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Transmuted Weibull Logistic Distribution

M. M. Nassar
Professor, Department of Mathematical Statistics & Mathematics,
Faculty of Science, Ain Shams University, Cairo, Egypt
S. S. Radwan
Assistant Professor, Department of Mathematical Statistics & Mathematics,
Faculty of Science (Girls' Section), Al-Azhar University, Cairo, Egypt
A. Elmasry
Assistant Lecturer, Department of Mathematical Statistics & Mathematics, Faculty of Science, Cairo, Egypt

Abstract:
Most of the time real data does not follow any of the classical or standard probability models, so more effort has been expended in the development of large classes of standard probability distribution by generalizing the standard distributions. A complete study of the transmuted Weibull Logistic distribution is proposed, introducing some basic properties of this distribution, such as Quantile function, mode, characteristic function and entropy, as well as the derivation of maximum likelihood estimators of the parameters and the information matrix. Real life data is used as an application to this distribution with a comparison with other distributions to illustrate the flexibility and ability to model lifetime data. Also, a simulation study is conducted to demonstrate the effect of the sample on the estimates of the parameters.

Keywords: *Weibull distribution, generalized distribution, Quantile function, entropy and maximum likelihood estimation.*

1. Introduction

Since the Weibull distribution has the ability to imagine the characteristics of many different types of distributions, it is a very popular model for modeling data in reliability, engineering and biological studies and is flexible enough to model a variety of data sets. The Weibull distribution can also model hazard functions that are decreasing, increasing or constant, allowing it to describe any phase of an item's lifetime.

Recently, there has been an increased interest among statisticians in defining new generators of distributions by adding one or more shape parameters to provide flexibility in modeling data in several areas such as finance, reliability, engineering, biological and medical studies. in the same vein as the extended Weibull Gurvich et al. (1997) and gamma Zografos and Balakrishnan (2009) families, using the Weibull generator applied to the odds ratio $G(x)/[1-G(x)]$.The term "generator" means that for each baseline distribution G we have a different distribution F. Shaw and Buckley (2007) introduced an interesting technique -via Quadratic Rank Transmutation Map- of adding a new parameter to an existing distribution called the transmuted family, which is a mixture of the baseline and exponentiated F with power 2. Several transmuted distributions have been investigated such as Aryal and Tsokos (2009, 2011), Merovci (2013 a& b), Merovci and Puka (2014) and Merovci et al. (2016).

In this paper we introduce a generalization of the Logistic distribution via Weibull-G family distribution and the Quadratic Rank Transmutation Map. This leads to Transmuted Weibull Logistic Distribution (TWL). Our aim is that this generalization is flexible enough to model different types of lifetime data important in reliability, engineering, marketing and in other areas of research.

Proposing a new model, the so-called Transmuted Weibull Logistic (TWL) distribution, the article is outlined as follows. In Section 2 we introduce the TWL distribution, provide plots of density function and cumulative distribution function, along with the hazard and survival function. Section 3 introduces some properties of the TWL distribution as well as a complete discussion in deducing the Quantile function, mode and the characteristic function followed by the deduction of Renyi and Shannon entropies. In Section 4 we derive the maximum likelihood estimators of the unknown parameters and the Fisher information matrix. We present, in Section 5, a simulation study, followed by an application to real data to illustrate the importance of the TWL distribution, in Section 6.

2. The Transmuted Weibull Logistic Distribution

A random variable is said to have the Weibull distribution if its cumulative distribution function (cdf) is given by

$$F(x) = 1 - e^{-ax^{b}}, \quad x > 0$$

with positive parameters a and b. Henceforth, let G be a continuous baseline distribution. For each G distribution, we define the Weibull-G distribution by introducing the generator

$G(x)/[1-G(x)]$ to obtain the cdf family given by

$$F(x;a,b,\lambda)=\int_{0}^{G(x;\lambda)/[1-G(x;\lambda)]} aby^{b-1}\,e^{-ay^{b}}\,dy$$

$$=1-e^{-a\left[\frac{G(x;\lambda)}{1-G(x;\lambda)}\right]^{b}},x\in D\subseteq R,a,b>0 \qquad (1)$$

where $G(x;\lambda)$ is a baseline cdf, which depends on a parameter vector λ. The family probability density function (pdf) reduces to

$$f(x;a,b,\varepsilon)=abg(x;\varepsilon)\frac{[G(x;\lambda)]^{b-1}}{[1-G(x;\lambda)]^{b+1}}\,e^{-a\left[\frac{G(x;\lambda)}{1-G(x;\lambda)}\right]^{b}}. \qquad (2)$$

If we take $G(x)$ to be the Type I Logistic distribution in Equation (1)

$$G(x)=\frac{1}{1+e^{-\lambda x}},\quad \lambda>0,\quad -\infty<x<\infty,$$

then the Weibull-Logistic distribution (cdf) is

$$F_{WL}(x;a,b,\lambda)=1-e^{-ae^{\lambda bx}},\quad a,b>0,\lambda>o,\qquad -\infty<x<\infty. \quad (3)$$

