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Mathematical Modelling of Groundwater Flow A Study of Flow of Water in Discharging an Aquifer

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Abstract:
In this Paper, a mathematical model is formulated and analysed to study the groundwater flow in an aquifer. The two-dimensional steady states of groundwater flow in a discharge of an aquifer in a well were considered. The model consisted of Poisson's and Laplace's partial differential equations coupled with a set of Neumann and Dirichlet conditions prescribed at the boundary of the aquifer.
The model was analysed using finite element method (FEM). The analysis showed that the flow of ground water depends on the distribution of hydraulic pressure in the aquifer and the pumping process of groundwater from an aquifer in a well lowers the hydraulic pressure at the well node and the immediate neighbourhood nodes; and, hence, causing the groundwater to flow towards the well. It was further found that more pumping of groundwater from the well causes over extraction of groundwater from the aquifer.
It is optimized that, since the results obtained in this study constitute an important first step in the study of groundwater flow in Tanzania, it therefore, makes it an important area of interest because it is concerned with the supply of water for domestic and industrial use.

1. Introduction

Groundwater is water in the saturated zone below the water table which has drained through surface layers of soil and rock until it reaches a layer of rock material through which it cannot pass, or can pass only very slowly. Groundwater is also found in the unsaturated zone above the water table and the degree of saturation changes with time due to the influence of the atmospheric cycle. The accumulated water is stored in gaps of rocks, or between the particles above the impermeable layer. The rocks and particles which retain water in this way are called aquifers (UK Groundwater Forum, 2009).

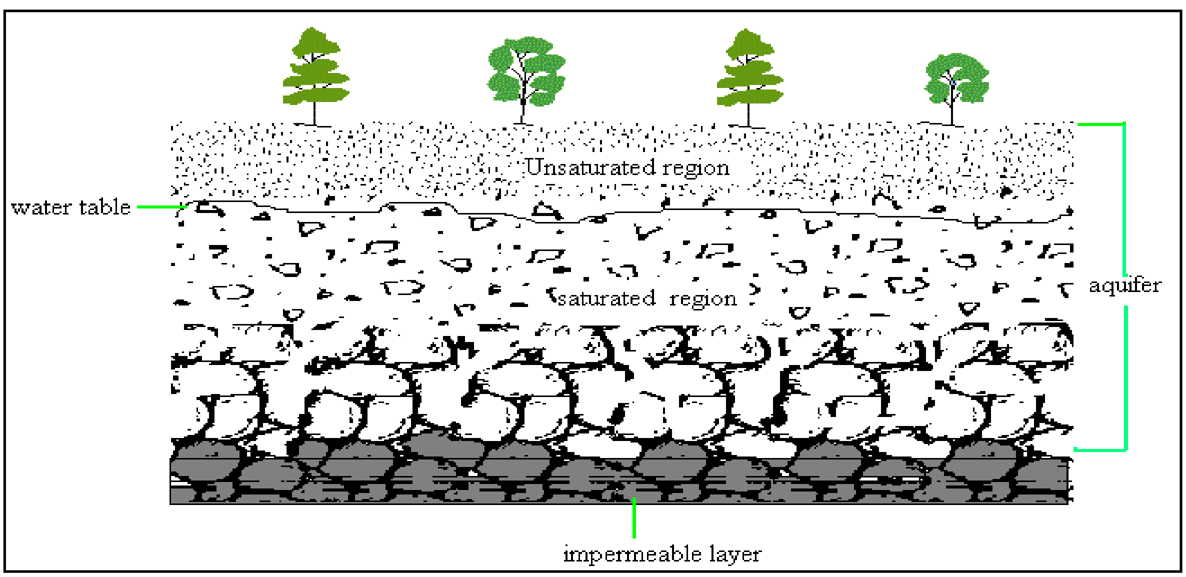


Figure 1: Structure of an Aquifer

Groundwater flows in sediments or rocks through the open spaces, which range from tiny imperfections along crystal boundaries in igneous rocks to huge caves in limestone (Schwartz and Zhang, 2003).

Groundwater is considered potentially good and useful for domestic purposes or used to augment surface water supplies in water schemes, irrigation, industrial and livestock uses (Otutu, 2010). People are drilling wells to obtain water as a solution to the problem of scarcity of surface water.

Drilling groundwater from aquifers results in groundwater deficit that has posed great threat to sustainability of shallow aquifers (Wijnen at el., 2002). Drilling of wells is being done while people have poor knowledge of groundwater flow. The United Republic of Tanzania, (2002) reported that all the on-going groundwater resources development in Tanzania is being carried out without sufficient knowledge of the resource potential.

The study aims to model groundwater flow at a discharging point (well) in two dimensions using the finite element method.

