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Dynamic Response to Moving Concentrated Masses of Highly Prestressed Orthotropic Rectangular Plate Resting on Pasternak Foundation

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Abstract:
In this paper, the response to moving concentrated masses of highly prestressed orthotropic rectangular plate resting on two parameter elastic sub-grades is investigated. In the equation governing the motion of the plate under the action of the travelling masses, it is observed that a small parameter multiplies the highest order operator which is peculiar of singular perturbation problems. From the literatures, problems in which the small parameter multiplies the highest derivatives in the governing differential equation do not have closed-form solution. Problems such as this, defile conventional mathematical methods used to obtain approximate solution, since the small parameter affects the problem in such a way that the solution varies rapidly in some regions of the domain of definition of the problem and slowly in the other parts. It is a well-known fact that when the problem's small parameter multiplies its highest order derivatives in the governing differential equation, it is only amenable to the methods of singular perturbation. Since of all the singular perturbation methods, the method of matched asymptotic expansions is more user-friendly, it is adopted for the solution of this plate problem. Analyses of the solution obtained for this plate dynamical problem are shown in plotted curves. These reveal that increase in the value of prestress or shear modulus leads to increase in the critical speed of the orthotropic rectangular plate traversed by moving concentrated mass. Also, as the rotatory inertia or mass ratio increases, the critical speed decreases. As observed, there is more than one resonance condition in this dynamical system which involves plate flexure under moving concentrated masses, and then, the smaller the mass ratio, the better the improvement in critical speed.

Keywords: *Bending rigidity, in-plane loading, critical speed, resonance, shear modulus, orthotropy*

1. Introduction

Several researchers in engineering, applied physics and applied mathematics have shown keen interest in the analysis of the dynamic response of elastic structures under moving loads. This is due to the great impact the vibration of such structures have on the environment and the riding comfort of the moving load. Foremost amongst researchers of a moving load on a plate is the work of Willis [1] who investigated the effects of weights travelling over bars with different velocities. Others are Stokes [2], Timoshenko [3], Lowan [4], Bondar [5], Reissmann [6, 7] and the monograph of Fryba [8] to mention but few. The operation of these moving loads – cars, heavy-duty vehicles, trucks, railways, steam and gas engines, etc. – introduce additional dynamic stresses on the plate structure. The stresses induced in the plate are dependent not only on the magnitude of the loads, but also strongly upon their speed of propagation. From literatures Fryba [8] and Leissa [9] have examined this phenomenon for simply-supported rectangular plates. Also Holl [10] and Livesley [11] considered the case of an infinite plate resting on elastic foundation and traversed by moving load. In Fryba [8], Leissa [9] and Holl [10], critical speeds of propagation are shown to exist. The study of the behaviour of solid bodies subjected to moving loads has been and still continues to be the concern of several researchers. Among these are the works of Stanisic *et al*. [12, 13, 14], Milormir *et al*. [15], Sadiku and Leipholz [16], Oni [17], Gbadeyan and Oni [18] to mention a few. The aforementioned researchers worked on one-dimensional dynamical beam problems. Among the earliest work on moving load plate problem is the work of Holl [10]. He solved the problem of a rectangular plate under the action of a uniform moving load. He indicated that a critical velocity existed for each vibration mode. Much later Stanisic *et al.* [19, 20] studied the two-dimensional problems of flexural vibration of plates under the action of loads, paying more attention to moving mass. Only the inertial term that measures the effect of local acceleration in the direction of the deflection was considered. Also Aiyesimi [21] studied the dynamic response of an elastic, isotropic rectangular non-Mindlin plate resting on a visco-elastic foundation under the action of a force moving with variable velocity. He solved this problem for simply-supported end conditions only. Ramkumar *et al* [22] treated the vibration of highly prestressed anisotropic plates using numerical-perturbation approach to obtain the displacement response of the circular plate under the action of general in-plane forces. The work in Stanisic *et al*. [14] was taken up much later by Gbadeyan and Oni [23] who studied the dynamic analysis of an elastic plate continuously supported by an elastic Pasternak foundation traversed by an arbitrary number of concentrated masses. The deflection of the plate was calculated for several values of the foundation moduli and shown graphically as a function of time. Worthy of mention in this area of study are the works of Shadnam *et al*. [24, 25]. Oni and Tolorunshagba [26] assessed the rotatory inertia influence on the highly prestressed orthotropic rectangular plate under the action of moving loads. They employed the method of composite expansion (MCE) in conjunction with the method of integral transformation and Cauchy residue theorem to obtain an approximate uniformly valid solution in the entire domain of definition of the rectangular plate. Much recently, Awodola and Omolofe [27], Awodola [28], Awodola and Oni [29], worked on the forced response of non-prestressed plate under moving masses.

In all the aforementioned studies, no consideration has been given to bending effects at the boundaries. In particular, when a plate structure is highly prestressed, a small parameter multiplies the highest derivative in the governing differential equation. Thus, the methods of solution in the works of authors under review breakdown. This is so because while dealing with a highly prestressed rectangular plate of moderate thickness, bending effects must be duly taken into account. In particular, the domain far from the boundaries can generally be regarded as obeying the reduced (order) theory, whereas, close to the boundary bending effects become significant and may even dominate the deformation pattern. Thus, a solution valid in the domain far from the boundary breaks down near and at the boundaries, while the solutions at the boundaries breakdown far away into the domain. This phenomenon is analogous to boundary layer in fluid mechanics, the edge layer in solid mechanics and skin layer in electrodynamics, Erich [30]. The solution in this region is usually termed the inner solution and the solution valid away from this sharp-change region is termed outer solution. The procedure whereby solutions valid in the boundary layers that are identified with the perturbation series solution valid in the so called outer region is often called the matching process.

This class of plate dynamical problems in which a small parameter multiplies the highest derivative in the governing differential equation, is not common in literature when the plate is subjected to a moving load. However, this class of plate problems has been solved when the plate is executing free vibration or when static load is acting on such plate. Even in exceptional cases in which a small parameter multiplies the highest derivative in the governing differential equation of the plate problem, only the gravitational effect of the moving concentrated load is considered, while the inertial effect of the mass of the moving load was seen to be infinitesimally small and so neglected on one hand, and on another hand the material properties of the plate structure were taken to be independent of a direction.

Singular perturbation has, to date, seen relatively little use in solid mechanics but it has, nonetheless, been successfully used by Cole [31, 32]. In particular, Hutter and Olunloyo [33] have employed it in investigating circular membranes with small bending stiffness, while Hutter and Olunloyo [34] treated, among other things, the vibration of a thick strip-like membrane under anisotropic prestress.

Another application in membrane theory is by Schneider [35] who considers the vibration of isotropically prestressed rectangular plates with built-in edges. In his paper, he constructed outer (core) and inner (boundary layer) solutions which are valid in partly disjointed domains. These solutions are then matched in an intermediate domain where both asymptotic expansions are both valid. Much later Olunloyo and Hutter [36] studied the response of thin isotropic, prestressed rectangular plate for the case when the ratio of bending rigidity to the applied in-plane loading is small. He used the method of composite expansion (MCE) to construct solutions for various boundary conditions. Oyediran and Gbadeyan [37] considered the case when the clamped highly prestressed rectangular plate exhibits natural material orthotropy. The problem was solved using the method of matched asymptotic expansions (MMAE). In another paper, Gbadeyan and Oyediran [38] compared the two singular perturbation techniques (MCE and MMAE) for initially stressed thin rectangular plate. They found that the results of the MMAE agree with those obtained using the generalized MCE and specialized version of MCE when the effect of shearing deformation is $O(\varepsilon)$. Another work worthy of mention is the work of Olunloyo and Hutter [39] who investigated the dynamic response of prestressed rectangular membrane to certain external time- dependent forces when the effect of bending rigidity is small using the MCE. After an earlier work by Oni [40] where he studied the dynamic response to a moving load, using MMAE, of a fully clamped prestressed orthotropic rectangular plate, Oni and Tolorunshagba [41] took up the problem of assessing the rotatory inertia influence on the response of the highly prestressed orthotropic rectangular plate to a travelling load using the Method of Composite Expansions (MCE), an alternate singular perturbation technique, in conjunction with the method of integral transformation and Cauchy residue theorem to obtain an approximate uniformly valid analytical solution in the entire domain of definition of the rectangular plate. Analysis showed that the critical velocity of the dynamical system increases with an increase in prestress and rotatory inertia values. Shortly after, Oni and Ogunbamike [42] worked on the transverse vibration of a highly prestressed isotropic rectangular plate resting on a bi-parametric sub-grade and traversed by a moving load. Also, Oni and Adedowole [43] examined the influence of prestress on the response to moving loads of isotropic rectangular plates incorporating rotatory inertia correction factor.

