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Recursive Least Square Estimation for Economic Growth Prediction

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Abstract:
Recursive Least Square is a popular method used to identify model parameters using a given set of data. This process could be carried out either on-line when each time a new data point becomes available or off-line using past known data. A continuous-time state-space model is z-transformed into a sampled-data transfer function before being converted to an Auto Regression Moving Average model in the backward shift operator z^{-1} with a random stochastic disturbance process included. In this study, Recursive Least Square Estimation is used to identify a plant representation to model the economic growth in the Southeastern United States. A 6th-order algorithm is developed with a parameter estimator and a Kalman filter to predict future economic growth based on some known past growths. Economic growth is estimated from information on new orders, production, employment, supplier delivery time and finished inventory.
To estimate the unknown least square parameters that will minimize the J-loss function, a weighing factor is included in the performance index in order to weight new data more heavily than older data. A positive-definite diagonal matrix is used to measure the estimation error of the parameter estimates.

Keywords: *Recursive least square, auto-regression moving average, parameter estimation, economic growth of southeastern us*

1. Introduction

Most processes are well represented by transfer function models of n^{th} order which may be expressed either in continuous or sampled-data variables. Let's look at a second-order transfer function as a review.

1.1. Continuous Time Model

Many processes are characteristically second order time lag and can be expressed by the following transfer function:

$$G(s) = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1} \qquad (1\text{-}1)$$

where

τ = time constant of the second-order process
$\omega_n = \frac{1}{\tau}$ = natural frequency of the process
ζ = damping ratio
K_p = dynamic gain.

1.2. Discrete Time Model

Although most processes are inherently continuous in nature, the systems used to control them are mostly based on digital computers and apply a sampled-data control algorithm. That is, control is implemented at discrete intervals of time denoted by the sampling period *T*. If the computer control technique is model-based, a discrete time model of the process is required for control output calculation. An equivalent discrete transfer function representation of the first- and second-order continuous time transfer function can be calculated using sampled-data techniques. This method cascades a zero-order hold (*ZOH*) with the continuous time transfer function and samples the process at a set sampling period, T_s. The equivalent sampled-data transfer function is of the form given below:

$$H(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2} \quad . \qquad (1\text{-}2)$$

The discrete transfer functions are often expressed as a ratio of polynomials in the forward shift operator:

$$H(z) = \frac{B(z)}{A(z)} \qquad (1\text{-}3)$$

where

$$A(z) = z^{n_a} + a_1 z^{n_a-1} + a_2 z^{n_a-2} + \cdots + a_{n_a} \qquad (1\text{-}4)$$

$$B(z) = b_0 z^{n_b} + b_1 z^{n_b-1} + b_2 z^{n_b-2} + \cdots + b_{n_b} \, . \qquad (1\text{-}5)$$

1.3. Disturbance Model

The design of an optimal control law strongly depends on the type of disturbance. Control, in fact, is necessary because of disturbances. In stochastic control theory, controllers are derived under the assumption that the disturbances are stochastic in nature. In a process environment, however, the majority of the process upsets are caused by such deterministic disturbances as operator set-point changes, failure of a pump, loss of a coolant, rapid change in environmental conditions, and the like.

There are two common ways to model disturbances, namely step disturbances and exponential disturbances. Of these two, step disturbances is simpler and the disturbance process, $N(t)$,is modeled in terms of the backward shift operator z^{-1} as follows:

$$N(t) = \frac{1}{1-z^{-1}} \zeta(t) \, . \qquad (1\text{-}6)$$

If the random variable $\zeta(t)$ is zero most of the times but takes nonzero values at discrete instances, then random steps occurs at these instances.

1.4. General Linear Difference Equation Models

Linearization of the process model is a generally accepted method in control theory to describe process behavior and thus has been the basis of most adaptive control algorithm.

Consider a continuous time system with input u, output y, and stochastic disturbanceζ, as shown in Figure 1below.