A random variable X is said to have a transmuted distribution *T(x)* if its cumulative distribution function (cdf) defined by Shaw and Buckley (2007) is given by

$$T(x)=(1+t)F(x)\big(1-tF(x)\big),\qquad |t|\leq 1. \qquad (4)$$

From (3) and (4) the cdf and pdf of Transmuted Weibull Logistic (TWL) distribution are defined as follows

$$F_{TWL}(x)=\left[1-e^{-ae^{\lambda bx}}\right]\left[1+te^{-ae^{\lambda bx}}\right],\ a,b>0,\lambda>o\ and\ \ |t|\leq 1,-\infty<x<\infty.\ (5)$$

$$f_{TWL}(x)=ab\lambda e^{\left(\lambda bx-ae^{\lambda bx}\right)}\left[1-t+2te^{-ae^{\lambda bx}}\right],$$

$$a,b>0,\lambda>o\ and\ |t|\leq 1,-\infty<x<\infty.\ (6)$$

Plots of the pdf and cdf of the TWL for different values of the parameters are given in Figures (1) and (2).

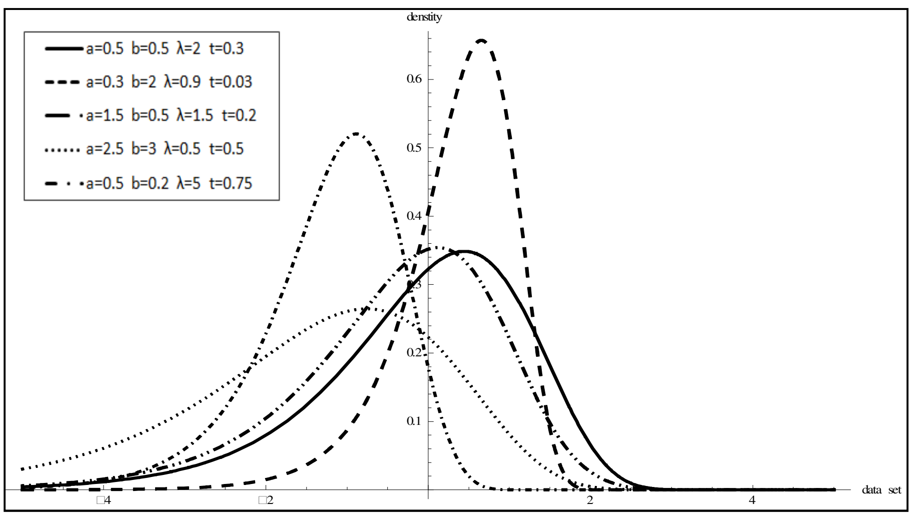

Figure 1: The pdf of the TWL distribution for different values of the parameters

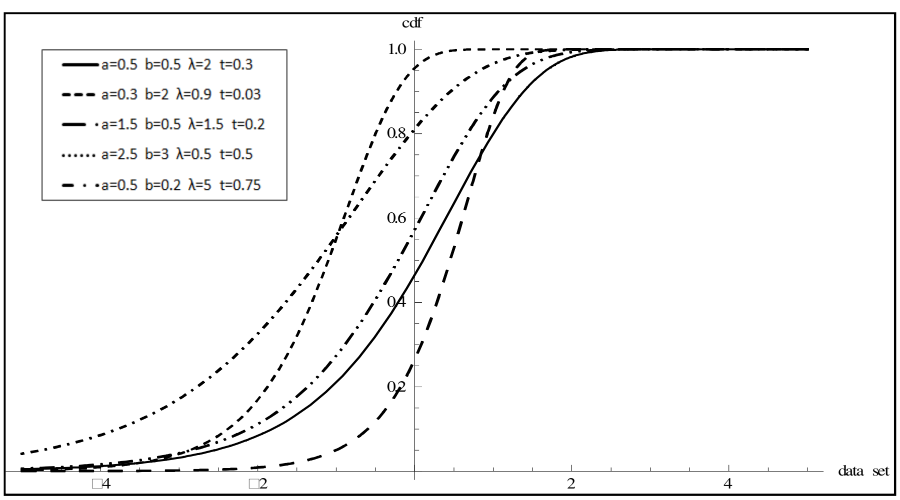

Figure 2: The cdf of the TWL distribution for different values of the parameters

The survival (reliability) function of the TWL, is defined as

$$S(x) = e^{-ae^{\lambda bx}}\left(1 - t + te^{-ae^{\lambda bx}}\right), \quad (7)$$

Also, the hazard rate function of the TWL is given by

$$h(x) = \frac{f(x)}{S(x)} = \frac{ab\lambda e^{\lambda bx}\left[1 - t + 2te^{-ae^{\lambda bx}}\right]}{\left[1 - t + te^{-ae^{\lambda bx}}\right]}. \quad (8)$$

Figure (3) illustrates the shape of the hazard rate function for different values of the parameters.