1.1. Governing Equations for Groundwater Flow

The standard equations that govern groundwater flow are derived using the principle of continuity and Darcy's law. Henry Darcy, a civil engineer in the mid 1800s, experimented on groundwater flow and found that for the flow to occur there must be a difference in hydraulic head creating a hydraulic gradient, where the hydraulic gradient is given by;

$$\text{Hydraulic gradient} = -\frac{dh}{dl} = \frac{h_1 - h_2}{\Delta l}$$

where h_1 and h_2 are the hydraulic heads at points 1 and 2 in meters, respectively, and Δl is the distance between the points. Essentially, the hydraulic gradient defines the direction of groundwater flow from a region of higher hydraulic head to a regional of lower hydraulic head (Schwartz and Zhang, 2003).

Darcy (1958) further experimented and showed that the velocity of groundwater flow is directly proportional to the hydraulic gradient

$$v = -K\frac{dh}{dl} \qquad (1.1)$$

where K is the permeability coefficient expressed in velocity units (m/s).

The hydraulic head or pressure head h (also known as piezometric head) is expressed in terms of water pressure by the equation

$$h = \frac{p}{\rho g} + z \qquad (1.2)$$

where p = water pressure, ρ = water density, g = acceleration due to gravity and z = elevation head.

The ground water flow in two dimensions under the steady state condition is governed by

$$K_x\frac{\partial^2 h}{\partial x^2} + K_y\frac{\partial^2 h}{\partial y^2} = 0. \qquad (1.3)$$

1.2. Discharging an Aquifer at a Well

Discharging happens when groundwater leaves an aquifer at discharge points. When water is discharged from a well, the water level in the well falls below the water level in the rock and soil. The water drawn down (discharge) a well in the process of removing water from the confined aquifer in a steady state condition is governed by the Poisson equation

$$K_x\frac{\partial^2 h}{\partial x^2} + K_y\frac{\partial^2 h}{\partial y^2} + Q = 0 \qquad (1.4)$$

where K_x and K_y are the coefficients of permeability (m/day), h is the piezo-metric head measured in meters from the bottom of the unconfined aquifer, and Q is the volume of discharge per unit aquifer area (Willy, 1984).

2. Derivation of the Numerical Model

This work seeks a finite element approximate solution of the differential equation (1.4) under specified boundary conditions. The choice of the finite element method rather than its rival, the finite difference method, is its ability to give a piecewise continuous solution over the domain rather than a discrete solution defined only at the element node points.

2.1. Galerkin's Weak Formulation of the Governing Differential Equation

In solving a boundary value problem governed by a differential equation

$$Lh(x,y) = f(x,y), \quad (x,y) \in \Omega \qquad (2.1)$$

one seeks a function $h(x,y)$ which satisfies equation (3.8) at all points inside the domain Ω, and also satisfies a set of given conditions along the boundary $\partial\Omega$ of the domain. Boundary value problems of this type have the property that the solution $h(x,y)$ being sought is the unique function that minimizes a certain functional. The task of solving the original boundary value problem is thus equivalent to that of finding a function $h(x,y)$ in some function space P which minimizes the particular functional. This latter problem is called the variational or weak formulation of the original boundary value problem. There are a number of approaches of finding the unique minimizing function $h(x, y)$, and this implies that there are several weak forms to a given boundary value problem. In this research, Galerkin weak formulation will be used due to its popularity in connection with the finite element method.

For the standard Galerkin finite element method, the solution $h(x, y)$ of the problem described by equation (1.3) will be sought in a function space P known as the trial space. If h and V are any two functions in P one forms the inner product

$$\iint_{\Omega}\left(K_x \frac{\partial^2 h(x, y)}{\partial x^2} + K_y \frac{\partial^2 h(x, y)}{\partial y^2}\right)V(x, y)dxdy = 0$$

and by applying Green's theorem to the integral (integration by parts) one obtains the corresponding Galerkin weak formulation. Find the unique function $h(x, y) \in P$ such that

$$B(h, V) = 0 \qquad \forall_V \in P$$

where $B(h, V)$ is a bilinear form defined by

$$B(h, V) = \int_{\partial\Omega}\left(K_x \frac{\partial h}{\partial x} + K_y \frac{\partial h}{\partial y}\right)V(x, y)dS - \iint_{\Omega}\left(K_x \frac{\partial h}{\partial x}\frac{\partial V}{\partial x} + K_y \frac{\partial h}{\partial y}\frac{\partial V}{\partial y}\right)dxdy \qquad (2.2)$$

Equation (3.9) can be also recast in the form

$$\iint_{\Omega}\left(K_x \frac{\partial h}{\partial x}\frac{\partial V}{\partial x} + K_y \frac{\partial h}{\partial y}\frac{\partial V}{\partial y}\right)dxdy = \int_{\partial\Omega}\left(K_x \frac{\partial h}{\partial x} + K_y \frac{\partial h}{\partial y}\right)V(x, y)dS \qquad (2.3)$$

Since the integral is a bilinear function, linear in both h and V and symmetrical, we make the substitution $h = \sum_{n=1}^{N} h_n Q_n$ into (3.10).