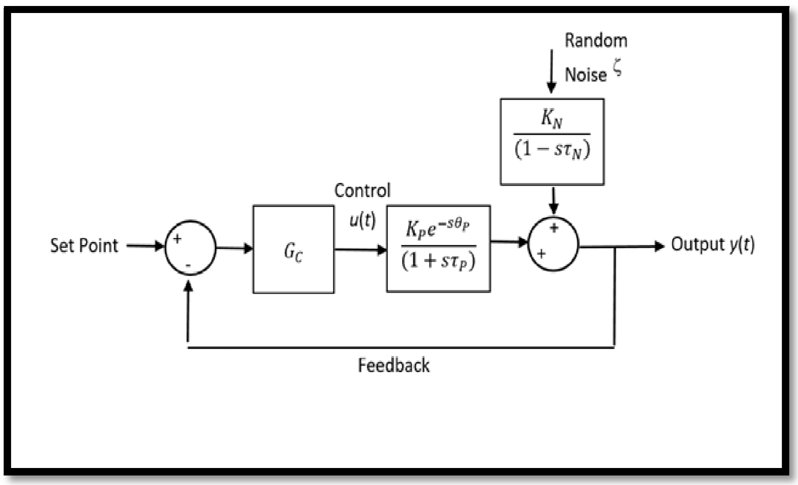

Figure 1: System Representation in Laplace Transformation

The transfer function shown in the block diagram of Figure 1 may represents first- or second-order models. The system representation in discrete time is shown in Figure 2 and is called ARMA (Autoregressive Moving Average) model.

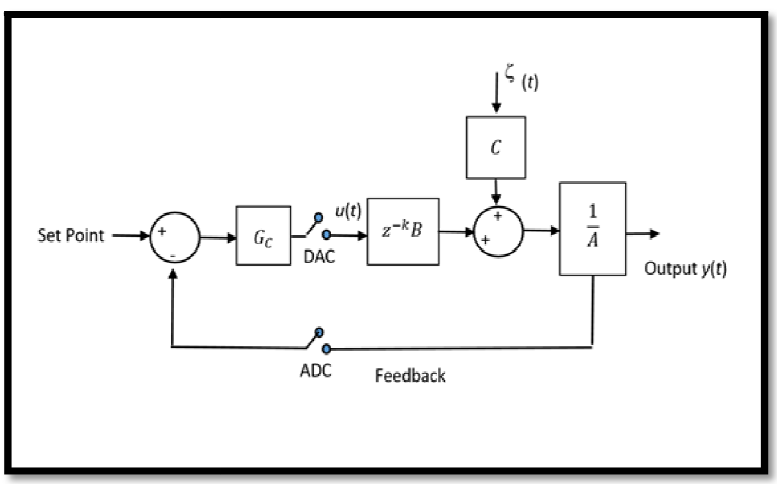

Figure 2: System Representation discrete time

The system transfer function is given by:

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})} z^{-k} u(t) + N(t) \; . \qquad (1\text{-}7)$$

$y(t)$ is the process output at time t; z^{-1} is the backward shift operator, $z^{-1}y(t)$ = y(t-1); and the$A$, B and C polynomials are defined by

$$A(z^{-1}) = 1 + \sum_{i=1}^{n_a} a_i\, z^{-i} \quad (1\text{-}8)$$
$$B(z^{-1}) = \sum_{i=0}^{n_b} b_i\, z^{-i} \quad (1\text{-}9)$$
$$C(z^{-1}) = 1 + \sum_{i=1}^{n_c} c_i\, z^{-i}\ . \quad (1\text{-}10)$$

The noise model in equation (1-6) only represents stationary noise and is given by

$$N(t) = \frac{C(z^{-1})}{A(z^{-1})}\zeta(t) \quad . \quad (1\text{-}11)$$

To design a control law with integral action that is capable of dealing with nonstationary disturbances, a differencing operator Δ is used to deal with nonstationary disturbances. This gives the following ARIMA (Autoregressive Integrating Moving Average) plant representation:

$$A(z^{-1})\Delta y(t) = z^{-k}B(z^{-1})\Delta u(t) + C(z^{-1})\zeta(t)\ . \quad (1\text{-}12)$$

Equation (1-12) can easily be written in the form of a difference equation as follows:

$$\Delta y(t) + a_1\Delta y(t-1) + \cdots + a_{n_a} y(t-n_a) = b_0\Delta u(t-k) + b_1\Delta u(t-k-1) + \cdots$$
$$+ b_{n_b}\Delta u(t-k-n_b) + \zeta(t)$$
$$+ c_1\zeta(t-1) + \cdots + c_{n_c}\zeta(t-n_c)$$
$$(1\text{-}13)$$

In equation (1-13), n_a , n_b and n_c are the degrees of polynomials A, B and C, respectively. Most process dynamics can be represented by first- or second-order model thus giving the said polynomials not more than three terms. Although the form of this model is linear, in adaptive control, the model coefficients are time varying and estimated in real time. Linear discrete time models are preferred in adaptive control because they lead to algorithms that can be easily implemented on a digital computer.