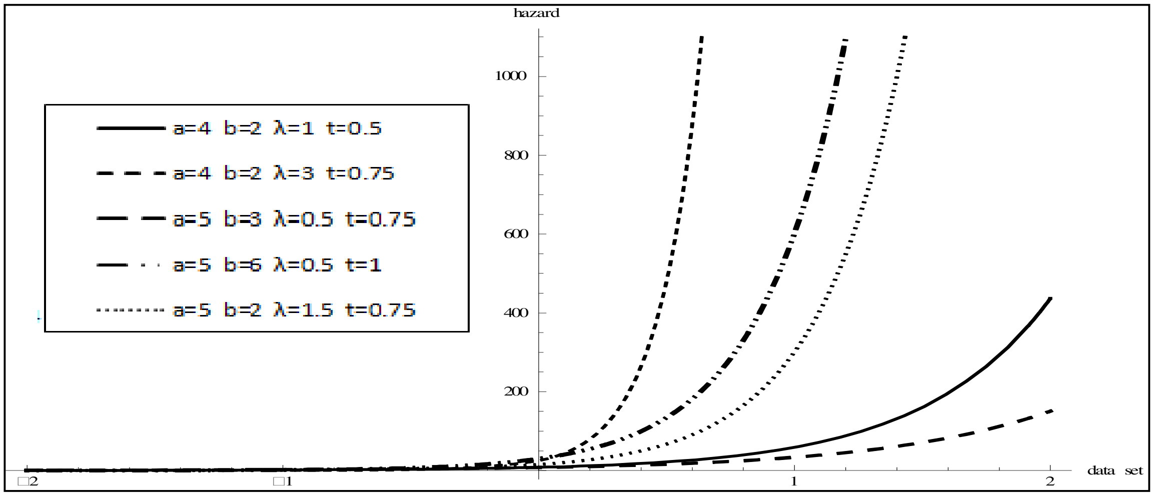

Figure 3: The hazard rate of the TKL distribution for selected values of the parameters

Section 3 introduces some properties of the $TWL(a,b,\lambda,t)$ as well as a complete discussion in deducing an explicit form for the Quantile function, numerical values for mode at different values of parameters and characteristic function followed by the deduction of Renyi and Shannon entropies. Section 4 we discuss maximum likelihood estimation and determine the observed information matrix and expected fisher information matrix. Section 5 we present a simulation study. Applications to real life data are used in Section 6 to illustrate our work.

3. Properties of the BGL Distribution

3.1. Quantile Function

Theorem 1:

Let X be a random variable following $TWL(a,b,\lambda,t)$ distribution and let $u \in (0,1)$.

A value of x such that $F(x) = u$ is called a quantile of order u for the distribution. A quantile of order u is at the following approximate value

$$x \approx \frac{1}{\lambda b}\log\left[\frac{u}{a(1+t)}\right]. \quad (9)$$

Proof:

Since $F(x)$ is continuous and strictly increasing, then the quantile function $x = F^{-1}(u), u \in (0,1)$ can be straightforward computed by inverting (5) to obtain

$$u = \left[1 - e^{-ae^{\lambda bx}}\right]\left[1 + te^{-ae^{\lambda bx}}\right]$$

$$1 - u = (1-t)e^{-ae^{\lambda bx}} + te^{-2ae^{\lambda bx}}$$

$$1 - u = (1-t)\sum_{k=0}^{\infty}(-1)^k\frac{a^k}{k!}\left[e^{\lambda bxk}\right] + t\sum_{k=0}^{\infty}(-1)^k\frac{(2a)^k}{k!}\left[e^{\lambda bxk}\right]$$

The summation on the right-hand side converges absolutely for $\left|\frac{1}{1+e^{-\lambda x}}\right| < 1$. Using the approximation technique, then the second approximation yields

$$1 - u \cong (1-t)\left[1 - ae^{\lambda bx}\right] + t\left[1 - 2ae^{\lambda bx}\right]$$

$$u \simeq (1+t)ae^{\lambda bx}$$

Therefore, an approximate Quantile function of order u of the TWL distribution is given by (9). In particular, the median of the TKL distribution is given by

$$Median \approx \frac{1}{\lambda b} \log \left[\frac{u}{a(1+t)}\right]. \qquad (10)$$

The random sample can also be easily generated from (6) by taking U as a uniform random variable in (0, 1).

3.2. Mode

Mode is one of the most important characteristic features for the distribution.

The mode of the $TWL(a, b, \lambda, t)$ is deduced by differentiating the pdf (6).

$$f^{/}(a,b,\lambda,t) = a(\lambda b)^2 e^{\lambda bx} e^{-ae^{\lambda bx}} \left[1 - t - 4ate^{\lambda bx} e^{-ae^{\lambda bx}} - ae^{\lambda bx}(1-t) + 2te^{-ae^{\lambda bx}}\right]$$

This implies that

$$\left[1 - t - 4ate^{\lambda bx} e^{-ae^{\lambda bx}} - ae^{\lambda bx}(1-t) + 2te^{-ae^{\lambda bx}}\right] = 0$$

But we cannot obtain an explicit form so we calculate the mode numerically for different values of parameters, using Maple software package.

