)^2 dx - \text{q}\int_0^{\text{L}} \text{H dx} - \text{Nx.A}\int_0^L \left(\frac{dH}{dx}\right)^2 dx - \rho \text{a}\lambda^2.A\int_0^L (H)^2\,dx \quad (11)$$

$$0 = \text{EIA}\int_0^L \frac{d^4H}{dx^4}.\text{H dx} - \text{q}\int_0^{\text{L}} \text{H dx} + \text{Nx.A}\int_0^{\text{L}} \frac{d^2H}{dx^2}.H\,dx - \rho \text{a}\lambda^2.A\int_0^L (H)^2\,dx \quad (12)$$

$$0 = -\text{EIA}\int_0^L \frac{d^3H}{dx^3}.\frac{dH}{dx}\text{dx} - \text{q}\int_0^{\text{L}} \text{H dx} + \text{Nx.A}\int_0^L \frac{d^2w}{dx^2}.H\,dx - \rho a.A\int_0^L (H)^2\,dx \quad (13)$$

These three equations can be used for pure bending, buckling and free vibration analyses. For pure bending analysis, Nx and λ are equal to zero and we obtain from equations (6), (9) and (10):

$$A = \frac{\text{q}\int_0^{\text{L}} \text{H dx}}{\text{EI}\int_0^L \left(\frac{d^2H}{dx^2}\right)^2 dx} \quad (14)$$

$$A = \frac{\text{q}\int_0^{\text{L}} \text{H dx}}{\text{EI}\int_0^L \frac{d^4H}{dx^4}.\text{H dx}} \quad (15)$$

$$A = \frac{-\text{q}\int_0^{\text{L}} \text{H dx}}{\text{EI}\int_0^L \frac{d^3H}{dx^3}.\frac{dH}{dx}\text{dx}} \quad (16)$$

For buckling analysis, q and λ are equal to zero and from equations (6), (9) and (10) we obtain:

$$N_x = \frac{\text{EI}\int_0^L \left(\frac{d^2H}{dx^2}\right)^2 dx}{\int_0^L \left(\frac{dH}{dx}\right)^2 dx} \quad (17)$$

$$N_x = \frac{-\text{EI}\int_0^L \frac{d^4H}{dx^4}.\text{H dx}}{\int_0^{\text{L}} \frac{d^2H}{dx^2}.H\,dx} \quad (18)$$

$$N_x = \frac{\text{EI}\int_0^L \frac{d^3H}{dx^3}.\frac{dH}{dx}\text{dx}}{\int_0^L \frac{d^2H}{dx^2}.H\,dx} \quad (19)$$

For free vibration analysis, q and Nx are equal to zero and we obtain from equations (6), (9) and (10):

$$\lambda^2 = \frac{\mathrm{EI}\int_0^L \left(\frac{d^2H}{dx^2}\right)^2 dx}{\rho a \int_0^L (H)^2\, dx} \qquad (20)$$

$$\lambda^2 = \frac{\mathrm{EI}\int_0^L \frac{d^4H}{dx^4}.\mathrm{H\, dx}}{\rho a \int_0^L (H)^2\, dx} \qquad (21)$$

$$\lambda^2 = \frac{-\mathrm{EI}\int_0^L \frac{d^3H}{dx^3}.\frac{dH}{dx}\mathrm{dx}}{\rho a \int_0^L (H)^2\, dx} \qquad (22)$$

5. New Energy (Work) Functional for Classical Plate Continuum

From the foregoing, one can write a new work functional for a classical plate. Note, the maximum derivative of strain energy in Ritz energy functional is two. In the works of Ibearugbulem, the maximum derivative of the internal work in the work functional is four. Herein, the internal work of this new functional shall have a maximum derivative of three (hence, third order energy functional) as:

$$\Pi = \frac{\mathrm{D}}{2}\int_0^a\int_0^b \left(\frac{\partial^3 w}{\partial x^3}.\frac{\partial w}{\partial \mathrm{x}} + 2\frac{\partial^3 w}{\partial x^2\,\partial \mathrm{y}}.\frac{\partial w}{\partial \mathrm{y}} + \frac{\partial^3 w}{\partial y^3}.\frac{\partial w}{\partial \mathrm{y}}\right)\partial \mathrm{x}\,\partial \mathrm{y}$$
$$+\int_0^a\int_0^b \mathrm{q.w}\ \partial \mathrm{x}\,\partial \mathrm{y} + \frac{\rho h\lambda^2}{2}\int_0^L w^2\,\partial \mathrm{x}\,\partial \mathrm{y}$$
$$-\frac{1}{2}\int_0^a\int_0^b \left(\mathrm{N_x}\frac{\partial^2 w}{\partial x^2}.w + 2\mathrm{N_{xy}}\frac{\partial^2 w}{\partial \mathrm{x}\,\partial \mathrm{y}}.w + \mathrm{N_y}\frac{\partial^2 w}{\partial x^2}.w\right)\partial \mathrm{x}\,\partial \mathrm{y} \qquad (23)$$

$$\Pi = \frac{\mathrm{D}}{2}\int_0^a\int_0^b \left(\frac{\partial^3 w}{\partial x^3}.\frac{\partial w}{\partial \mathrm{x}} + 2\frac{\partial^3 w}{\partial \mathrm{x}\,\partial y^2}.\frac{\partial w}{\partial \mathrm{x}} + \frac{\partial^3 w}{\partial y^3}.\frac{\partial w}{\partial \mathrm{y}}\right)\partial \mathrm{x}\,\partial \mathrm{y}$$
$$+\int_0^a\int_0^b \mathrm{q.w}\ \partial \mathrm{x}\,\partial \mathrm{y} + \frac{\rho h\lambda^2}{2}\int_0^L w^2\,\partial \mathrm{x}\,\partial \mathrm{y} - \frac{1}{2}\int_0^a\int_0^b \left(\mathrm{N_x}\frac{\partial^2 w}{\partial x^2}.w + 2\mathrm{N_{xy}}\frac{\partial^2 w}{\partial \mathrm{x}\,\partial \mathrm{y}}.w + \mathrm{N_y}\frac{\partial^2 w}{\partial x^2}.w\right)\partial \mathrm{x}\,\partial \mathrm{y} \qquad (24)$$

Applying direct variational calculus on equations (23) and (24) and rearranging the resulting weak equilibrium equations for pure bending, buckling and free vibration, we obtain:

For pure bending

$$A = \frac{-\mathrm{q}\int_0^a\int_0^b \mathrm{H}\,\partial \mathrm{x}\,\partial \mathrm{y}}{\mathrm{D}\int_0^a\int_0^b \left(\frac{\partial^3 H}{\partial x^3}.\frac{\partial H}{\partial \mathrm{x}} + 2\frac{\partial^3 H}{\partial x^2\,\partial \mathrm{y}}.\frac{\partial H}{\partial \mathrm{y}} + \frac{\partial^3 H}{\partial y^3}.\frac{\partial H}{\partial \mathrm{y}}\right)\partial \mathrm{x}\,\partial \mathrm{y}} \qquad (25)$$

$$A = \frac{-\mathrm{q}\int_0^a\int_0^b \mathrm{H}\,\partial \mathrm{x}\,\partial \mathrm{y}}{\mathrm{D}\int_0^a\int_0^b \left(\frac{\partial^3 H}{\partial x^3}.\frac{\partial H}{\partial \mathrm{x}} + 2\frac{\partial^3 H}{\partial \mathrm{x}\,\partial y^2}.\frac{\partial H}{\partial \mathrm{x}} + \frac{\partial^3 H}{\partial y^3}.\frac{\partial H}{\partial \mathrm{y}}\right)\partial \mathrm{x}\,\partial \mathrm{y}} \qquad (26)$$