$$B(h_1\varphi_1(x,y)+h_2\varphi_2(x,y)+h_3\varphi_3(x,y)+...+h_n\varphi_n(x,y)),V) = \int_{\partial\Omega}(K_x\bar{q}_1 + K_y\bar{q}_2)\varphi_n dS \qquad (2.4)$$

where $\bar{q}_1 = \frac{\partial h}{\partial x}$ and $\bar{q}_2 = \frac{\partial h}{\partial y}$.

$$B(\varphi_1, V)h_1 + B(\varphi_2, V)h_2 + B(\varphi_3, V)h_3 + ... + B(\varphi_n, V)h_n = \int_{\partial\Omega}(K_x\bar{q}_1 + K_y\bar{q}_2)\varphi_n dS \qquad (2.5)$$

Once V is chosen then each $B(\varphi_n, V)$ is a constant, and equation (2.5) is a single equation with n unknown $h_1, h_2, h_3, h_4, ..., h_n$. Using Galerkin method we choose V to be $\varphi_1, \varphi_2, \varphi_3, ... \varphi_n$.

Then, equation (2.5) can be written in matrix form

$$\begin{bmatrix} B(\varphi_1,\varphi_1) & B(\varphi_2,\varphi_1) & B(\varphi_3,\varphi_1) & \cdots B(\varphi_n,\varphi_1) \\ B(\varphi_1,\varphi_2) & B(\varphi_2,\varphi_2) & B(\varphi_3,\varphi_2) & \cdots B(\varphi_n,\varphi_2) \\ B(\varphi_1,\varphi_3) & B(\varphi_2,\varphi_3) & B(\varphi_3,\varphi_3) & \cdots B(\varphi_n,\varphi_3) \\ \vdots & \vdots & \vdots & \cdots \quad \vdots \\ B(\varphi_1,\varphi_n) & B(\varphi_2,\varphi_n) & B(\varphi_3,\varphi_n) & \cdots B(\varphi_n,\varphi_n) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} \int_{\partial\Omega}\bar{q}\varphi_1 dS \\ \int_{\partial\Omega}\bar{q}\varphi_2 dS \\ \int_{\partial\Omega}\bar{q}\varphi_3 dS \\ \vdots \\ \int_{\partial\Omega}\bar{q}\varphi_n dS \end{bmatrix} \qquad (2.6)$$

where $\bar{q} = K_x\bar{q}_1 + K_y\bar{q}_2$.

This system of equations represents the finite element mathematical model of the groundwater flow. Moreover, the system matrix equation (2.6) can also be written as

$$M\,\mathbf{h} = \mathbf{f} \qquad (2.7)$$

where M is the global stiffness matrix

h is a column vector whose elements are the nodal solution f is the vector in which the boundary conditions are incorporated.

2.1.1. Finite Element Discretization of the Domain

The second step in implementing the finite element method is subdivision of the domain Ω into smaller sub-regions generally referred to as finite elements. In this study, Ω is the rectangle $R: 0 \le x \le X,\ \ 0 \le y \le Y$, and its sub-regions are right-angled triangles for which the sides enclosing the right angle are parallel to $x-$ and $y-$coordinate axes. One assigns node numbers to the vertices, each element having three nodes. Without going into any details, it suffices to observe that generation of the elements, numbering the elements and their respective nodes is far from trivial.

2.1.2. Selecting the Element Shape Functions

At each vertex node, a linear element shape function is defined. The aim is to obtain the hydraulic head h^e (x, y) at each node. Consider the element shown in Figure 2.

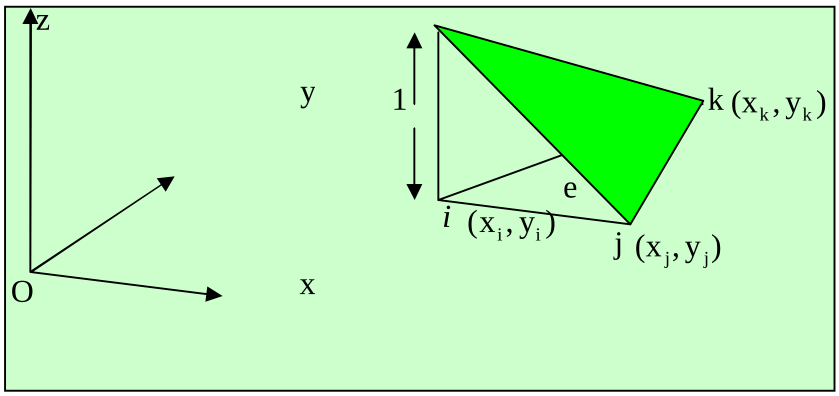


Figure 2: Triangular element

We define the trial solution h^e (x, y) throughout the triangular element e by linear Lagrange interpolation function,

$$h^e(x, y) = \gamma + \alpha x + \beta y \qquad (2.8)$$

where α, β and γ are coefficient to be determined. The equations for the three nodes are

$$h^e(x_i, y_i) = \gamma + \alpha x_i + \beta y_i = h^e_i \qquad (2.9a)$$
$$h^e(x_j, y_j) = \gamma + \alpha x_j + \beta y_j = h^e_j \qquad (2.9b)$$
$$h^e(x_k, yk) = \gamma + \alpha x_k + \beta y_k = h^e_k \qquad (2.9c)$$