Parameters	**Mode**
$a = 0.5 \quad b = 0.5 \quad \lambda = 2 \quad t = 0.3$	0.4446572576
$a = 0.3 \quad b = 2 \quad \lambda = 0.9 \quad t = 0.03$	0.6563830551
$a = 1.5 \quad b = 0.5 \quad \lambda = 1.5 \quad t = 0.2$	-0.7568152066
$a = 2.5 \quad b = 3 \quad \lambda = 0.5 \quad t = 0.5$	-0.8844701292
$a = 0.5 \quad b = 0.2 \quad \lambda = 5 \quad t = 0.75$	0.1202212746

Table (1): f Mode for sum chosen different values of parameters.

3.3. Characteristic Function

In this subsection, we derive the characteristic function of TWL distribution.

The characteristic function (cf) of the TWL (a, b, λ, t) distribution can be deduced to yield

$$\Phi(\Theta)_{TWL} = a^{\frac{i\Theta}{\lambda b}} \Gamma\left(1 - \frac{i\Theta}{\lambda b}\right) \left[1 - t + 2^{\frac{i\Theta}{\lambda b}} t\right], \; a, b > 0, \lambda > o \; and |t| \leq 1 \quad (11)$$

From equation (12) we find the first mean, second mean and variance as follows

$$E(X) = \frac{1}{\lambda b} [\log a + t \log 2 + \gamma]. \qquad (12)$$

$$E(X^2) = \frac{1}{(\lambda b)^2} \left[(\log a)^2 + t \log 2 \left(\log 2 - 2\Gamma'(1) + 2 \log a\right) + \Gamma''(1) - 2\Gamma'(1) \log a\right]. \qquad (13)$$

$$V(x) = \frac{1}{(\lambda b)^2} [t(1-t)(\log 2)^2 + \psi'(1)]. \qquad (14)$$

where $\psi(x) = \frac{d}{dx} \log \Gamma(x)$ is known as digamma function

and

$\psi'(x) = \frac{d}{dx} \psi(x)$ is known as polygamma function.

In fact, $\gamma = -\psi(1) = 0.577215$ is called the Euler's constant.

3.4. Reni and Shannon Entropies

The notion of entropy is of fundamental importance in different areas such as physics, probability and statistics, communication theory, and economics. Since the entropy of a random variable is a measure of variation of the uncertainty, the Renyi entropy can be deduced to yield

$$I_X(\xi) = \frac{1}{1-\xi}\log[(\lambda b)^{\xi-1}\sum_{m=0}^{\infty}(\frac{1-t}{\xi+m})^{\xi}\binom{\xi}{m}(\frac{2t}{1-t})^{m}\Gamma(\xi)] \qquad \xi \geq 0, \xi \neq 1 \quad .(15)$$

A special case, defined in Shannon's (1948) pioneering work on the mathematical theory of communication, given by Shannon entropy - a major tool in information theory and in almost every branch of science and engineering is

$$h_{sh}(f_{TWL}) = -\left[2\log a + \log b + \log\lambda + \gamma + t\log 2 - \left(1-\frac{t}{2}\right) + \log(1-t) + \sum_{l=1}^{\infty}(-1)^{l}(\frac{2t}{1-t})^{l}\left[\frac{1-t}{l+1}+\frac{2t}{l+2}\right]\right]. \qquad (16)$$

4. Maximum Likelihood Estimation

Here, we consider the maximum likelihood estimators (MLE) of the TWL (a, b, λ, t) distribution given in (7). Let $X=(X_1, X_2, ..., X_n)$ be a random sample of size n from this distribution. The log-likelihood function can be written as follows

$$\log L = n\log a + n\log b + n\log\lambda + \lambda b\sum_{i=1}^{n}x_i - a\sum_{i=1}^{n}e^{\lambda b x_i} + \sum_{i=1}^{n}\log\left[1-t+2te^{-ae^{\lambda b x_i}}\right].$$

Differentiating with respect to a, b, λ and t we obtain the following equations

$$\frac{\partial\log L}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}e^{\lambda b x_i} - \sum_{i=1}^{n}\frac{2te^{\lambda b x_i}e^{-ae^{\lambda b x_i}}}{\left(1-t+2te^{-ae^{\lambda b x_i}}\right)},$$

$$\frac{\partial\log L}{\partial b} = \frac{n}{b} + \lambda\sum_{i=1}^{n}x_i - a\lambda\sum_{i=1}^{n}x_ie^{\lambda b x_i} - \sum_{i=1}^{n}\frac{2ta\lambda x_ie^{\lambda b x_i}e^{-ae^{\lambda b x_i}}}{\left(1-t+2te^{-ae^{\lambda b x_i}}\right)},$$

$$\frac{\partial\log L}{\partial\lambda} = \frac{n}{\lambda} + b\sum_{i=1}^{n}x_i - ab\sum_{i=1}^{n}x_ie^{\lambda b x_i} - \sum_{i=1}^{n}\frac{2tabx_ie^{\lambda b x_i}e^{-ae^{\lambda b x_i}}}{\left(1-t+2te^{-ae^{\lambda b x_i}}\right)},$$

$$\frac{\partial\log L}{\partial t} = \sum_{i=1}^{n}\frac{2e^{-ae^{\lambda b x_i}}-1}{\left[1-t+2te^{-ae^{\lambda b x_i}}\right]},$$