More recently Are *et al* [44] researched on the vibrational response of damped simply-supported orthotropic rectangular plate, resting on Winkler foundation, to moving loads.

It is remarked at this juncture that, to the best of the author's knowledge while the effect of small bending rigidity has been investigated for free and static load plate problems aside the works of Oni [40], Oni and Tolorunshagba [41], calculations for this class of moving load plate problems do not exist in literature. However, in the work of Oni [40], the effect of rotatory inertia correction factor is neglected while in the work of Oni and Tolorunshagba [41] the inertia effect of the mass of the moving load is neglected. Novel as the works of Awodola and Omolofe [27], Oni and Adedowole [43] who worked on prestressed plates are, the plates they considered are not highly prestressed. So, they used the conventional methods of solution to solve the dynamical plate problem in their research work. Thus, in this study, the dynamic analysis of a highly prestressed orthotropic rectangular plate incorporating the effects of rotatory inertia correction factor, under the influence of gravitational and inertial forces (moving mass) is examined.

2. Mathematical Formulation

Consider an orthotropically prestressed rectangular plate resting on a two parameter subgrade and occupying the domain $\Omega[0 \le u \le 1, 0 \le v \le b]$which is held along the four edges and traversed by a moving concentrated load which has mass commensurable with the mass of the plate, when the rotatory inertia correction factor is incorporated. Neglecting the effects of shearing prestress and shear deformation of the governing relation, when properly non-dimensionalized, is of the form

$$\varepsilon^2\left[\frac{\partial^4 W(x,y,t)}{\partial x^4} + 2\alpha_1^2\frac{\partial^4 W(x,y,t)}{\partial x^2\partial y^2} + \alpha_2^2\frac{\partial^4 W(x,y,t)}{\partial y^4}\right] - \beta_1^2\frac{\partial^2 W(x,y,t)}{\partial x^2} - \beta_2^2\frac{\partial^2 W(x,y,t)}{\partial y^2} + \frac{\partial^2 W(x,y,t)}{\partial \bar{t}^2} - \alpha_{0t}\left[\frac{\partial^4 W(x,y,t)}{\partial t^2\partial x^2} + \frac{\partial^4 W(x,y,t)}{\partial t^2\partial y^2}\right] - G\left(\frac{\partial^2 V(x,y,t)}{\partial x^2} + \frac{\partial^2 V(x,y,t)}{\partial y^2}\right) + KV(x,y,t) + \Gamma_0\delta(x-ct)\delta(y-y_0)\left[\frac{\partial^2 W(x,y,t)}{dt^2} + 2c\frac{\partial^2 W(x,y,t)}{\partial t\partial x} + c^2\frac{\partial^2 W(x,y,t)}{\partial x^2}\right] = M_0 g\delta(x-ut)\delta(y-y_0) \quad (1)$$

where it is assumed that the small parameter $0 < \varepsilon \ll 1$ is defined by the relation

$$\varepsilon^2 = \frac{D_{xx}}{N_0 L^2} \quad (2)$$

D_{xx} is the flexural rigidity in x- direction, N_0 is reference prestress, L is the characteristic length which normalizes the spatial coordinates and displacement response, x, y are the spatial coordinates, t is the time coordinate, $\alpha_{0t}$is the rotatory inertia correction factor, μ is the mass of the plate per unit area, $W(x,y,t)$ is the displacement response of the plate, M_0 is the mass of the applied external moving load, G and K are respectively the shear and foundation moduli, $\Gamma_{0=\frac{M_0}{\mu}}$ is the mass ratio, g is the acceleration due to gravity, $\delta(x-a)$ is the unit concentrated force, acting at a point $x = a$, called the Dirac delta function with the property

$$\int_a^b f(x)\delta(x-\zeta) = \begin{cases} 0 & x > b \\ f(\zeta) & a < x < b. \\ 0 & x < a \end{cases} \quad (3)$$

Since the plate is assumed to be fully clamped, the boundary conditions (in non-dimensional form) are

$$\left.\begin{array}{ll} x = 0, & 0 \le y \le b \\ x = 1 & 0 \le y \le b \end{array}\right\} W(x,y,t) = 0, \quad \frac{\partial W(x,y,t)}{\partial x} = 0 \quad (4.1)$$

$$\left.\begin{array}{ll} y = 0, & 0 \le x \le 1 \\ y = b & 0 \le x \le 1 \end{array}\right\} W(x,y,t) = 0, \quad \frac{\partial W(x,v,t)}{\partial y} = 0 \quad (4.2)$$

For simplicity, the plate-structure is assumed to be at rest prior to the arrival of the moving mass, and so the initial conditions (also in non-dimensional form) are

$$W(x,y,0) = 0, \quad \frac{\partial W(x,y,0)}{\partial t} = 0 \quad (5)$$

3. Operational Simplification

It is observed that a small parameter multiplies the highest derivatives in equation (1). For such problems a regular perturbation lowers the order of the differential equation –except in these regions of rapid change (often called boundary layer) where the high value of the derivative cancels the effects of the multiplying small parameter- which, in turn, means the solution cannot satisfy all the boundary conditions. A special treatment is, therefore, needed in the region near as well as at the boundary where its boundary condition is yet to be satisfied. And as such, the problem is only amenable to singular perturbations; in particular the method of matched asymptotic expansions (MMAE) is adopted. However, equation (1) is considerably simplified by introducing the Laplace transformation defined by

$$W(x,y,s) = \int_0^\infty W(x,y,t)e^{-st}dt, s > 0, t \ge 0 \quad (6)$$

With the properties

$$\{W(x,y,t)\} = W(x,y,s) \quad (7.1)$$

$$\{\delta(t-t_0)\} = e^{-st_0} \quad (7.2)$$

Where is the Laplace transform operation symbol. Taking t as the principal variable makes equation (1) to become

$$\varepsilon^2\left[\frac{\partial^4 w(x,y,s)}{\partial x^4} + 2\alpha_1^2\frac{\partial^4 w(x,y,s)}{\partial x^2\partial y^2} + \alpha_2^2\frac{\partial^4 w(x,y,s)}{\partial y^4}\right] - \beta_1^2\frac{\partial^2 w(x,y,s)}{\partial x^2} - \beta_2^2\frac{\partial^2 w(x,y,s)}{\partial y^2} + s^2 W(x,y,s) - \alpha_{0t} s^2\left[\frac{\partial^4 w(x,y,s)}{\partial x^2} + \frac{\partial^4 w(x,y,t)}{\partial y^2}\right] - G\left[\frac{\partial^4 w(x,y,s)}{\partial x^2} + \frac{\partial^4 w(x,y,t)}{\partial y^2}\right] + KW(x,y,s) + \Gamma_0\delta(y-y_0)\{I_a + 2c^* I_b + c^{*2} I_c \quad (8)$$

where

$$I_a = \int_0^\infty \delta(x-ct)\frac{\partial^2 w(x,y,t)}{\partial t^2}e^{-st}dt \quad (9.1)$$

$$I_b = \int_0^\infty \delta(x-ct)\frac{\partial^2 w(x,y,t)}{\partial t\partial x}e^{-st}dt \quad (9.2)$$

$$I_c = \int_0^\infty \delta(x-ct)\frac{\partial^2 w(x,y,t)}{\partial x^2}e^{-st}dt \qquad (9.3)$$

$$I_d = \int_0^\infty \delta(x-ct)\,e^{-st}dt \qquad (9.4)$$

The integrals (9) cannot be easily evaluated and so use is made of trigonometrical series representation of the Dirac delta function obtained from the Fourier series expansion of the function as

$$\delta(x-ct) = 1 + 2\sum_{r=1}^{\infty}[\cos 2\pi rct\cos 2\pi rcx + \sin 2\pi rct\, sin2\pi rcx] \qquad (10)$$