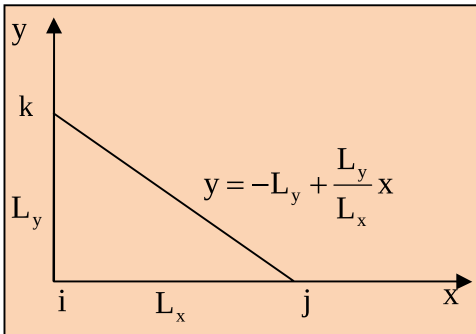
Solving equations (2.9) for the coefficients α, β and γ and substituting them in equation (2.8) we get,

$$h^e(x, y) = V_i^e(x, y)h^e{}_i + V_j^e(x, y)h^e{}_j + V_k^e(x, y)h^e{}_k \qquad (2.10)$$

where $V_i^e(x, y)$, $V_j^e(x, y)$, $V_k^e(x, y)$ are the element shape functions

$$\text{Note that } V_i^e(x, y) = \begin{cases} 1, & \text{at node i} \\ 0, & \text{at node j} \\ 0, & \text{at node k} \end{cases}$$

Element shape functions are calculated according to the dimensions of the domain. If the domain in discharging of the aquifer is discretized into non - isosceles right angles triangles, then the element shape functions are as given in Figure 3



$$V_i^e(x,y) = \frac{L_xL_y - L_yL_x - L_xy}{L_xL_y}$$
$$V_j^e(x,y) = \frac{x}{L_x}$$
$$V_k^e(x,y) = \frac{y}{L_y} \qquad (2.11)$$

Figure 3: Prototype General Right Angled Triangular Element

2.1.3. Determining the Element Stiffness Matrices

The individual element stiffness matrix is found by evaluating the double integrals given in equation (2.3) over each element "e".

$$\iint_e \{K_x(\frac{\partial V_i^e}{\partial x}h_i + \frac{\partial V_j^e}{\partial x}h_j + \frac{\partial V_k^e}{\partial x}h_k)\frac{\partial V_n^e}{\partial x} + K_y(\frac{\partial V_i^e}{\partial y}h_i + \frac{\partial V_j^e}{\partial y}h_j + \frac{\partial V_k^e}{\partial y}h_k)\frac{\partial V_n^e}{\partial y}\}dxdy \qquad (2.12)$$

where n = i, j and k.

If A^e is the area of the element, then

$$\iint_e \left(K_x\frac{\partial h^e}{\partial x}\frac{\partial V_n^e}{\partial x} + K_Y\frac{\partial h^e}{\partial y}\frac{\partial V_n^e}{\partial x}\right)dxdy = A^e(K_x\frac{\partial V_i^e}{\partial x}\frac{\partial V_n^e}{\partial x} + K_y\frac{\partial V_i^e}{\partial y}\frac{\partial V_n^e}{\partial y})h_i +$$

$$A^e(K_x\frac{\partial V_j^e}{\partial x}\frac{\partial V_n^e}{\partial x} + K_y\frac{\partial V_j^e}{\partial y}\frac{\partial V_n^e}{\partial y})h_j + A^e(K_x\frac{\partial V_k^e}{\partial x}\frac{\partial V_n^e}{\partial x} + K_y\frac{\partial V_k^e}{\partial y}\frac{\partial V_n^e}{\partial y})h_k \qquad (2.13)$$

The quantities h_i, h_j and h_k seen in equation (2.9) are the column entries along row n of the element stiffness matrix that is

$$m_{n,i}^e = A^e(K_x\frac{\partial V_i^e}{\partial x}\frac{\partial V_n^e}{\partial x} + K_y\frac{\partial V_i^e}{\partial y}\frac{\partial V_n^e}{\partial y})$$

$$m_{n,j}^e = A^e(K_x\frac{\partial V_j^e}{\partial x}\frac{\partial V_n^e}{\partial x} + K_y\frac{\partial V_j^e}{\partial y}\frac{\partial V_n^e}{\partial y})$$

$$m_{n,k}^e = A^e(K_x\frac{\partial V_k^e}{\partial x}\frac{\partial V_n^e}{\partial x} + K_y\frac{\partial V_k^e}{\partial y}\frac{\partial V_n^e}{\partial y}) \qquad (2.14)$$

The element stiffness matrix for domain in discharging an aquifer is calculated from the element shape functions from equation (2.11) and the expression (2.14) to get

$$M^e = \frac{1}{2}\begin{bmatrix} K_x\left(\frac{L_y}{L_x}\right) + K_y\left(\frac{L_x}{L_y}\right) & -K_x\left(\frac{L_y}{L_x}\right) & -K_y\left(\frac{L_x}{L_y}\right) \\ -K_x\left(\frac{L_y}{L_x}\right) & K_x\left(\frac{L_y}{L_x}\right) & 0 \\ -K_y\left(\frac{L_x}{L_y}\right) & 0 & K_y\left(\frac{L_x}{L_y}\right) \end{bmatrix} \qquad (2.15)$$