For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The Fisher information matrix $K = K(\theta), \theta = (a, b, \lambda, t)^T$, is

$$K = \begin{pmatrix} K_{a,a} & K_{a,b} & K_{a,\lambda} & K_{a,t} \\ K_{b,a} & K_{b,b} & K_{b,\lambda} & K_{b,t} \\ K_{\lambda,a} & K_{\lambda,b} & K_{\lambda,\lambda} & K_{\lambda,t} \\ K_{t,a} & K_{t,b} & K_{t,\lambda} & K_{t,t} \end{pmatrix}$$

whose elements are

$$K_{t,t} = E\left(-\frac{\partial^2\log L}{\partial t^2}\right) = \frac{n}{1-t}\sum_{j=1}^{\infty}(-1)^{j}(\frac{2t}{1-t})^{j}[\frac{1}{1+j}+\frac{4}{3+j}-\frac{4}{2+j}]$$

$$K_{t,a} = E\left(-\frac{\partial^2\log L}{\partial t\,\partial a}\right) = \frac{-2nt}{a(1-t)}\sum_{j=1}^{\infty}(-1)^{j}(\frac{2t}{1-t})^{j}[\frac{2}{(3+j)^2}-\frac{1}{(2+j)^2}] + \frac{n}{2a},$$

$$K_{t,b} = E\left(-\frac{\partial^2\log L}{\partial t\,\partial b}\right) = \frac{-2tn}{b(1-t)}\sum_{j=1}^{\infty}(-1)^{j}(\frac{2t}{1-t})^{j}\{\frac{-\Gamma(2)}{(j+2)^2}[\psi(2)-\log(a(j+2))] + \frac{2\Gamma(2)}{(j+3)^2}[\psi(2)-\log(a(j+3))]\} + \frac{n\Gamma(2)}{2b}[\psi(2)-\log(2a)],$$

$$K_{t,\lambda} = E\left(-\frac{\partial^2\log L}{\partial t\,\partial\lambda}\right) = \frac{-2tn}{\lambda(1-t)}\sum_{j=1}^{\infty}(-1)^{j}(\frac{2t}{1-t})^{j}\{\frac{-\Gamma(2)}{(j+2)^2}[\psi(2)-\log(a(j+2))] + \frac{2\Gamma(2)}{(j+3)^2}[\psi(2)-\log(a(j+3))]\} + \frac{n\Gamma(2)}{2\lambda}[\psi(2)-\log(2a)],$$

$$K_{aa}=E\left(-\frac{\partial^2 \log L}{\partial a^2}\right)=\frac{n}{a^2}+\frac{8nt}{a^2(1-t)}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j+3)^3}(\frac{2t}{1-t})^j-\frac{nt}{2a^2},$$

$$K_{a,b}=E\left(-\frac{\partial^2 \log L}{\partial a\partial b}\right)=\frac{n}{ab}\left\{(1-t)[\psi(2)-\log a]+\frac{t}{2}\,[\psi(2)-\log 2a]\right\}$$

$$+\frac{8nt^2}{ab(1-t)}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j+3)^3}(\frac{2t}{1-t})^j\,[\psi(3)-\log(a(j+3))]+\frac{nt}{2ba}\,[\psi(2)-\log 2a]$$

$$-\frac{nt}{2ab}\,[\psi(3)-\log 2a],$$

$$K_{a,\lambda}=E\left(-\frac{\partial^2 \log L}{\partial a\partial \lambda}\right)=\frac{n}{a\lambda}\left\{(1-t)[\psi(2)-\log a]+\frac{t}{2}\,[\psi(2)-\log 2a]\right\}$$

$$+\frac{8nt^2}{a\lambda(1-t)}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j+3)^3}(\frac{2t}{1-t})^j\,[\psi(3)-\log(a(j+3))]+\frac{nt}{2\lambda a}\,[\psi(2)-\log 2a]$$

$$-\frac{nt}{2a\lambda}\,[\psi(3)-\log 2a],$$

$$K_{b,b}=E\left(-\frac{\partial^2 \log L}{\partial b^2}\right)=\frac{n}{b^2}$$

$$+\frac{n}{b^2}\{\,(1-t)\big([\psi(2)-\log a]^2+\zeta(2,2)\big)+\frac{t}{2}\,\big([\psi(2)-\log 2a]^2+\zeta(2,2)\big)\}$$

$$+\frac{8nt^2}{b^2(1-t)}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j+3)^3}(\frac{2t}{1-t})^j\,\big([\psi(3)-\log((a(j+3))]^2+\zeta(2,3)\big)-\frac{tn}{2b^2}\,\big([\psi(2)-\log 2a]^2+\zeta(2,2)\big)$$

$$+\frac{tn}{2b^2}\,\big([\psi(3)-\log 2a]^2+\zeta(2,3)\big),$$

where the Riemann's Zeta function is given by

$$\zeta(x,q)=\sum_{n=0}^{\infty}\frac{1}{(q+n)^x}\ ,\ \operatorname{Re}x>1,\quad q\neq 0.-1,-2,...$$

$$K_{\lambda,\lambda}=E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right)=\frac{n}{\lambda^2}$$