As is known, there is a loss of information when a continuous process is subject to sampling. This loss is insignificant as long as the sampling time Δt is about one tenth of the time constant [illegible]. Choosing a too small Δt can cause the control action to become excessive while if Δt is too large, the controller may response sluggishly to load and set-point changes. Generally, slower sampling will improve the robustness of the controller.

Stock and Watson (1991 and 1993) have shown how to construct a state-space model which can be used to estimate a state variable that captures the co-movements of a set of economic time series. The estimated state variables for the state-space model, when appropriate time series variables are employed, can be thought of as a measure of the unobservable state of the economy. Their study uses information from the purchasing managers survey of manufacturing activities for the state of Georgia to produce an economic indicator which represents the unobservable state of the economy for the entire South-Eastern United States, namely the states of Georgia, Alabama, Florida, North Carolina, South Carolina and Tennessee. The survey of manufacturing activities includes new orders, production, employment, supplier delivery time and finished inventory.

2. Recursive Least Squares Estimation

The purpose of process identification is to identify the model parameters using a given set of data. Although this activity could be carried out off-line, it is often done on-line each time a new data point becomes available. The most popular technique to accomplish this is recursive least squares (Gustavsson et al. 1977; Astrom and Wittenmark 1980; Young 1984).

For this purpose, the model is given by

$$\Delta y(t) = -\widehat{a_1}\Delta y(t-1) - \widehat{a_2}\Delta y(t-2) + \widehat{b_0}\Delta u(t-k) + \widehat{b_1}\Delta u(t-k-1) + \widehat{b_2}\Delta u(t-k-2) + \widehat{b_3}\Delta u(t-k-3) + \zeta(t)$$
$$+ \widehat{c_1}\zeta(t-1)\ .$$

When $\widehat{c_1}$ is assumed to be zero, Eq. (2-1) can be written as (2-1)

$$\Delta y(t) = \varphi^T(t-1)\hat{\theta}(t-1) + \epsilon(t) \quad . \quad (2\text{-}2)$$

$\epsilon(t)$ represents an error that is assumed to be statistically independent of the inputs and outputs. The regression vector [illegible]and the parameter vector $\hat{\theta}$ are defined as

$$\varphi(t-1) = \begin{bmatrix} -\Delta y(t-1) \\ -\Delta y(t-2) \\ \Delta u(t-k) \\ \Delta u(t-k-1) \\ \Delta u(t-k-2) \\ \Delta u(t-k-3) \end{bmatrix} \quad (2\text{-}3)$$

$$\widehat{\theta^T}(t-1) = \left[\widehat{a_1}\ , \widehat{a_2}, \widehat{b_0}\ , \widehat{b_1}\ , \widehat{b_2}\ , \widehat{b_3}\right] \quad (2\text{-}4)$$

in which

$$\Delta y(t-1) = y(t-1) - y(t-2)\,. \qquad (2\text{-}5)$$

The parameter estimation problem is to find estimates $\hat{\theta}$ of the unknown parameter □ that will minimize the loss function:

$$J[\theta] = \sum_{t=1}^{N} \gamma^{N-t}[\varphi^T(t-1)\theta - \Delta y(t)]^2, \quad 0 \le \gamma \le 1. \qquad (2\text{-}6)$$

An exponential forgetting or weighing factor □ is included in the performance index in order to weigh new data more heavily than old data. The smaller the value of □, the heavier weigh is placed on recent measurements than on older ones.

The recursive least squares algorithm can be expressed as:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + K(t)[\Delta y(t) - \Delta\hat{y}(t)] \qquad (2\text{-}7)$$

$$K(t) = \frac{P(t-1)\varphi(t-1)}{\gamma + \varphi^T(t-1)P(t-1)\varphi(t-1)} \qquad (2\text{-}8)$$

$$P(t) = \frac{1}{\gamma}[I - K(t)\varphi^T(t-1)]P(t-1) \qquad (2\text{-}9)$$

where $\Delta\hat{y}(t)$ follows from equation (2-2) for $\epsilon(t) = 0$; $P(t)$ is the covariance matrix of the estimation error of the parameter estimates with maximum dimension $(n_a + n_b) \times (n_a + n_b)$ which is, for this case, $(2 + 4) \times (2 + 4) = 6 \times 6$.