2.1.4. Determining (Assembling) the Global Stiffness Matrix

The global stiffness matrix is assembled using a MATLAB code from the element stiffness matrices. Element stiffness matrix contributes three rows and three columns to the global stiffness matrix. Then, the element contribution to the global matrix is summed. The global stiffness matrix for the domain in discharging of an aquifer is

$$M = \frac{1}{2}\begin{array}{c} \\ 1 \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ k \\ \vdots \\ N \end{array}\begin{array}{c} \begin{array}{ccccccccc} 1 & \cdots & i & \cdots & j & \cdots & k & \cdots & N \end{array} \\ \left[\begin{array}{ccccccccc} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & K_x\left(\frac{L_y}{L_x}\right)+K_y\left(\frac{L_x}{L_y}\right) & \cdots & -K_x\left(\frac{L_y}{L_x}\right) & \cdots & -K_y\left(\frac{L_x}{L_y}\right) & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -K_x\left(\frac{L_y}{L_x}\right) & \cdots & K_x\left(\frac{L_y}{L_x}\right) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -K_y\left(\frac{L_x}{L_y}\right) & \cdots & 0 & \cdots & K_y\left(\frac{L_x}{L_y}\right) & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{array}\right] \end{array} \qquad (2.16)$$

$$M = \sum_{e=1}^{e} M^{e}_{n,\,i}$$

For all n and i, note that $M_{n,i} = M_{i,n}$ is a symmetric matrix

2.1.5. Incorporating the Boundary Conditions

The flow boundary conditions are incorporated into the column vector **f** of the system of equation (2.7). For all interior nodes or nodes on a no- flow boundary, f_n = 0. Our domain also included the specified heads from the Dirichlet and Neumann boundary conditions that provide values of the hydraulic head and water flux at the boundaries, respectively.

The boundary nodes on a specified head, f_n is evaluated from the integral given on right hand side of equation (2.3)

$$\int_{\partial\Omega}\left(K_x\frac{\partial h}{\partial x}+K_y\frac{\partial h}{\partial y}\right)V(x,y)dS = \int_{\partial\Omega}\left(K_x\overline{q}_1+K_y\overline{q}_2\right)V(x,y)dS$$

$$= \int_{\partial\Omega}K_x\overline{q}_1V(x,y)dS+\int_{\partial\Omega}K_y\overline{q}_2V(x,y)dS$$

where

$\overline{q}_1$ and $\overline{q}_2$ are piezomericheads or water flux at the boundary.

The components of the vector f are given by

$$f_n^{\ e} = \int_{\partial\Omega}\overline{q}\varphi_n(x,y)dS \quad \text{where} \quad \begin{cases}\overline{q}dS \text{ if } n \text{ is on } \partial\Omega \\ 0 \quad \text{if} \quad \text{otherwise}\end{cases} \qquad (2.17)$$

when n is a boundary node and $\overline{q} = K_x\overline{q}_1 + K_y\overline{q}_2$.

The condensed force vector with the positions of its term in the global force can be written in matrix form as

$$f = \begin{bmatrix} \int_{\partial\Omega}\overline{q}\,\varphi_1 dS \\ \int_{\partial\Omega}\overline{q}\,\varphi_2 dS \\ \vdots \\ \int_{\partial\Omega}\overline{q}\,\varphi_n dS \end{bmatrix} \qquad (2.18)$$

2.1.6. Solving the System of Equations for the Unknown Solution

The system of equation (2.7) is assembled using a MATLAB code from the global stiffness matrices equation 2.17 and condensed force vector equation (2.18). The MATLAB codes also used to find the unknown values of the hydraulic heads in column vector **h** and graphical solution of groundwater flow in the domains.

3. Numerical Simulations of the Model

3.1. Material Properties of an Aquifer
The study considered the case of two-dimensional groundwater flow in an aquifer where the material properties (permeability), hydraulic heads and fluxes of the area of the discharging of an aquifer at a well, were approximated. In the study two cases were considered

i Flow of groundwater in an aquifer without well
ii Flow of Groundwater in Discharging an Aquifer with a well (Pumping)

3.2. Flow of Groundwater in an Aquifer without Well (pumping)
A rectangular plot of land of length 3600m and width 1500m is taken as the domain. The length sides are bounded by hard rocks which are impermeable. The two width sides allow water to infiltrate at the piezometric head (Q) at the rates 200m per day and 150m per day, respectively. Measurements show that the permeability coefficients in both the x and y directions is $30m^3$/day/m.