$$+\frac{n}{\lambda^2}\{\,(1-t)\big([\psi(2)-\log a]^2+\zeta(2,2)\big)+\frac{t}{2}\,\big([\psi(2)-\log 2a]^2+\zeta(2,2)\big)\}$$

$$+\frac{8nt^2}{\lambda^2(1-t)}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j+3)^3}(\frac{2t}{1-t})^j\,\big([\psi(3)-\log(a(j+3))]^2+\zeta(2,3)\big)-\frac{tn}{2\lambda^2}\,\big([\psi(2)-\log 2a]^2+\zeta(2,2)\big)$$

$$+\frac{tn}{2\lambda^2}\,\big([\psi(3)-\log 2a]^2+\zeta(2,3)\big),$$

$$K_{b,\lambda}=E\left(-\frac{\partial^2 \log L}{\partial b\partial \lambda}\right)=\frac{-n}{\lambda b}\,[\log a+t\log 2+\gamma]$$

$$+\frac{n}{\lambda b}\left[(1-t)[\psi(2)-\log a]+\frac{t}{2}\,[\psi(2)-\log 2a]\right]+\frac{n}{\lambda b}\left[(1-t)\big([\psi(2)-\log a]^2+\zeta(2,2)\big)+\frac{t}{2}\,\big([\psi(2)-\log 2a]^2+\zeta(2,2)\big)\right]$$

$$+\frac{nt}{2\lambda b}\,[\psi(2)-\log 2a]+\frac{24nt^2}{\lambda b(1-t)}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j+3)^3}(\frac{2t}{1-t})^j\,\big([\psi(3)-\log(a(j+3))]^2+\zeta(2,3)\big)$$

$$+\frac{nt}{2\lambda b}\,\big([\psi(2)-\log 2a]^2+\zeta(2,2)\big)-\frac{3nt}{2\lambda b}\,\big([\psi(3)-\log 2a]^2+\zeta(2,3)\big),$$

where, $\gamma=-\psi(1)=0.577215$ is called the Euler's constant.

The MLE $\hat{\theta}=(\hat{a}_{ML},\hat{b}_{ML},\hat{\lambda}_{ML},\hat{t}_{ML})^T$ of θ is determined from the solution of the nonlinear system of equations given earlier. Under conditions that are fulfilled for the parameter θ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $[\sqrt{n}(\hat{a}_{ML}-a),\sqrt{n}(\hat{b}_{ML}-b),\sqrt{n}(\hat{\lambda}_{ML}-\lambda),\sqrt{n}(\hat{t}_{ML}-t)]^T$ is $N_4(0,K^{-1}(a,b,\lambda,t)^T)$. The asymptotic normal $N_4\left(0,K^{-1}(\hat{a}_{ML},\hat{b}_{ML},\hat{\lambda}_{ML},\hat{t}_{ML})^T\right)$ distribution of $\hat{\theta}=(\hat{a}_{ML},\hat{b}_{ML},\hat{\lambda}_{ML},\hat{t}_{ML})^T$ can be used to construct confidence regions for some parameters and for the hazard and survival functions. In fact, a $100(1-\gamma)\%$ asymptotic confidence interval (*ACI*) for each parameter is given by

$$ACI_a=\big(\hat{a}_{ML}-z_{\gamma/2}\sqrt{K_{11}},\hat{a}_{ML}+z_{\gamma/2}\sqrt{K_{11}}\big),$$

$ACI_b = \left(\hat{b}_{ML} - z_{\gamma/2}\sqrt{K_{22}}, \hat{b}_{ML} + z_{\gamma/2}\sqrt{K_{22}}\right),$

$ACI_\lambda = \left(\hat{\lambda}_{ML} - z_{\gamma/2}\sqrt{K_{33}}, \hat{\lambda}_{ML} + z_{\gamma/2}\sqrt{K_{33}}\right),$

$ACI_t = \left(\hat{t}_{ML} - z_{\gamma/2}\sqrt{K_{44}}, \hat{t}_{ML} + z_{\gamma/2}\sqrt{K_{44}}\right).$

where K_{ii} denotes the ith diagonal element of $K^{-1} = (\hat{a}_{ML}, \hat{b}_{ML}, \hat{\lambda}_{ML}, \hat{t}_{ML})^T$ for i= 1, 2, 3, 4 and $z_{\gamma/2}$ is the ($1-\gamma/2$)of the standard normal distribution.

5. Simulation Study

We conducted Monte Carlo simulation studies to assess the finite sample behavior of the TWL (a $, b, \lambda, t$). All results were obtained from 1000 Monte Carlo replication simulations. The TWL random number generation was performed using the inversion method. In each replication, random sample of size n is drawn from the TWL(a, b, λ, t) distribution and the maximum likelihood estimates (MLEs) of the parameters were obtained. The mean, variance, bias and mean squared error (MSE) for each parameter was computed under different sample size n=10, 25, 75, 100, and 200.