$P(t)$ is a positive definite measure of the estimation error of the parameter estimates and its elements tend to decrease as t increases. Equation (2-7) requires an initial estimate of the parameter vector $\hat{\theta}$, and equations (2-8) and (2-9) require an initial estimate $P(0)$. I is the identity matrix and $K(t)$ the Kalman filter gain, which multiplies the prediction error to yield the correction term for the model parameter vector.

Usually, $P(0)$ is chosen as a diagonal matrix. Large values of the diagonal elements of $P(0)$, e.g. $> 10^4$, indicates that the confidence in $\hat{\theta}(0)$ is poor. Large values of the diagonal elements of $P(0)$ will cause rapid changes in $\hat{\theta}(t)$ initially.

For many real-time applications, the covariance matric P is factored as

$$P = UDU^T \qquad (2\text{-}13)$$

where U is an upper triangular and D is a diagonal matrix.

There are various mechanisms to resetting the covariance. One can observe the estimation error $\Delta y(t) - \Delta\hat{y}(t)$ and reset the covariance only if the absolute value of the estimation error exceeds user-specified limits. The size of the elements of P is indicated by the trace of P.

3. Economic Growth Rates of Southeastern United States

The recursive least square algorithm (Roffel, Vermeer and Chin) given in equations (2-7), (2-8) and (2-9) with supporting equations (2-1) to (2-5) are used to predict the economic growth rates of Alabama and South Carolina from the year 1993 to 2008 using the known growth rates of 1991 and 1992 as initial points.The noise function ζ is set to zero and ϵ is set as a constant random number. The degree of the zero polynomial, n_b is 4 and the degree of the pole polynomial, n_a is 2, hence the dimension of the matrix P is 6×6. The matrix P is chosen as $20{,}000 * I_6$and the weighing factor □ is set at 0.5 .

Table 1 shows the actual economic growth rates of the states of Alabama and South Carolina from the year 1991 to 2008 (Harris, 1991). The predicted growth rates using the above-mentioned algorithm are shown in the adjacent columns.

	Alabama	**Alabama**	**S. Carolina**	**S. Carolina**
Year	Actual	Predicted	Actual	Predicted
1991	2.514	2.514	0.596	0.596
1992	4.233	4.233	2.339	0.339
1993	1.177	2.725	3.125	2.676
1994	3.561	2.182	4.961	5.106
1995	3.127	3.669	3.507	4.735
1996	2.977	2.909	2.691	2.027
1997	3.361	3.154	5.076	3.676
1998	2.877	3.137	3.607	4.040
1999	3.311	2.846	3.464	3.324
2000	0.128	0.137	1.443	0.960
2001	0.889	1.154	1.360	0.676
2002	2.212	3.137	1.443	2.040
2003	2.819	2.846	3.330	3.324
2004	5.040	4.363	0.195	0.960
2005	3.386	4.154	2.406	1.676
2006	2.011	1.637	1.970	2.040
2007	0.891	0.846	0.866	1.324
2008	0.709	0.363	0.591	0.960

Table 1: Actual and Predicted Economic Growth Rate of Alabama and South Carolina

Figures 1 and 2 present the information in Table 1 in graphical form. In these graphs, the dotted lines represent the actual growth rates while the solid lines show the predicted rates.

From the graphs one can see that the growth rates predicted by the recursive least square algorithm are reasonably close to the actual values and, for most years, the upward and downward trends are also correctly forecasted.