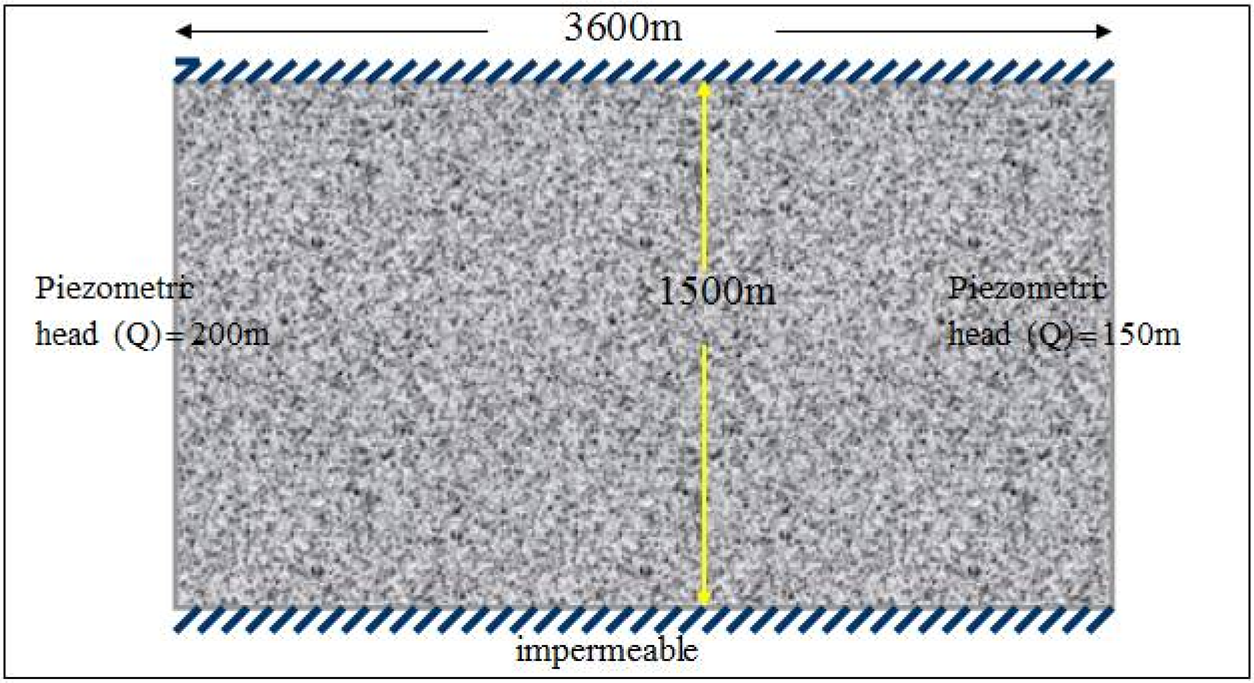


Figure 4: Domain of an aquifer

The researcher investigated the groundwater flow pattern in the domain. With reference to the contents in section 2.0, the conceptual system developed was simulated using a mathematical model that is solved numerically using a MATLAB code. This involves solving Poisson's equation (1.4) and subject to the prescribed boundary conditions using the finite element method. Graphical representation of the solutions at the global nodes obtained using the MATLAB programs, the results are as shown in Figures 5 and 6.