For buckling load

$$\mathrm{N_x} = \frac{\mathrm{D}\int_0^a\int_0^b \left(\frac{\partial^3 H}{\partial x^3}.\frac{\partial H}{\partial \mathrm{x}} + 2\frac{\partial^3 H}{\partial x^2\,\partial \mathrm{y}}.\frac{\partial H}{\partial \mathrm{y}} + \frac{\partial^3 H}{\partial y^3}.\frac{\partial H}{\partial \mathrm{y}}\right)\partial \mathrm{x}\,\partial \mathrm{y}}{\int_0^a\int_0^b \left(\frac{\partial^2 H}{\partial x^2}.H + 2\frac{\mathrm{N_{xy}}}{\mathrm{N_x}}\frac{\partial^2 H}{\partial \mathrm{x}\,\partial \mathrm{y}}.H + \frac{\mathrm{N_y}}{\mathrm{N_x}}\frac{\partial^2 H}{\partial y^2}.H\right)\partial \mathrm{x}\,\partial \mathrm{y}} \qquad (27)$$

$$\mathrm{N_x} = \frac{\mathrm{D}\int_0^a\int_0^b \left(\frac{\partial^3 H}{\partial x^3}.\frac{\partial H}{\partial \mathrm{x}} + 2\frac{\partial^3 H}{\partial \mathrm{x}\,\partial y^2}.\frac{\partial H}{\partial \mathrm{x}} + \frac{\partial^3 H}{\partial y^3}.\frac{\partial H}{\partial \mathrm{y}}\right)\partial \mathrm{x}\,\partial \mathrm{y}}{\int_0^a\int_0^b \left(\frac{\partial^2 H}{\partial x^2}.H + 2\frac{\mathrm{N_{xy}}}{\mathrm{N_x}}\frac{\partial^2 H}{\partial \mathrm{x}\,\partial \mathrm{y}}.H + \frac{\mathrm{N_y}}{\mathrm{N_x}}\frac{\partial^2 H}{\partial y^2}.H\right)\partial \mathrm{x}\,\partial \mathrm{y}} \qquad (28)$$

For free vibration

$$\lambda^2 = \frac{-\mathrm{D}\int_0^a\int_0^b \left(\frac{\partial^3 H}{\partial x^3}.\frac{\partial H}{\partial \mathrm{x}} + 2\frac{\partial^3 H}{\partial x^2\,\partial \mathrm{y}}.\frac{\partial H}{\partial \mathrm{y}} + \frac{\partial^3 H}{\partial y^3}.\frac{\partial H}{\partial \mathrm{y}}\right)\partial \mathrm{x}\,\partial \mathrm{y}}{\rho h\int_0^a\int_0^b H^2\,\partial \mathrm{x}\,\partial \mathrm{y}} \qquad (29)$$

$$\lambda^2 = \frac{-\mathrm{D}\int_0^a\int_0^b\left(\frac{\partial^3 H}{\partial x^3}.\frac{\partial H}{\partial \mathrm{x}} + 2\frac{\partial^3 H}{\partial \mathrm{x}\,\partial y^2}.\frac{\partial H}{\partial \mathrm{x}} + \frac{\partial^3 H}{\partial y^3}.\frac{\partial H}{\partial \mathrm{y}}\right)\partial \mathrm{x}\,\partial \mathrm{y}}{\rho \mathrm{h}\int_0^a\int_0^b H^2\,\partial \mathrm{x}\,\partial \mathrm{y}} \quad (30)$$

6. Numerical Problems

The **third-order-work functional** is required for pure bending, buckling and free vibration analyses of the following continuums:

i. SS line continuum with $w = A(R - 2R^3 + R^4)$
ii. CC line continuum with $w = A(R^2 - 2R^3 + R^4)$
iii. CS line continuum with $w = A(1.5R^2 - 2.5R^3 + R^4)$
iv. CSSS plate continuum with $w = A(R - 2R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$