In view of equation (10) the complete Laplace transformation of equation (8) is

$$\varepsilon^2\left[\frac{\partial^4 W(x,y,s)}{\partial x^4} + 2\alpha_1^2\frac{\partial^4 W(x,y,s)}{\partial x^2\partial y^2} + \alpha_2^2\frac{\partial^4 W(x,y,s)}{\partial y^4}\right] - \beta_1^2\frac{\partial^2 W(x,y,s)}{\partial x^2} - \beta_2^2\frac{\partial^2 W(x,y,s)}{\partial y^2} +$$
$$S^2 W(x,y,s) - \alpha_{0t}S^2\left[\frac{\partial^4 W(x,y,s)}{\partial x^2} + \frac{\partial^2 W(x,y,t)}{\partial y^2}\right] - G\left[\frac{\partial^4 W(x,y,s)}{\partial x^2} + \frac{\partial^2 W(x,y,t)}{\partial y^2}\right] + KW(x,y,s) + \qquad (11)$$
$$\Gamma_0\delta(y-y_0)[s^2 W(x,y,s) + 2c^* sW_x(x,y,s) + c^{*2}W_{xx}(x,y,s)] = \frac{M_0}{u}g\delta(y-y_0)e^{-s\frac{x}{C}}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} x=0, & 0\le y\le b \\ x=1 & 0\le y\le b\end{array}\right\} W(x,y,s) = 0 = \frac{\partial w(x,y,s)}{\partial x} \qquad (12.1)$$

$$\left.\begin{array}{ll} y=0, & 0\le x\le 1 \\ y=b & 0\le x\le 1\end{array}\right\} W(x,y,s) = 0 = \frac{\partial w(x,y,s)}{\partial y} \qquad (12.2)$$

together with the initial conditions

$$W(x,y,0) = 0 = \frac{\partial w(x,y,0)}{\partial t} \qquad (13)$$

4. Solution Procedure

In equation (11), an exact uniformly valid solution in the entire domainΩ is not possible since it is observed that a small parameter multiplies the highest derivatives in the governing differential equation in accordance with the informal principle that the behaviour of solution is governed primarily by the highest order terms. This is due to the bending effects at the boundaries. Consequently, solution valid away from the boundaries breaks down near as well as at the boundaries. Thus, only approximate solutions are possible. The two but equivalent approaches, that could be used to tackle this type of problem, are the method of composite expansions (MCE) and the method of matched asymptotic expansions (MMAE). In this work, MMAE is used. This technique provides an approximate solution to the given problem in terms of two separate expansions which are valid in a closed interval $\Omega[0\le x\le 1, 0\le y\le b]$. The two expansions are called inner and outer; neither of them is uniformly valid but their domain of validity together covers the interval Ω. The method of matched asymptotic expansions (MMAE) requires that the asymptotic solution of equation (11) be of the form

$$W(x,y,t) = W_o(x,y,t) + \varepsilon W_1(x,y,t) \qquad (14)$$

Substitution of equation (14) into equation (11) gives, after rearranging and equating coefficients of the powers of ε, the following recurrence relations

$$H^*{}_v(x,y,s) = \begin{cases} \frac{M_o g}{c}\delta(y-y_o)e^{-sx/u} & ,v=0 \\ 0 & ,v=1 \\ D\nabla^4 W_{v-2}(x,y,s) & ,v\ge 2\end{cases} \qquad (15)$$

where

$$H^*{}_v(x,y,s) = -\beta_1^2 \frac{\partial^2 W_v(x,y,s)}{\partial x^2} - \beta_2^2 \frac{\partial^2 W_v(x,y,s)}{\partial y^2} + s^2 W_v(x,y,s) -$$
$$\alpha_{ot} s^2 \left[\frac{\partial^2 W_v(x,y,s)}{\partial x^2} + \frac{\partial^2 W_v(x,y,s)}{\partial y^2}\right] - G\left[\frac{\partial^2 W_v(x,y,s)}{\partial x^2} + \frac{\partial^2 W_v(x,y,s)}{\partial y^2}\right] + \quad (16.1)$$
$$KW_v(x,y,s) + \Gamma_o \partial(y-y_o)\left(s^2 W_v(x,y,s) + 2cs\frac{\partial W_v(x,y,s)}{\partial x} + c^2 \frac{\partial^2 W_v(x,y,s)}{\partial x^2}\right)$$

and

$$D\nabla^4 W_{\nu-2}(x,y,s) = -\left(\frac{\partial^4 W_{\nu-2}(x,y,s)}{\partial x^4} + 2\alpha_1^2 \frac{\partial^4 W_{\nu-2}(x,y,s)}{\partial x^2 \partial y^2} + \alpha_2^2 \frac{\partial^4 W_{\nu-2}(x,y,s)}{\partial y^4}\right) \quad (16.2)$$

Here the subscript of $W(x,y,s)$ denote the order of ε and $\nabla^4 = \nabla^2 \cdot \nabla^2$ (∇^2 is the Laplacian operator), while D is as earlier defined.

Subject to the transformed conditions at the boundaries

$$Wj(x,y,s)\big|_{x=0,1} = 0 \quad j = 0, 1, 2, \ldots \quad (17.1)$$
$$\left.\frac{\partial W_j(x,y,s)}{\partial x}\right|_{x=0,1} = 0 \quad j = 0, 1, 2, \ldots$$
$$Wj(x,y,s)\big|_{y=0,b} = 0 \quad j = 0, 1, 2, \quad (17.2)$$
$$\left.\frac{\partial W_j(x,y,s)}{\partial x}\right|_{y=0,b} = 0 \quad j = 0, 1, 2, \ldots$$

Consider the inner solution, fashioned after (14), of the form

$$W^i(X,y,s) = W_0^i(X,y,s) + \varepsilon W_1^i(X,y,s) \quad (18)$$

valid at x = 0 where the inner variable is set as $X = \frac{x}{\varepsilon}$,
where superscript i refers to inner solution. Equation (18) is also valid near $x = 1$, where the inner variable is set as $X = \frac{1-x}{\varepsilon}$. Expressions similar to (18) can be written down for the solutions near $y = 0$ and $y = b$, where the inner variables are set respectively as $Y = \frac{y}{\varepsilon}$ and $Y = \frac{b-y}{\varepsilon}$, thus

$$W^i(x,Y,s) = W_0^i(x,Y,s) + \varepsilon W_1^i(x,Y,s) \quad (19)$$

Substitution of equation (18) into equation (16.1) near either $x = 0$ or $x = 1$ produces respectively

$$\frac{\partial^4 W_v^i(X,y,s)}{\partial X^4} - \left[\beta_1^2 + \alpha_{ot}s^2 + G_0 - c^2\Gamma_O\,\delta(y-y_o)\right]\frac{\partial^2 W^i_{v(X,y,s)}}{\partial X^2} = \left[\beta_2^2 + \alpha_{ot}s^2 + G_0\right]\frac{\partial^2 W^i_{v-2(X,y,s)}}{\partial y^2} -$$
$$2\alpha_2^2 \frac{\partial^4 W^i_{v-2(X,y,s)}}{\partial X^2 \partial y^2} - \left[K_0 + s^2 + s^2 r_{0\delta(y-y_0)}\right]W^i_{v-2(X,y,s)} - 2cs\Gamma_O\,\delta(y-y_o)\frac{\partial W^i_{v-1}(X,y,s)}{\partial X} + \begin{cases} 0, & v = 0,1,3,4,\ldots \\ M_0 g\delta(y-y_0)e^{\frac{-s}{c}x}, & v = 2 \end{cases} \quad (20)$$

or

$$\frac{\partial^4 W_v^i(X,y,s)}{\partial X^4} - \beta_1^2 \frac{\partial^2 W_v^i(X,y,s)}{\partial X^2} - \alpha_{ot}s^2 \frac{\partial^2 W_v^i(X,y,s)}{\partial X^2} + c^2\Gamma_0\delta(y-y_0)\frac{\partial^2 W_v^i(X,y,s)}{\partial X^2} + G_0 \frac{\partial^2 W_v^i(X,y,s)}{\partial X^2}$$
$$= 2cs\Gamma_0\,\delta(y-y_0)\frac{\partial^2 W^i_{v-1}(X,y,s)}{\partial X^2} - 2\alpha_1^2 \frac{\partial^4 W^i_{v-2}(X,y,s)}{\partial X^2 \partial y^2} + \beta_2^2 \frac{\partial^4 W^i_{v-2}(X,y,s)}{\partial y^2} + s^2 W^i_{v-2}(X,y,s)$$
$$+ \alpha_{0t}s^2 \frac{\partial^2 W^i_{v-2}(X,y,s)}{\partial y^2} - s^2\Gamma_0\,\delta(y-y_0)W^i_{v-2}(X,y,s) + \frac{M_0 g}{u} e^{sx/u}, \quad v = 2. \quad (21)$$