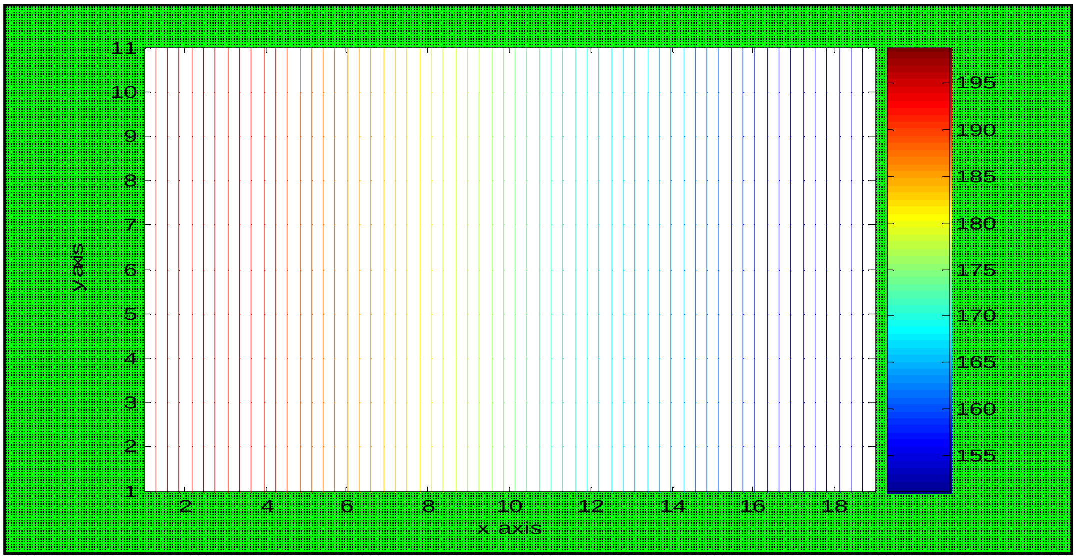


Figure 5: Equipotential lines for the domain

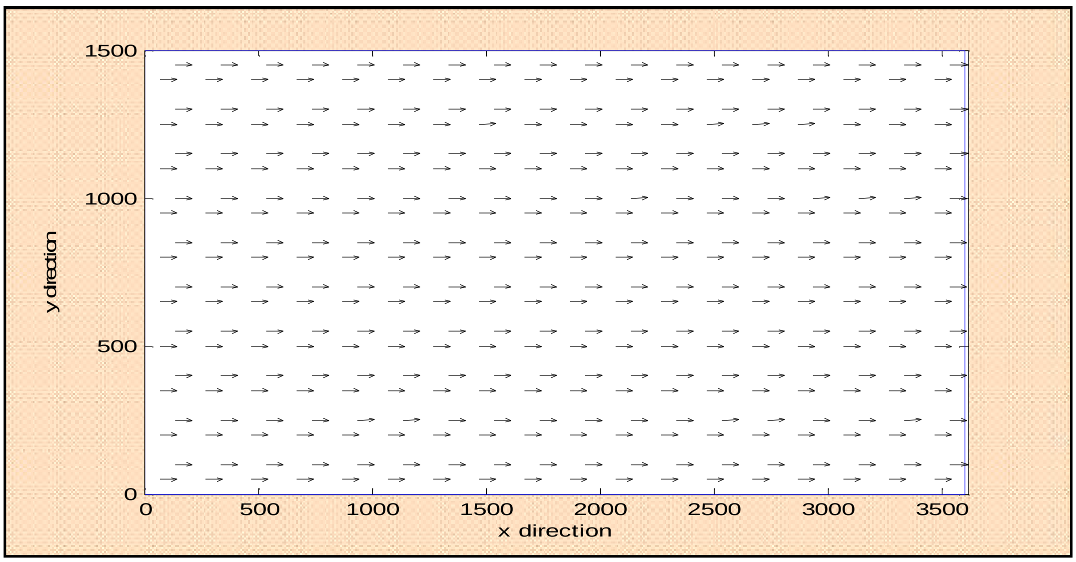


Figure 6: The streamlines flow before wells installations

Figure 5 shows the equipotential lines of flow which are the lines of equal head in the aquifer before well was installed. The equipotential lines are seen to be higher from the left to right of Figure 5. This is due to the higher piezometric head of 200m on the right side compared to the lower piezometric head of 150m measured from the left side of the domain. The Figure 6 represents the streamlines flow that indicate the path followed by the particle of water as it moves through the aquifer in decreasing head.

3.3. Flow of Groundwater in Discharging an Aquifer by Pumping

Consider if a well-constructed at the center of the domain (Figure 7), pumping water at the rate of 1000 m^3/day while the permeability and boundaries as given in Figure 1 remain unchanged.

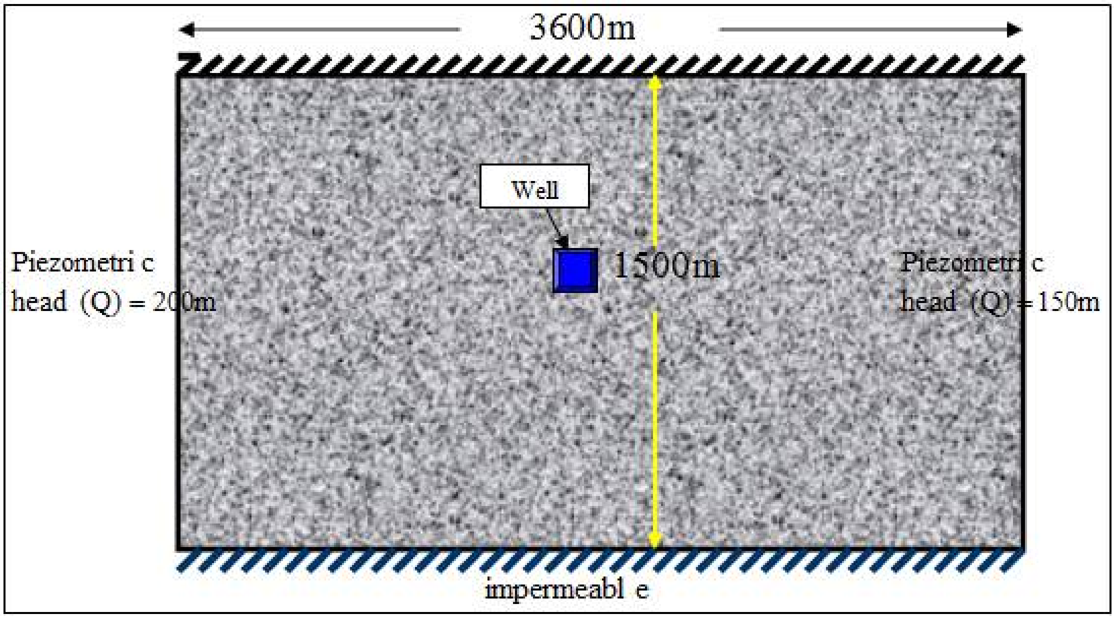


Figure 7: Domain of the aquifer with a well

We can observe the flow pattern of the groundwater from the graphical solution obtained using the programs given in the graphs as shown in Figures 8 and 9