The following integrals from the deflection equation of the plate continuum are obtainable:

$$\int_0^a\int_0^b \frac{\partial^3 H}{\partial x^3}.\frac{\partial H}{\partial \mathrm{x}}\,\partial \mathrm{x}\,\partial \mathrm{y} = -\frac{19\mathrm{b}}{525\mathrm{a}^3} \quad (31)$$

$$\int_0^a\int_0^b \frac{\partial^3 H}{\partial y^3}.\frac{\partial H}{\partial \mathrm{y}}\,\partial \mathrm{x}\,\partial \mathrm{y} = -\frac{31\mathrm{a}}{350\mathrm{b}^3} \quad (32)$$

$$\int_0^a\int_0^b \frac{\partial^3 H}{\partial \mathrm{x}\,\partial y^2}.\frac{\partial H}{\partial \mathrm{x}}\,\partial \mathrm{x}\,\partial \mathrm{y} = \int_0^a\int_0^b \frac{\partial^3 H}{\partial \mathrm{x}^2\,\partial \mathrm{y}}.\frac{\partial H}{\partial \mathrm{y}}\,\partial \mathrm{x}\,\partial \mathrm{y} = -\frac{51}{1225\mathrm{ab}} \quad (33)$$

$$\int_0^a\int_0^b H^2\,\partial \mathrm{x}\,\partial \mathrm{y} = \frac{5\mathrm{ab}}{13477} \quad (34)$$

$$\int_0^a\int_0^b \mathrm{H}\,\partial \mathrm{x}\,\partial \mathrm{y} = \frac{3\mathrm{ab}}{200} \quad (35)$$

$$\int_0^a\int_0^b \frac{\partial^2 H}{\partial x^2}.H\,\partial \mathrm{x}\,\partial \mathrm{y} = -\frac{495\mathrm{b}}{135167\mathrm{a}} \quad (36)$$

7. Results and Discussions

Substitutions of the integrals contained on table 2 into equation (16) shall yield the values of A for SS, CC and CS continuums as $qL^4/(24EI)$. This implies that the deflection equations for SS, CC and CS continuums are respectively:

$$w = \frac{qL^4}{24EI}(R - 2R^3 + R^4)$$

$$w = \frac{qL^4}{24EI}(R^2 - 2R^3 + R^4)$$

$$w = \frac{qL^4}{24EI}(1.5R^2 - 2.5R^3 + R^4)$$

Substitutions of the integrals contained on table 2 into equations (19) and (22) shall yield the values of Nx and λ for SS, CC and CS continuums as presented on table 3. The values of deflections at the center (when R = ½) of the continuums are also presented on table 3. Close observation of table 3 reveals that the present result is approximately the same as the past result. Substituting the values of the integrals of equations (31), (32), (33) and (35) into equation (25 or 26), we obtain the value of coefficient of deflection, A and the corresponding deflection equation, w for CSSS plate continuum as:

$$A = \frac{-\frac{3\mathrm{ab}}{200}\mathrm{q}}{-\mathrm{D}\left(\frac{19\mathrm{b}}{525\mathrm{a}^3} + \frac{102}{1225\mathrm{ab}} + \frac{31\mathrm{a}}{350\mathrm{b}^3}\right)}$$

$$\mathrm{w} = \frac{-\frac{3\mathrm{ab}}{200}\mathrm{q}\,(R - 2R^3 + R^4)(1.5R^2 - 2.5R^3 + R^4)}{-\mathrm{D}\left(\frac{19\mathrm{b}}{525\mathrm{a}^3} + \frac{102}{1225\mathrm{ab}} + \frac{31\mathrm{a}}{350\mathrm{b}^3}\right)}$$

Substituting the values of the integrals of equations (31), (32), (33), (34) and (36) into equations (27 or 28) and (29 or 30) where appropriate gives the values of buckling load, Nx (when Ny = Nxy = 0) and natural frequency, λ^2:

$$\mathrm{N_x} = \frac{-\mathrm{D}\left(\frac{19\mathrm{b}}{525\mathrm{a}^3} + \frac{102}{1225\mathrm{ab}} + \frac{31\mathrm{a}}{350\mathrm{b}^3}\right)}{-\frac{495\mathrm{b}}{135167\mathrm{a}}}$$

$$\lambda^2 = \frac{D\left(\frac{19b}{525a^3} + \frac{102}{1225ab} + \frac{31a}{350b^3}\right)}{\frac{5ab}{13477}}$$