Subject to boundary conditions

$$W_v^i(X,y,s) = 0 = \frac{\partial W_v^i(X,y,s)}{\partial X}, \quad v = 0,1,3,4,\ldots \quad (22)$$

Similarly, near $y = 0$ or $y = b$, one obtains the differential equations

$$\alpha_2^2 \frac{\partial^4 W_{v(x,Y,s)}}{\partial Y^4} - \left(\beta_2^2 + \alpha_{0t}s^2 + G_0\right)\frac{\partial^2 W_{v(x,Y,s)}}{\partial Y^2}$$
$$= \left[\beta_1^2 + \alpha_{0t}s^2 + G_0 - c^2\Gamma_0\delta(y-y_0)\right]\frac{\partial^2 W_{v-2(x,Y,s)}}{\partial x^2} - \frac{\partial^4 W_{v-4}(x,Y,s)}{\partial x^4} - 2\alpha_1^2 \frac{\partial^4 W_{v-2(x,Y,s)}}{\partial x^2 \partial Y^2}$$

$$-2sc\Gamma_0\delta(y-y_0)\frac{\partial W_{v-2(x,Y,s)}}{\partial x}-\begin{bmatrix} s^2+K_0 \\ +s^2\Gamma_0\delta(y-y_0)\end{bmatrix}W_{v-2(x,Y,s)},\quad v=0,1,3,4,\ldots \qquad (23)$$

and

$$\alpha_2^2\frac{\partial^4 W_{v(x,Y,s)}}{\partial Y^4}-(\beta_2^2+\alpha_{0t}s^2+G_0)\frac{\partial^2 W_{v(x,Y,s)}}{\partial Y^2}$$
$$=[\beta_1^2+\alpha_{0t}s^2+G_0-c^2\Gamma_0\delta(y-y_0)]\frac{\partial^2 W_{v-2(x,Y,s)}}{\partial x^2}-\frac{\partial^4 W_{v-4}(x,Y,s)}{\partial x^4}-2\alpha_1^2\frac{\partial^4 W_{v-2(x,Y,s)}}{\partial x^2\partial Y^2}$$
$$-2sc\Gamma_0\delta(y-y_0)\frac{\partial W_{v-2(x,Y,s)}}{\partial x}-\begin{bmatrix} s^2+K_0 \\ +s^2\Gamma_0\delta(y-y_0)\end{bmatrix}W_{v-2(x,Y,s)}+M_0 g\,\delta(y-y_0)e^{\frac{-s}{c}x},\qquad v=2 \qquad (24)$$

Subject to the boundary conditions

$$W_v^i(x,Y,s)=0=\frac{\partial W_v^i(x,Y,s)}{\partial Y},\qquad v=0,1,2,3,\ldots \qquad (25)$$

4.1. Solution Process

The solutions of equations (20) for the function $W_v(x,y,s)$ and equations (20), (21), (23) and (24) for the functions $W_v(X,y,s)$ and $W_v(x,Y,s)$ subject to the respective boundary conditions (22) and (25) are sought using finite Fourier sine integral transformation method.

4.2. Leading Order Solution

Here the solutions of $W_0^o(x,y,s)$ and $W_0^i(x,y,s)$ are sought.

4.3. Solution for $W_0^o(x,y,s)$

Substitute v =0 in the recurrence equation (15), the governing differential equation for $W_0^o(x,y,s)$ is given as

$$\beta_1^2\frac{\partial^2 W_{0(x,y,s)}}{\partial x^2}+\beta_2^2\frac{\partial^2 W_{0(x,y,s)}}{\partial y^2}-s^2V_0+\alpha_{0t}s^2\left[\frac{\partial^2 W_{0(x,y,s)}}{\partial x^2}+\frac{\partial^2 W_{0(x,y,s)}}{\partial y^2}\right]-K_0W_0(x,y,s)+G_0\left[\frac{\partial^2 W_0(x,y,s)}{\partial x^2}+\frac{\partial^2 W_{0(x,y,s)}}{\partial y^2}\right]-$$
$$\Gamma_0\delta(y-y_0)\left[s^2W_{0(x,y,s)}+2sc\frac{\partial W_{0(x,y,s)}}{\partial x}+c^2\frac{\partial^2 W_{0(x,y,s)}}{\partial x^2}\right]=-M_0g\,\delta(y-y_0)e^{\frac{-s}{c}x} \qquad (26)$$

Now, one attempts equation (26) for the solution of $W_0^o(x,y,s)$ by introducing the finite Fourier sine transform defined as

$$W(m,y,s)=\int_0^1 W(x,y,s)\sin m\pi x\;dx \qquad (27)$$

With the inverse

$$W(x,y,s)=2\sum_{m=1}^{\infty}W(m,y,s)\sin m\pi x \qquad (28)$$

and $W(x,n,s)=\int_0^b W(x,y,s)\sin\frac{n\pi y}{b}dy$ (29)

$$W(x,y,s)=\frac{2}{b}\sum_{m=1}^{\infty}W(x,n,s)\sin\frac{n\pi y}{b} \qquad (30)$$

With the inverse

Thus, the transformation of (26) with respect to x is

$$\frac{\partial^2 W_0(m,y,s)}{\partial y^2}+\varphi_1^2W_0^0(m,y,s)=\tau_1\delta(y-y_0) \qquad (31)$$

Where

$$\varphi_1^2=\frac{\beta_1^2m^2\pi^2-s^2+m^2\pi^2\alpha_{0t}s^2-K_0+m^2\pi^2G_0-\Gamma_0\delta(y-y_0)+(s^2+c^2m^2\pi^2)}{[\beta_2^2+\alpha_{0t}s^2+G_0]} \qquad (32.1)$$

$$\tau_1=\frac{M_0gm\pi c^2\left[1-(-1)^m e^{\frac{-s}{c}}\right]}{(\beta_2^2+\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)} \qquad (32.2)$$

The general solution of (26) is

$$W_0^0(m,y,s)=G_1cos\varphi_1y+G_2sin\varphi_1y+\frac{\tau_1}{\varphi_1}sin\varphi_1(y-y_0) \qquad (33)$$

While the transformation with respect to y is

$$\frac{\partial^2 W_0^o(x,n,s)}{\partial x^2}+\varphi_2\frac{\partial W_0^o(x,n,s)}{\partial x}+\varphi_3W_0^o(x,n,s)=\tau_2e^{-sx/u} \qquad (34)$$

where

$$\eta_1=\beta_1^2+\alpha_{0t}s^2+G_0-\frac{\Gamma_0c^2}{b} \qquad (35.1)$$

$$\varphi_2=\frac{2\Gamma_0sc}{b}\Big/\eta_1 \qquad (35.2)$$

$$\varphi_3=\frac{\eta^2\pi^2}{b^2}\beta_2^2-s^2+\frac{\eta^2\pi^2}{b^2}\alpha_{0t}s^2-K_0+\frac{\eta^2\pi^2}{b^2}G_0-\frac{s^2\Gamma_0}{b}\Big/\eta_1 \qquad (35.3)$$

$$\tau_2 = \frac{M_0}{c} g \sin \frac{n\pi y_0}{b} \Big/ \eta_1 \qquad (35.4)$$

The complimentary solution of (34) is

$$W_{0\,c}(x,n,s) = G_3 e^{\gamma_1 x} + G_4 e^{\gamma_2 x} \qquad (36)$$

Where

$$\gamma_1 = \varphi_2 + \sqrt{\varphi_2^2 - 4\varphi_3} \qquad (37.1)$$

$$\gamma_2 = \varphi_2 - \sqrt{\varphi_2^2 - 4\varphi_3} \qquad (37.2)$$

Making use of the method of variation of parameters, the particular solution of (34) can be shown to be

$$\therefore W_{0\,p}(x,n,s) = \frac{\tau_2}{(\gamma_1-\gamma_2)}\left(\frac{c}{c\gamma_1+s}\right)\left[1 - e^{-\left(\frac{c\gamma_1+s}{c}\right)}\right] e^{\gamma_1 x} + \frac{\tau_2}{(\gamma_2-\gamma_1)}\left(\frac{c}{c\gamma_2+s}\right)\left[1 - e^{-\left(\frac{c\gamma_2+s}{c}\right)}\right] e^{\gamma_2 x} \qquad (38)$$