N	parameter	Mean	Variance	Bias	MSE
10	α	5.9699687	7.13282673	-0.0300	7.13373
	β	5.5676314	4.908126474	2.56763	11.5009
	λ	0.0317631	0.0000409187	0.00176	0.00044
	t	-0.4800461	0.646595999	-1.1801	2.03910
25	α	6.62644714	0.005892637	0.62645	0.3983286
	β	6.43122571	0.14099112	3.43123	11.914301
	λ	0.02860015	0.00000157753	-0.0014	0.0000021
	t	-0.4027871	0.002182292	-1.1028	1.2183120
75	α	6.53272286	0.730695427	0.532723	1.01448907
	β	6.35177743	0.008180484	3.351777	11.2425905
	λ	0.0292078	0.000010835	-0.00079	0.000011463
	t	-0.3365234	0.290009664	-1.03652	1.364390482
100	α	6.80724714	0.241429107	0.807246	0.89307475
	β	5.26421286	0.650199061	2.264213	5.776858924
	λ	0.02940053	0.000000483	-0.0006	0.000000820
	t	0.01146629	0.649386246	-0.68853	1.123464922
200	α	6.94612714	2.237077824	0.946127	3.132234395
	β	4.82563429	0.347052798	1.825634	3.679993344
	λ	0.02773161	0.0000669672	-0.00227	0.0000721127
	t	0.77389171	0.660652135	0.073892	0.666112121

Table 2: Mean estimates, bias, variance and mean square errors of the (MLEs) when α=6, b=3, λ=0.03, t=0.7.

We note that the MSE of the parameters a, b and t decrease as the sample size increases. The mean estimates of the parameters tend to be closer to the true parameter values. It is observed that for all values of n, the variance and MSE of the estimator of λ are small as expected.

6. Numerical Example

In this section, we study the transmuted Weibull logistic distributions and provide detailed mathematical treatment for this distribution. As applications, the first set due to Smith and Naylor (1987) consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67,1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. These data have also been analyzed by Smith and Naylor (1987).

For all data, we fit the Transmuted Weibull Logistic (TWL) distribution defined in (7) and compare it with Trans muted Logistic (TL) (for$-\infty < x < \infty$) and Weibull Logistic (WL) (for$-\infty < x < \infty$) models with corresponding densities:

$f_{TL} = \lambda e^{-\lambda * x}(1 + e^{-\lambda * x})^{-2}(1 + t - 2t(1 + e^{-\lambda * x})^{-1})$,

$f_{WL} = ab\lambda e^{\lambda bx} e^{-ae^{\lambda bx}}$,

Where $a, b > 0, \lambda > o\ and\ |t| \leq 1$

The maximum likelihood method is applied to estimate the parameters of the three models Transmuted Logistic (TL), Weibull Logistic (WL) and Transmuted Weibull Logistic (TWL) distribution. The resulting estimates with the negative of the likelihood function ($-\ell$).

Model	maximum likelihood estimates	$-\ell$
TL	$\hat{\lambda} = 2.079760$ $\hat{t} = 0.015320$	310.974
WL	$\hat{a} = 0.0624001$ $\hat{b} = 2.1150200$ $\hat{\lambda} = 1.0018969$	-29.46
TWL	$\hat{a} = 0.0051595$ $\hat{b} = 3.1058101$ $\hat{\lambda} = 1.0324516$ $\hat{t} = 0.3692410$	-129.27

Table 3: The maximum likelihood estimated and Log- likelihood function for the first data set.

The variance covariance matrix $I(\theta)^{-1}$ of the MLEs under the TKL distribution for the first data set is computed as

$$\begin{pmatrix} 0.000047 & -0.0049631 & -0.00164987 & 0.015919 \\ -0.0049631 & 0.618871 & 0.14461812 & -1.72172 \\ -0.00164987 & 0.144618 & 0.0683900 & -0.572345 \\ 0.015919 & -1.72172 & -0.572345 & 5.69709 \end{pmatrix}$$

Thus $var\ (\hat{a}\,) = 0.000047, var\ (\hat{b}) = 0.618871, var(\hat{\lambda}) = 0.0683900, var(\hat{t}\,) = 5.69709.$

There for, 95% confidence interval for a. b, λ and t are [0.003467, 0.006852], [3.0444, 3.1672], [0.9679, 1.0970], [-0.2202, 6.2864], respectively.

Model	$-\ell$	AIC	AICC	BIC
TL	310.974	625.948	626.148	625.547
WL	-29.46	-52.92	-52.5132	-53.522
TWL	-129.27	-250.54	-249.850	-251.343

Table 4: Criteria comparison for the first data set

The second data set were used by Birnbaum and Saunders (1969) and correspond to the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). The data are: 70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123,124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212.

Model	maximum likelihood estimates	$-\ell$
TL	$\hat{\lambda} = 0.00550992$ $\hat{t} = 0.0035320$	602.005
WL	$\hat{a} = 0.0931514$ $\hat{b} = 4.1730230$ $\hat{\lambda} = 0.00415942$	150.919
TWL	$\hat{a} = 0.0391714$ $\hat{b} = 3.1810230$ $\hat{\lambda} = 0.00615942$ $\hat{t} = 0.0511610$	109.662

Table 5: The maximum likelihood estimated and Log- likelihood function for the second data set.