For a square plate the values of A, Nx and λ respectively become:

$$A = \frac{-380000qa^4}{-D5269999} = \frac{0.072106qa^4}{D}$$

$$N_x = \frac{56.8049D}{a^2}$$

$$\left(\text{The value from Ibearugbulem et al., 2013 is } \frac{56.80234D}{a^2}\right)$$

$$\lambda = \frac{23.6795}{a^2}\sqrt{\frac{D}{\rho h}}$$

$$\left(\text{The value from Ibearugbulem et al., 2013 is } \frac{23.67718}{a^2}\sqrt{\frac{D}{\rho h}}\right)$$

At the center of the plate (R = Q =1/2), the deflection of CSSS plate continuum is obtained as:

$$w\left(\frac{1}{2},\frac{1}{2}\right) = \frac{0.002817qa^4}{D}$$

$$\left(\text{The value from Ibearugbulem et al., 2013 is } \frac{0.00282qa^4}{D}\right)$$

It is obvious herein that the values from this present study are approximately the same with the values from pasts studies.

Deflection function, w	$\left(\frac{d^2w}{dR^2}\right)^2$	$\frac{d^4w}{dR^4}.w$	$\frac{d^3w}{dR^3}.\frac{dw}{dR}$
A sin πR	$A^2\pi^4 Sin^2\pi R$	$A^2\pi^4 Sin^2\pi R$	$-A^2\pi^4 Cos^2\pi R$
A (1 - Cos2πR)	$16A^2\pi^4 Cos^2 2\pi R$	$16A^2\pi^4 Sin^2 2\pi R$	$-16A^2\pi^4 Sin^2 2\pi R$
$A(R - 2R^3 + R^4)$	$144A^2(R^4 - 2R^3 + R^2)$	$24A^2(R - 2R^3 + R^4)$	$12A^2(2R + 6R^2 - 16R^3 + 8R^4 - 1)$
$A(R^2 - 2R^3 + R^4)$	$4A^2(1-12R+48R^2-72R^3 + 36R^4)$	$24A^2(R^2 - 2R^3 + R^4)$	$-12A^2(2R -10R^2+16R^3 - 8R^4)$
$A(1.5R^2-2.5R^3 + R^4)$	$9A^2(1-10R+33R^2-40R^3+ 16R^4)$	$24A^2(1.5R^2-2.5R^3+R^4)$	$-3A^2(15R-61.5R^2 + 80R^3-32R^4)$

Table 1: Product of functions

Deflection function, w	$\left(\frac{dw}{dR}\right)^2$	$\frac{d^2w}{dR^2}.w$
A sin πR	$A^2\pi^2 Cos^2\pi R$	$-A^2\pi^2 Sin^2\pi R$
A (1 - Cos2πR)	$4A^2\pi^2 Sin^2 2\pi R$	$-4A^2\pi^2 Sin^2 2\pi R$
$A(R - 2R^3 + R^4)$	$A^2(1-12R^2+8R^3+36R^4 -48R^5+ 16R^6)$	$-12A^2(R^2 - R^3 -2R^4+ 3R^5-R^6)$
$A(R^2 - 2R^3 + R^4)$	$A^2(4R^2-24R^3+52R^4 -48R^5+ 16R^6)$	$2A^2(R^2 - 8R^3 + 19R^4 - 18R^5 + 6R^6)$
$A(1.5R^2-2.5R^3 + R^4)$	$A^2(9R^2-45R^3 + 80.25R^4 - 60R^5 + 16R^6)$	$3A^2(1.5R^2-10R^3+19.5R^4 -15R^5 + 4R^6)$