Consequently, the general solution of the ordinary differential equation (34) is

$$W_0^0\,(x,n,s) = G_3 e^{\gamma_1 x} += G_4 e^{\gamma_1 x} + \frac{\tau_2}{(\gamma_1-\gamma_2)}\left(\frac{c}{c\gamma_1+s}\right)\left[1 - e^{-\left(\frac{c\gamma_1+s}{c}\right)}\right] e^{\gamma_1 x} + \frac{\tau_2}{(\gamma_2-\gamma_1)}\left(\frac{c}{c\gamma_2+s}\right)\left[1 - e^{-\left(\frac{c\gamma_2+s}{c}\right)}\right] e^{\gamma_2 x} \qquad (39)$$

The inversion of (33) and (39) gives the general solution of the equation (26) as

$$W_o^o(x,y,s) = 2\left[G_1 cos\varphi_1 y + G_2 sin\varphi_1 y + \frac{\tau_1}{\varphi_1} sin\varphi_1(y-y_0)\right]\sin m\pi x + \frac{2}{b}\left[G_3 e^{\gamma_1 x} += G_4 e^{\gamma_1 x} + \frac{\tau_2}{(\gamma_1-\gamma_2)}\left(\frac{c}{c\gamma_1+s}\right)\left[1 - e^{-\left(\frac{c\gamma_1+s}{c}\right)}\right] e^{\gamma_1 x} + \frac{\tau_2}{(\gamma_2-\gamma_1)}\left(\frac{c}{c\gamma_2+s}\right)\left[1 - e^{-\left(\frac{c\gamma_2+s}{c}\right)}\right] e^{\gamma_2 x}\right]\sin\frac{n\pi y}{b} \qquad (40)$$

Where $G_{1,}G_{2,}G_3$ and G_4 are arbitrary constants that are yet to be determined by matching.

5. Leading Order Solution (Inner Problem)

The differential equation governing the inner solution ($near\ x = 0{,}1,$) in equation (15) where one neglects the terms with negative subscripts, one obtains for the leading order problem

$$\frac{\partial^4 w_0^i(X,y,s)}{\partial X^4} - [\beta_1^2 + \alpha_{0t}s^2 + G_0 - \mathrm{c}^2\Gamma_0\delta(y-y_0)]\frac{\partial^2 w(X,y,s)}{\partial X^2} = 0 \qquad (41)$$

Subject to

$$w_0^i(X,y,s) = 0 = \frac{\partial w_0^i(X,y,s)}{\partial X} \qquad (42)$$

Solving equation (40) subject to equation (41) produces

$$W_0^i(X,y,s) = \begin{cases} \breve{b}_0(y)\left[X + \frac{1}{\theta_1}e^{-\theta_1 X} - \frac{1}{\theta_1}\right], & near\ x = 0 \\ \breve{b}_0(y)\left[X + \frac{1}{\theta_1}e^{-\theta_1 X} - \frac{1}{\theta_1}\right], & near\ x = 1 \end{cases} \qquad (43)$$

where

$$\theta_1^2 = \beta_1^2 + \alpha_{0t}s^2 + G_0 - \mathrm{c}^2\Gamma_0\delta(y-y_0)$$

Similarly, the differential equation governing the inner solution (near $y = 0, b$) in equation (15), when one neglects the terms with negative subscript, one obtains for the leading order problem

$$\frac{\partial^4 W_0^i(x,Y,s)}{\partial Y^4} - \theta_2^2\frac{\partial^2 W(x,Y,s)}{\partial Y^2} = 0 \qquad (44)$$

Subject to

$$W_0^i(x,Y,s) = \frac{\partial W_0^i(x,Y,s)}{\partial Y} \qquad (45)$$

Where

$$\theta_2^2 = \frac{(\beta_1^2 + \alpha_{0t}s^2 + G_0)}{\alpha_2^2} \qquad (46)$$

Solving equation (44) subject to equation (45) produces

$$W_0^i(x,Y,s) = \begin{cases} \check{f}_0(x)\left[Y + \frac{1}{\theta_2}e^{-\theta_2 Y} - \frac{1}{\theta_2}\right], & near\ y = 0 \\ \check{\check{f}}_0(x)\left[Y + \frac{1}{\theta_2}e^{-\theta_2 Y} - \frac{1}{\theta_2}\right], & near\ y = b \end{cases} \qquad (47)$$

Thus, the leading order solution of the inner problem $(20 - 22)$ can be written down as

$$\tilde{b}_0(y)\left[X + \frac{1}{\theta_1}e^{-\theta_1 X} - \frac{1}{\theta_1}\right] \qquad near\ x = 0$$

$$\tilde{\tilde{b}}_0(y)\left[X + \frac{1}{\theta_1}e^{-\theta_1 X} - \frac{1}{\theta_1}\right] \qquad near\ x = 1 \qquad (48)$$

$$\tilde{f}_0(x)\left[Y + \frac{1}{\theta_2}e^{-\theta_2 Y} - \frac{1}{\theta_2}\right] \qquad near\ y = 0$$

$$\tilde{\tilde{f}}_0(x)\left[Y + \frac{1}{\theta_2}e^{-\theta_2 Y} - \frac{1}{\theta_2}\right] \qquad near\ y = b$$

where exponentially growing terms have been discarded as unmatchable. The functions $\tilde{b}_0$, $\tilde{\tilde{b}}_0$, $\tilde{f}_0$ and $\tilde{\tilde{f}}_0$ are yet to be determined. The unknowns in (40) and (48) will be determined by matching inner and outer solutions. To this end, Van Dyke's matching principle, which requires m-term inner expansion of (the n-term outer expansion) equals the n-term outer expansion of (the m-term inner expansion), is adopted. Thus, matching one-term outer expansion written in inner variable (40) with one term inner expansion written in outer variable (48) (1-1 matching) immediately produces

$$\bar{b}(x) = \bar{\bar{b}}(x) = \bar{f}(y) = \bar{\bar{f}}(y) = 0 \tag{49}$$

$$G_1 = \frac{M_0 gm\pi c^2\left[1-(-1)^m e^{\frac{-s}{c}}\right]}{(\beta_2^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)} sin\varphi_1 y_0 \tag{50.1}$$

$$G_2 = \frac{\tau_1}{\varphi_1} sin\varphi_1 y_0 cot\varphi_1 b - \frac{\tau_1}{\varphi_1} cos\varphi_1 y_0 - \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}] sin\varphi_1 y_0 cot\varphi_1 b}{(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)} \tag{50.2}$$

$$G_3 = \frac{\tau_2}{(\gamma_1-\gamma_2)}\left(\frac{c}{c\gamma_1+s}\right)\left[1-e^{-\left(\frac{c\gamma_1+s}{c}\right)}\right] \tag{50.3}$$

$$G_4 = \frac{\tau_2}{(\gamma_1-\gamma_2)}\left(\frac{c}{c\gamma_2+s}\right)\left[1-e^{-\left(\frac{c\gamma_2+s}{c}\right)}\right] \tag{50.4}$$

In view of equations (49) and (50) equation (40) becomes

$$W_0(x,y,s) = \frac{2M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]}{(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)} sin\varphi_1 y_0 cos\varphi_1 y sinm\pi x + 2\left[\frac{\tau_1}{\varphi_1} sin\varphi_1 y_0 \frac{cos\varphi_1 b sin\varphi_1 y}{sin\varphi_1} - \frac{\tau_1}{\varphi_1} cos\varphi_1 y_0 sin\varphi_1 y - \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]}{(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)} sin\varphi_1 y_0 cos\varphi_1 y sin\varphi_1 y\right] sinm\pi x + 2\frac{\tau_1}{\varphi_1} sin\varphi_1(y-y_0) sinm\pi x + \left\{\frac{2}{b}e^{\gamma_2 x}\frac{\tau_2}{(\gamma_2-\gamma_1)}\left(\frac{c}{c\gamma_1+s}\right)\left[1-e^{-\left(\frac{c\gamma_1+s}{c}\right)}\right] + \frac{2}{b}e^{\gamma_1 x}\frac{\tau_2}{(\gamma_1-\gamma_2)}\left(\frac{c}{c\gamma_1+s}\right)\left[1-e^{-\left(\frac{c\gamma_2+s}{c}\right)}\right] + \frac{2}{b}e^{\gamma_2 x}\frac{\tau_2}{(\gamma_2-\gamma_1)}\left(\frac{c}{c\gamma_2+s}\right)\left[1-e^{-\left(\frac{c\gamma_2+s}{c}\right)}\right]\right\} sin\frac{n\pi y}{b} \tag{51}$$