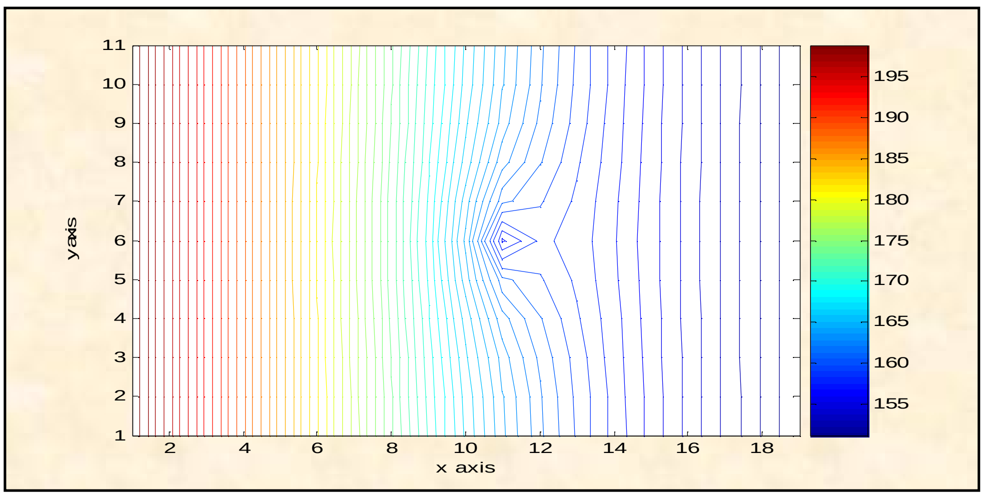


Figure 8: Equipotential lines after well construction

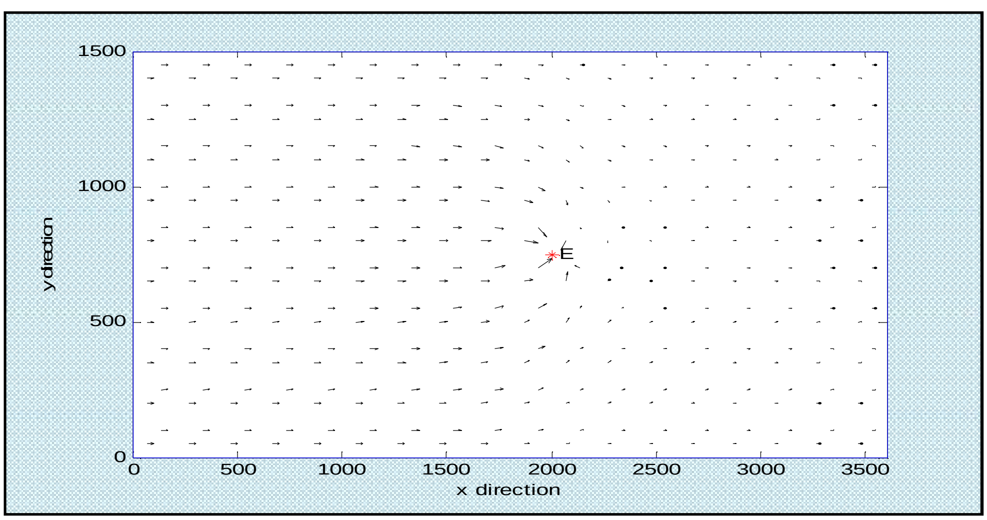


Figure 9: Streamline flow after well construction

Figure 8 shows the equipotential lines; lines joining all points with equal head in the aquifer. We observe that the presence of discharging well pumping water at a rate of 1000 m^3/day at the center of the domain affects the flow of groundwater at nodes in their immediate neighborhood and in the whole domain compared to the situation depicted in Figure 3 in the absence of a discharging well. Figure 9 show that the streamlines flow is towards a well placed at the center of the domain. We observe that the pumping process deflects the direction of groundwater flow in and near a well is due to the lowered hydraulic pressure created by the pumping process.

4. Discussion

The flow of groundwater in discharging an aquifer at a well is governed by Poison's equation with Dirichlet and Neumann Boundary Conditions. A two-dimensional domain of length 3600m and width 1500m was considered. Using the finite element method, the distribution of hydraulic heads in the domain was calculated using self-developed MATLAB codes. It has been shown that if groundwater in aquifer is pumped from a well, the hydraulic pressure around the well is lowered, thus causing the groundwater to flow towards the well. This is supported by the results in Figures 5, 6, 8 and 9.

5. Conclusion

A mathematical model for investigating groundwater flow has been developed and tested; Equations (1.3) and (1.4) are derived from the well-established principles of continuity and Darcy's law and it has been shown that when groundwater is pumped from a well, the hydraulic pressure near the well decreases and causes the groundwater to flow towards the well due to the lowered hydraulic pressure from pumping process.

6. Recommendations

The results obtained in this study constitute an important first step in the study of groundwater flow in Tanzania using. It is an important area of interest because it is concerned with the study of supply of water for domestic industrial and agricultural purposes.

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