Table 1: Product of functions continued

Deflection function, w	w^2
A sin πR	$A^2 Sin^2 \pi R$
A (1 - Cos2πR)	$A^2(1 - 2 Cos2\pi R + Cos^2 2\pi R)$
$A(R - 2R^3 + R^4)$	$A^2(R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7+ R^8)$
$A(R^2 - 2R^3 + R^4)$	$A^2(R^4 - 4R^5 + 2R^6 + 4R^6 - 4R^7+ R^8)$
$A(1.5R^2-2.5R^3 + R^4)$	$A^2(2.25R^4 - 7.5R^5 + 9.25R^6 - 5R^7 + R^8)$

Table 1: Product of functions continued

Deflection function, w	$\int_0^1 \left(\frac{d^2w}{dR^2}\right)^2 dR$	$\int_0^1 \frac{d^4w}{dR^4}.w\,dR$	$\int_0^1 \frac{d^3w}{dR^3}.\frac{dw}{dR}dR$	$\int_0^1 wdR$
A sin πR	$0.5A^2\pi^4$	$0.5A^2\pi^4$	$-0.5A^2\pi^4$	2A/π
A (1 - Cos2πR)	$8A^2\pi^4$	$8A^2\pi^4$	$-8A^2\pi^4$	A
$A(R - 2R^3 + R^4)$	$4.8A^2$	$4.8A^2$	$-4.8A^2$	A/5
$A(R^2 - 2R^3 + R^4)$	$0.8A^2$	$0.8A^2$	$-0.8A^2$	A/30
$A(1.5R^2-2.5R^3 + R^4)$	$1.8A^2$	$1.8A^2$	$-1.8A^2$	3A/40

Table 2: Values of integrals

Deflection function, w	$\int_0^1 \left(\frac{dw}{dR}\right)^2 dR$	$\int_0^1 \frac{d^2w}{dR^2}.w\,dR$	$\int_0^1 w^2 dR$
A sin πR	$0.5A^2\pi^2$	$-0.5A^2\pi^2$	$A^2/2$
A (1 - Cos2πR)	$2A^2\pi^2$	$-2A^2\pi^2$	$3\ A^2/2$
$A(R - 2R^3 + R^4)$	$\frac{17A^2}{35}$	$-\frac{17A^2}{35}$	$31\ A^2/630$
$A(R^2 - 2R^3 + R^4)$	$\frac{2A^2}{105}$	$-\frac{2A^2}{105}$	$A^2/630$
$A(1.5R^2-2.5R^3 + R^4)$	$\frac{3A^2}{35}$	$-\frac{3A^2}{35}$	$570\ A^2/75600$

Table 2: Values of integrals continued

Continuums	Deflection at center $w\left(\frac{1}{2},\frac{1}{2}\right)$		Buckling Load, Ncr		Natural frequency, λ	
	present	Ibearugbulem et al., 2013	present	Ibearugbulem et al., 2013	present	Ibearugbulem et al., 2013
SS	$\frac{5qL^4}{384EI}$	$\frac{5qL^4}{384EI}$	$\frac{168EI}{17L^2}$	$\frac{9.8824EI}{L^2}$	$\frac{1.001\pi^2}{L^2}\sqrt{\frac{EI}{\rho a}}$	$\frac{9.8767}{L^2}\sqrt{\frac{EI}{\rho a}}$
CC	$\frac{qL^4}{384EI}$	$\frac{qL^4}{384EI}$	$\frac{42EI}{L^2}$	$\frac{42EI}{L^2}$	$\frac{2.275\pi^2}{L^2}\sqrt{\frac{EI}{\rho a}}$	$\frac{22.4521}{L^2}\sqrt{\frac{EI}{\rho a}}$
CS	$\frac{2qL^4}{384EI}$	$\frac{2qL^4}{384EI}$	$\frac{21EI}{L^2}$	$\frac{21EI}{L^2}$	$\frac{1.566\pi^2}{L^2}\sqrt{\frac{EI}{\rho a}}$	$\frac{15.4511}{L^2}\sqrt{\frac{EI}{\rho a}}$

Table 3: Values of integrals continued

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