Where

$$\varphi_1^2 = \frac{[\beta_1^2 m^2\pi^2 - s^2 + m^2\pi^2\alpha_{0t}s^2 - K_0 + m^2\pi^2 G_0 - \Gamma_0\delta(y-y_0)(s^2+c^2m^2\pi^2)]}{\beta_2^2+\alpha_{0t}s^2+G_0}, \quad \varphi_2 = \frac{2\Gamma_0 sc}{\beta_2^2+\alpha_{0t}s^2+G_0-\frac{\Gamma_0 c^2}{b}}$$

$$\varphi_3 = \frac{n^2\pi^2\beta_2^2 - b^2s^2 + n^2\pi^2\alpha_{0t}s^2 - b^2K_0 + n^2\pi^2G_0 - bs^2\Gamma_0}{b^2\beta_1^2 + b^2\alpha_{0t}s^2 + b^2G_0 - b\Gamma_0 c^2}, \quad \tau_1 = \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]}{(\beta_1^2+\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)}, \tau_2 = \frac{M_0 g e^{-sx/c} sin\frac{n\pi y_0}{b}}{\beta_1^2+\alpha_{0t}s^2+G_0-\frac{\Gamma_0 c^2}{b}} \tag{52}$$

$$\gamma_1 = \varphi_2 + \sqrt{\varphi_2^2 - 4\varphi_3}, \quad \gamma_2 = \varphi_2 - \sqrt{\varphi_2^2 - 4\varphi_3},$$

$$\beta_1^2 - \alpha_{0t}s^2 + G_0 = -\alpha_{0t}\left[s^2 - \frac{\beta_1^2}{\alpha_{0t}} - \frac{G_0}{\alpha_{0t}}\right] - \alpha_{0t}[s^2 - w_1^2] \tag{53}$$

Further simplification of equation (51) produces

$$W(x,y,s) = 2M_\circ \text{g}m\pi c^2 \sin m\pi x \Big\{\frac{-[1-(-1)^m e^{-s}/_c]}{\alpha_{ot}(s^2+\Lambda_5)(s^2+\Lambda_2)} \sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y_o \text{Cos}\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y + \frac{[1-(-1)^m e^{-s}/_c]\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] b \sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y}{\Lambda_3(s^2+\Lambda_4)(s^2+\Lambda_2)\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] b}$$
$$\frac{-[1-(-1)^m e^{-s}/_c]}{\alpha_{ot}(s^2+\Lambda_5)(s^2+\Lambda_2)}\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y_o \text{Cos}\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y + \frac{[1-(-1)^m e^{-s}/_c]}{2\Lambda_3(s^2+\Lambda_4)(s^2+\Lambda_2)}\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y_\circ -$$
$$\frac{[1-(-1)^m e^{-s}/_c]}{\Lambda_3(s^2+\Lambda_4)(s^2+\Lambda_2)}\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y_\circ \Big\} + \frac{2M_\circ \text{g}m\pi c^2 \sin\frac{n\pi y}{b}}{b\alpha_{ot}}\Big\{\left(\frac{c}{cr_1+s}\right)\frac{[1-e^{-(cr_1+s/c]}\, e^{r_1 x}}{r_1-r_2}\left[\frac{1}{s^2+\Lambda_6} - \frac{1}{s^2+\Lambda_1}\right]\Big\} \tag{54}$$

The Laplace inversion of (54) is defined as

$$W_0^o(x,y,t) = P_{b_1}\{F_1(x,y,t) + F_2(x,y,t) - F_3(x,y,t) + F_4(x,y,t) - F_5(x,y,t)\} + P_{b_2}\{F_6(x,y,t) - F_7(x,y,t)\} \tag{55}$$

where

$$P_{b_1} = 2M_\circ \text{g}m\pi c^2 \sin m\pi x \tag{56.1}$$

$$P_{b_2} = \frac{2M_\circ \text{g}m\pi c^2}{b\alpha_{ot}}\sin\frac{n\pi y}{b} \tag{56.2}$$

$$F_1(x,y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{[1-(-1)^m e^{-s}/_c]}{\alpha_{ot}(s^2+\Lambda_5)(s^2+\Lambda_2)}\text{Sin}\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y_o \text{Cos}\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \, ds \tag{56.3}$$

$$F_2(x,y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{[1-(-1)^m e^{-s}/_c]}{\Lambda_3(s^2+\Lambda_4)(s^2+\Lambda_2)}\frac{\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] b \sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y}{\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] b} ds \tag{56.4}$$

$$F_3(x,y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{[1-(-1)^m e^{-s}/_c]}{\alpha_{ot}(s^2+\Lambda_5)(s^2+\Lambda_2)}\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \text{ Cos}\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \, ds \tag{56.5}$$

$$F_4(x,y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{[1-(-1)^m e^{-s}/_c]}{\Lambda_3(s^2+\Lambda_4)(s^2+\Lambda_2)}\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \text{ Cos}\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y_o \, ds \tag{56.6}$$

$$F_5(x,y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{[1-(-1)^m e^{-s}/_c]}{\Lambda_3(s^2+\Lambda_4)(s^2+\Lambda_2)}\text{Cos}\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y \sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right] y_o \, ds \tag{56.7}$$

$$F_6(x,y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{c\,[1-e^{-(cr_1+s/c]}\, e^{r_1 x}}{(cr_1+s)(r_1-r_2)(s^2+\Lambda_6)}\, ds \tag{56.8}$$

$$F_7(x,y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{c\,[1-e^{-(cr_1+s/c]}\, e^{r_1 x}}{(cr_1+s)(r_1-r_2)(s^2+\Lambda_1)}\, ds \tag{56.9}$$

In order to evaluate the integrals in (56), the Cauchy residue theorem is employed. The singularities in the integrals are poles. In particular the denominators of the integrands of $F_1(x,y,t), F_2(x,y,t), F_3(x,y,t), F_4(x,y,t)$, and $F_5(x,y,t)$ have simple poles at $S = \pm i\Lambda_5, S = \pm i\Lambda_2, S = \pm i\Lambda_4, S = \pm i\Lambda_{21}$ It is straightforward to show that

$$F_1(x,y,t) = \frac{\mathrm{Sin}\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_5^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_5^2\right)}\right] y_o \,\mathrm{Cos}[\frac{\Lambda_3\left(\Lambda_4-\Lambda_5^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_5^2\right)}]\,\mathrm{y}}{2\mathrm{i}\Lambda_5\left(\Lambda_2-\Lambda_5^2\right)\alpha_{ot}} \{\, e^{i\Lambda_5 t}[1-(-1)^m e^{-i\Lambda_5}/_c] - e^{-i\Lambda_5 t}[1-(-1)^m e^{i\Lambda_5}/_c]\,\} + \frac{\mathrm{Sin}\,[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}]\, y_o \,\mathrm{Cos}[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}]\,\mathrm{y}}{2\mathrm{i}\Lambda_2\left(\Lambda_5-\Lambda_2^2\right)\alpha_{ot}} \{\, e^{i\Lambda_2 t}\left[1-(-1)^m e^{-i\Lambda_2}/_c\right] - e^{-i\Lambda_2 t}\left[1-(-1)^m e^{i\Lambda_2}/_c\right]\} \quad (57.1)$$