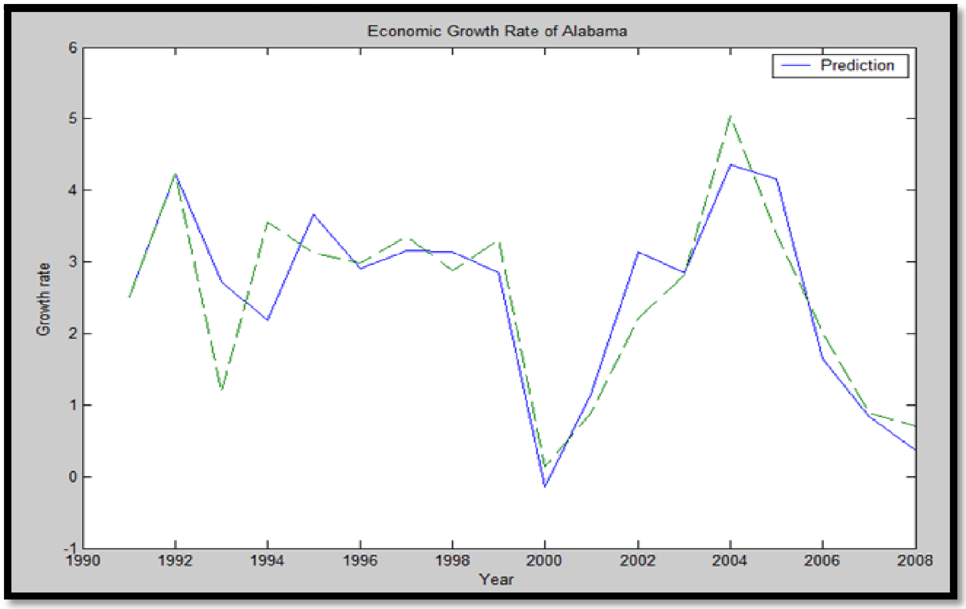

Figure 3: Actual and Predicted Economic Growth Rate of Alabama

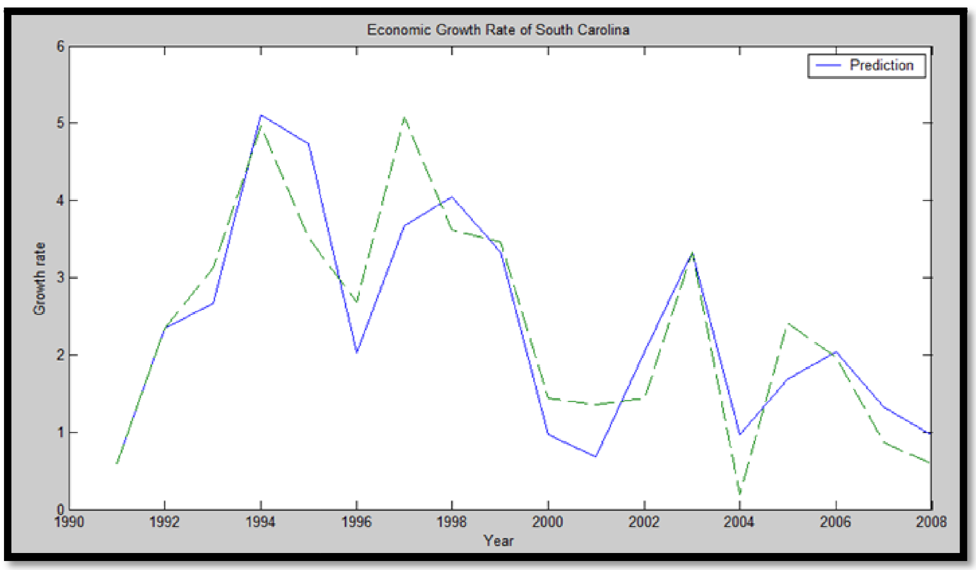

Figure 4: Actual and Predicted Economic Growth Rate of South Carolina

Although the prediction graphs in both cases do follow the general trend and look pretty much in agreement with the actual graphs, however, it must be pointed out that the prediction error is actually quite large. For instance, for Alabama in 1999, the prediction error is

$$p.e. = \frac{3.311-2.846}{3.311} \times 100\% = 14\%.$$

For South Carolina in 2002, the prediction error is

$$p.e. = \frac{1.443-2.040}{1.443} \times 100\% = -41\%.$$

This shows that the algorithm has to be fine-tuned with some pre-set parameter values altered and assumptions reexamined.

5. Conclusion

The recursive least square algorithm used in this paper predicted the economic growths of the states of Alabama and South Carolina from the year 1993 to 2008 to a satisfactory level. The predicted growth rates for both states are reasonably close to the actual values and, for most years, the upward and downward trends are also correctly forecasted.

6. Discussion

As pointed out above, although the prediction graphs in both cases do follow the general trend and look pretty much in agreement with the actual graphs, the percentage prediction error is quite large. When looking at economic growth rate, one can expect the graph to turn sharply either upwards or downwards and this fact usually present a challenge to any prediction algorithm.

For a more accurate prediction, we believe that the algorithm used in this paper has to be fine-tuned with some pre-set parameter values altered and assumptions reexamined and a non-zero noise function ζ added. This should be a challenging but rewarding further research.

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