The variance covariance matrix $I(\theta)^{-1}$ of the MLEs under the TKL distribution for the second data set is computed as

$$\begin{pmatrix} 0.0000373 & -0.00057279 & -0.0000011 & 0.0001101 \\ -0.00057279 & 0.03462102 & 0.00003699 & 0.0101324 \\ -0.0000011 & 0.00003699 & 1.2980\times10^{-7} & 0.000019 \\ 0.0001101 & 0.010132400 & 0.000019 & 0.02274 \end{pmatrix}$$

Thus $var\ (\hat{a}\ ) = 0.0000373, var\ (\hat{b}) =\ 0.03462102, var(\hat{\lambda}) = 1.2980 \times 10^{-7}, var(\hat{t}\ ) = 0.02274.$
There for, 95% confidence interval for a. b, λ and t are [0.03798, 0.040363], [3.1447, 3.21731], [0.00609, 0.00623], [0.02175,0.08057], respectively.

Model	$-\ell$	AIC	AICC	BIC
TL	602.005	1208.01	1208.132	1208.019
WL	150.919	307.838	308.085	307.851
TWL	109.662	47.324	47.741	227.341

Table 6: Criteria comparison for the second data set

In order to compare the three distributions, we consider criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC the Bayesian information criterion, for the first data set given by Smith and Naylor (1987) and the second data set given by Birnbaum and Saunders (1969). As shown in table (4) and table (6), the better distribution corresponds to smaller $-\ell$, AIC, AICC and BIC values
where
$AIC = 2K - 2\ell,$
$\text{AICC} = AIC + \frac{2k(k+1)}{n-k-1},$
$BIC = k\log n - 2\ell,$
Here k is the number of parameters and n is the number of observations. The values of the parameters' estimates are used to plot the pdf for the three distributions TL, WL and TWL in Fig (4) and Fig (5)

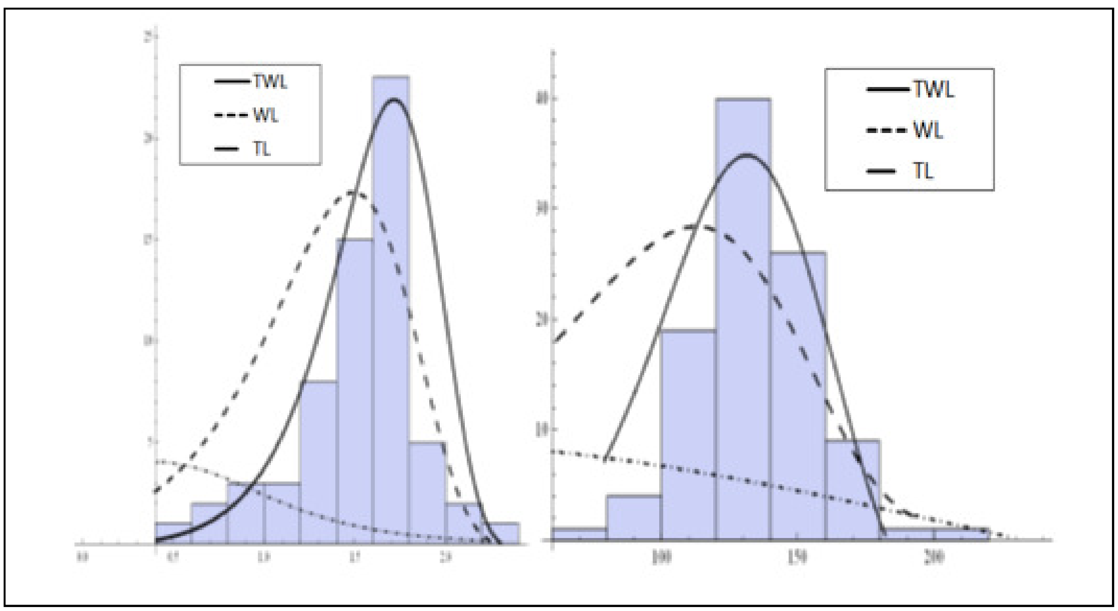

Figure 4: Estimated densities of the models for the first data set. *Figure 5: Estimated densities of the models for the second data set.*

7. Concluding Remarks

In this paper, we proposed a new distribution, named the transmuted Weibull Logistic distribution which extends the Weibull Logistic distribution. Several properties of the new distribution were investigated, including moments, median, mode, Rényi and Shannon entropy. The model parameters are estimated by maximum likelihood and the information matrix is derived. An application of the transmuted Weibull Logistic distribution (TWL) to real data is considered. The results of our study indicate that the TWL distribution has the lowest AIC, AICC and BIC statistics among all the sub-models. From the plots of the fitted densities and histogram, cleary, the TWL distribution provides a closer fit to the histogram than the other Weibull Logistic and Transmuted Logistic model. Therefore, the new TWL model can be used quite effectively in analyzing data. Also, we note that the Monte Carlo simulation indicate that the performance of the maximum likelihood estimation are quite satisfactory. Finally, the application to the real data sets shows that the fit of the new model is superior to the fits of its main sub- models. We hope that the proposed model can be used effectively as a competitive model to fit real data.

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