$$F_2(x,y,t) = \frac{\mathrm{Cos}[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}]\,\mathrm{b}\ \mathrm{Sin}\,[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}]\,\mathrm{y}}{2\,\Lambda_3\Lambda_4\left(\Lambda_2-\Lambda_4^2\right)(\Lambda_{21}^2+\Lambda_4^2)} \{\, ie^{i\Lambda_4 t}\left[1-(-1)^m e^{-i\Lambda_4}/_c\right] - ie^{\Lambda_4 t}\left[1-(-1)^m e^{i\Lambda_4}/_c\right]\} + \frac{\mathrm{Cos}\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] b\ \mathrm{Sin}\,[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}]\, y}{2\,\Lambda_2\Lambda_3\left(\Lambda_2-\Lambda_2^2\right)(\Lambda_{21}^2-\Lambda_2^2)} \{\, ie^{-i\Lambda_2 t}\left[1-(-1)^m e^{i\Lambda_2}/_c\right] - ie^{i\Lambda_2 t}\left[1-(-1)^m e^{-i\Lambda_2}/_c\right]\} + \frac{\mathrm{Cos}[\frac{\Lambda_3\left(\Lambda_{21}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{21}^2+\Lambda_1\right)}]\,\mathrm{b}\ \mathrm{Sin}\left[\frac{\Lambda_3\left(\Lambda_{21}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{21}^2+\Lambda_1\right)}\right] y}{2\,\Lambda_{21}\Lambda_3\left(\Lambda_{21}^2+\Lambda_4\right)\left(\Lambda_{21}^2+\Lambda_2\right)} \{\, e^{\Lambda_{21} t}\left[1-(-1)^m e^{-\Lambda_{21}}/_c\right] + \{\, e^{-\Lambda_{21} t}\left[1-(-1)^m e^{\Lambda_{21}}/_c\right]\} . \quad (57.2)$$

$$F_3(x,y,t) = \frac{\mathrm{Sin}\,[\frac{\Lambda_3\left(\Lambda_4-\Lambda_5^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_5^2\right)}]\ y_o\,\mathrm{Cos}\,[\frac{\Lambda_3\left(\Lambda_4-\Lambda_5^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_5^2\right)}]\,\mathrm{y}\ \mathrm{Sin}\,[\frac{\Lambda_3\left(\Lambda_4-\Lambda_5^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}]\,\mathrm{y}}{2i\alpha_{ot}\Lambda_5\ (\,\Lambda_2-\Lambda_5^2)} \{e^{i\Lambda_5 t}\left[1-(-1)^m e^{-i\Lambda_5}/_c\right] e^{-i\Lambda_5 t}\left[1-(-1)^m e^{i\Lambda_5}/_c\right]\} + \sin\,[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}]\ y_o \cos\,[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}]\,\mathrm{y}\ \mathrm{Sin}\,[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}]\,\mathrm{y}\,\{\, e^{i\Lambda_2 t}\left[1-(-1)^m e^{-i\Lambda_2}/_c\right] - e^{-i\Lambda_2 t}\left[1-(-1)^m e^{i\Lambda_2}/_c\right]\} \quad (57.3)$$

$$F_4(x,y,t) = \frac{\sin\,[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}]\, y \cos\,[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}]\, y_o}{2\Lambda_3\Lambda_4(\Lambda_4^2-\Lambda_4)} \{ie^{i\Lambda_4 t}\left[1-(-1)^m e^{-i\Lambda_4}/_c\right]\} - \{ie^{i\Lambda_2 t}\left[1-(-1)^m e^{-i\Lambda_2}/_c\right] - \{ie^{-i\Lambda_2 t}\left[1-(-1)^m e^{i\Lambda_2}/_c\right]\} \quad (57.4)$$

$$F_5(x,y,t) = \cos\,[\frac{\Lambda_3(\Lambda_4-\Lambda_4^2)}{\alpha_{ot}(\Lambda_1-\Lambda_4^2)}]\,\mathrm{y}\ \sin\ [\frac{\Lambda_3(\Lambda_4-\Lambda_4^2)}{\alpha_{ot}(\Lambda_1-\Lambda_4^2)}] y_o\,\{ie^{i\Lambda_4 t}\left[1-(-1)^m e^{-i\Lambda_4}/_c\right] - \{ie^{-i\Lambda_4 t}\left[1-(-1)^m e^{i\Lambda_4}/_c\right] + \cos\,[\frac{\Lambda_3(\Lambda_4-\Lambda_4^2)}{\alpha_{ot}(\Lambda_1-\Lambda_4^2)}]\,\mathrm{y}\ \sin\,[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}] y_o\,. \{ie^{i\Lambda_2 t}\left[1-(-1)^m e^{-i\Lambda_2}/_c\right] - ie^{-i\Lambda_2 t}\left[1-(-1)^m e^{i\Lambda_2}/_c\right]\} \quad (57.5)$$

$$f_6(x,y,x) = \frac{e^{\Lambda_{11}t}[1-e^{-\alpha_1}]e^{k_1 x}\sqrt{(\Lambda_{11}^2+\Lambda_1)(\Lambda_{11}^2+\Lambda_9)}}{(\Lambda_{11}-\Lambda_{17})(\Lambda_{11}-\Lambda_{18})(\Lambda_{11}-\Lambda_{19})(\Lambda_{11}-\Lambda_{20})(\Lambda_{11}^2-\Lambda_6)\sqrt{(\Lambda_{11}-\Lambda_{12})(\Lambda_{11}-\Lambda_{13})(\Lambda_{11}-\Lambda_{14})}} . \frac{e^{\Lambda_{12}t}[1-e^{-\alpha_2}]e^{k_2 x}\sqrt{(\Lambda_{12}^2+\Lambda_1)(\Lambda_{12}^2+\Lambda_9)}}{(\Lambda_{12}-\Lambda_{17})(\Lambda_{12}-\Lambda_{18})(\Lambda_{12}-\Lambda_{19})(\Lambda_{12}-\Lambda_{20})(\Lambda_{12}^2-\Lambda_6)\sqrt{(\Lambda_{12}-\Lambda_{11})(\Lambda_{12}-\Lambda_{13})(\Lambda_{12}-\Lambda_{14})}} + \frac{e^{\Lambda_{13}t}[1-e^{-\alpha_3}]e^{k_3 x}\sqrt{(\Lambda_{13}^2+\Lambda_1)(\Lambda_{13}^2+\Lambda_9)}}{(\Lambda_{13}-\Lambda_{17})(\Lambda_{13}-\Lambda_{18})(\Lambda_{13}-\Lambda_{19})(\Lambda_{13}-\Lambda_{20})(\Lambda_{13}^2-\Lambda_6)\sqrt{(\Lambda_{13}-\Lambda_{11})(\Lambda_{13}-\Lambda_{13})(\Lambda_{13}-\Lambda_{14})}} + \frac{e^{\Lambda_{14}t}[1-e^{-\alpha_4}]e^{k_4 x}\sqrt{(\Lambda_{14}^2+\Lambda_1)(\Lambda_{14}^2+\Lambda_9)}}{(\Lambda_{14}-\Lambda_{17})(\Lambda_{14}-\Lambda_{18})(\Lambda_{14}-\Lambda_{19})(\Lambda_{14}-\Lambda_{20})(\Lambda_{14}^2-\Lambda_6)\sqrt{(\Lambda_{14}-\Lambda_{11})(\Lambda_{14}-\Lambda_{13})(\Lambda_{14}-\Lambda_{13})}} \quad (57.6)$$

$$f_7(x,y,t) = \frac{e^{\Lambda_{11}t}[1-e^{-\alpha_1}]e^{k_1 x}}{2\Lambda_{15}(\Lambda_{11}-\Lambda_{17})(\Lambda_{11}-\Lambda_{18})(\Lambda_{11}-\Lambda_{19})(\Lambda_{11}-\Lambda_{20})(\Lambda_{11}^2-\Lambda_1)\sqrt{(\Lambda_{11}-\Lambda_{12})(\Lambda_{11}-\Lambda_{13})(\Lambda_{11}-\Lambda_{14})}} + \frac{e^{\Lambda_{12}t}[1-e^{-\alpha_2}]e^{k_2 x}}{2\Lambda_{15}(\Lambda_{12}-\Lambda_{17})(\Lambda_{12}-\Lambda_{18})(\Lambda_{12}-\Lambda_{19})(\Lambda_{12}-\Lambda_{20})(\Lambda_{12}^2-\Lambda_1)\sqrt{(\Lambda_{12}-\Lambda_{11})(\Lambda_{12}-\Lambda_{13})(\Lambda_{12}-\Lambda_{14})}} + \frac{e^{\Lambda_{13}t}[1-e^{-\alpha_3}]e^{k_3 x}}{2\Lambda_{15}(\Lambda_{13}-\Lambda_{17})(\Lambda_{13}-\Lambda_{18})(\Lambda_{13}-\Lambda_{19})(\Lambda_{13}-\Lambda_{20})(\Lambda_{13}^2-\Lambda_1)\sqrt{(\Lambda_{13}-\Lambda_{11})(\Lambda_{13}-\Lambda_{12})(\Lambda_{13}-\Lambda_{14})}} + \frac{e^{\Lambda_{14}t}[1-e^{-\alpha_4}]e^{k_4 x}}{2\Lambda_{15}(\Lambda_{14}-\Lambda_{17})(\Lambda_{14}-\Lambda_{18})(\Lambda_{14}-\Lambda_{19})(\Lambda_{14}-\Lambda_{20})(\Lambda_{14}^2-\Lambda_1)\sqrt{(\Lambda_{14}-\Lambda_{11})(\Lambda_{14}-\Lambda_{12})(\Lambda_{14}-\Lambda_{13})}} \quad (57.7)$$

The combination of the results (57.1 – 57.7) substituted into (55) yields the desired leading order solution of (26) which represents the uniformly valid solution of the entire domain of definition of the given plate.

5.1. First Order Correction

5.1.1. Solution for $W_1^o(x,y,t)$

The next corrections in outer solution are obtained by setting $\nu = 1$ in equation (15). The governing equation for $W_1^o(x,y,t)$ is given as

$$-\gamma_1^2 \frac{\partial^2 W_1}{\partial x^2}(x,y,s) - \gamma_2^2 \frac{\partial^2 W_1}{\partial y^2}(x,y,s) + s^2 W_1(x,y,s) - \alpha_{0t} s^2 \left[\frac{\partial^2 N_1}{\partial x^2}(x,y,s) + \frac{\partial^2 W}{\partial y^2}(x,y,s)\right] +$$
$$\Gamma_0 \delta \left[s^2 W_1(x,y,s) + 2us \frac{\partial W_1}{\partial x}(x,y,s) + u^2 \frac{\partial^2 W_1}{\partial x^2}(x,y,s)\right] = 0 \quad (58)$$

Now one attempts the solution of $W_1(x,y,s)$ by introducing the finite Fourier sine transform in (27) in equation (58) with respect to x, and after simplification and slight re-arrangement, produces

$$W_{1,yy}(m,y,s) + \eta^2 W_1(m,y,s) = 0 \quad (59)$$

where

$$\eta^2 = \left[\frac{[s^2 + m^2\pi^2\alpha_{0t}s^2 - m^2\pi^2\gamma_1^2 + \Gamma_0\delta(y-y_0)(s^2 - (m\pi u)^2)]}{\alpha_{0t}s^2 - \gamma_2^2}\right] \quad (60)$$

The homogeneous solution of (59) gives

$$W_1(m,y,s) = B_1 \cos \eta y - B_2 \sin \eta y \quad (61)$$

Similarly, if equation (58) is subjected to finite Fourier sine transform (29) with respect to y, and after simplification slight arrangement, one obtains

$$\frac{\partial^2 W_{1(x,n,s)}}{\partial x^2} + \eta_2^2 W_1(x,n,s) = 0 \quad (62)$$

Where

$$\eta_2^2 = \frac{\left(\frac{n\pi\gamma_2}{b}\right) + s^2 - \alpha_{0t}\left(\frac{n\pi}{b}\right)^2 s^2}{-\gamma_1^2 - \alpha_{0t}s^2 + \frac{u^2\Gamma_0}{b}} \quad (63)$$

The solution of which is

$$W_1(x,n,s) = B_3 \cos \eta_2 x - B_4 \sin \eta_2 x \quad (64)$$

The finite Fourier sine inversion of equation (61) together with equation (64) gives

$$W_1(x,y,s) = 2\sum_{m=1}^{\infty}[B_1 \cos \eta y - B_2 sin\eta y]\, sin m\pi x + \frac{2}{b}\sum_{n=1}^{\infty}[B_3 \cos \eta_2 x - B_4 \sin \eta_2 x] \sin \frac{n\pi y}{b} \quad (65)$$

where $B_1,\ B_2,\ B_3\ and\ B_4$ are unknown constants to be determined by matching.

6. First Order Correction (Inner Problem)

The first order correction is obtained by setting $\nu = 1$ in the differential equations (20) and (23). Doing this and neglecting terms with negative subscripts yields

$$\frac{\partial^4 W_1^i}{\partial x^4}(x,y,s) - [\gamma_1^2 + \alpha_{0t}s^2 - u^2\Gamma_0\delta(y-y_0)]\frac{\partial^2 W_1^i}{\partial x^2}(x,y,s) = 0$$

Rewritten as

$$\frac{\partial^4 W^i_{1(x,y,s)}}{\partial x^4} - \varphi_1^2 \frac{\partial^4 W^i_{1(x,y,s)}}{\partial x^4} = 0 \quad (66)$$

Where

$$\varphi_1^2 = [\gamma_1^2 + \alpha_{0t}s^2 - u^2\Gamma_0\delta(y-y_0)] \quad (67)$$

Subject to the boundary conditions

$$W_1^i(x,y,s) = 0 = \frac{\partial W^i_{1(x,y,s)}}{\partial x} \quad (68)$$

Following usual argument in equations (41) and (44), the first order correction of the inner problem can be written as

$$W_1^i(x,y,s) = \begin{cases} \hat{b}_1(y)\left[x + \frac{1}{\varphi_1}e^{-\varphi_1 x} - \frac{1}{\varphi_1}\right] & \text{near } x = 0 \\ \hat{\hat{b}}_1(y)\left[x + \frac{1}{\varphi_1}e^{-\varphi_1 x} - \frac{1}{\varphi_1}\right] & \text{near } x = 1 \\ \hat{f}_1(x)\left[y + \frac{1}{\varphi_2}e^{-\varphi_2 y} - \frac{1}{\varphi_2}\right] & \text{near } y = 0 \\ \hat{\hat{f}}_1(x)\left[y + \frac{1}{\varphi_2}e^{-\varphi_2 y} - \frac{1}{\varphi_2}\right] & \text{near } y = b \end{cases} \quad \mathbf{(69)}$$

Here exponentially growing terms have been neglected as unmatchable. The function $\hat{b}_1(y),\ \hat{\hat{b}}_1(y),\ \hat{f}_1(x)\ and\ \hat{\hat{f}}_1(x)$ will be determined by matching. By matching one term outer solution with two terms inner solution expansion written in outer variable, produces the following.

$$\hat{b}_1(y) = 2m\pi\left\{\frac{M_0 g m\pi c^2[1-(-1)^m e^{-s/c}]sin\varphi_1 y_0}{(\beta_1^2 - \alpha_{0t}s^2 + G_0)(s^2 + c^2m^2\pi^2)}[cos\varphi_1 y - cot\varphi_1 b\ sin\varphi_1 y] + \frac{\tau_1}{\varphi_1}sin\varphi_1 y_0[cot\varphi_1 b\ sin\varphi_1 y - cos\varphi_1 y]\right\} \quad (70)$$

$$\hat{\hat{b}}_1(y) = -2(-1)^m\, m\pi\left\{\frac{M_0 g m\pi c^2[1-(-1)^m e^{-s/c}]}{((\beta_1^2 - \alpha_{0t}s^2 + G_0)(s^2 + c^2m^2\pi^2)}sin\varphi_1 y_0(cos\varphi_1 y - cot\varphi_1 b\ sin\varphi_1 y) + \frac{\tau_1}{\varphi_1}sin\varphi_1 y_0(cot\varphi_1 b\ sin\varphi_1 y - cos\varphi_1 y) - -2\frac{\tau_1}{\varphi_1}sin\varphi_1 y cos\varphi_1 y_0\right\} \quad (71)$$

$$\hat{f}_1(x) = 2\left\{\tau_1 - \frac{M_0 g m\pi c^2[1-(-1)^m e^{-s/c}]\varphi_1}{(\beta_1^2 - \alpha_{0t}s^2 + G_0)(s^2 + c^2m^2\pi^2)}\right\}sin\varphi_1 y_0 cot\varphi_1 b sin m\pi x \quad (72)$$

$$\hat{\hat{f}}_1(x) = 2\left\{\varphi_1 \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]}{(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)} sin\varphi_1 y_0 sin\varphi_1 b\ sinm\pi x - \tau_1 cos\varphi_1 y_0 cos\varphi_1 b\right\} \quad (73)$$

One seeks an asymptotic outer solution of the form

$$W_1^o(x,y,s) = 2[B_1 \cos\eta y - B_2 sin\eta y]sinm\pi x + \frac{2}{b}[B_3 \cos\eta_2 x - B_4 \sin\eta_2 x]\sin\frac{n\pi y}{b} \quad (74)$$

By matching two terms outer solution with two terms inner solution (2-2 matching) of equation (68) as $\varepsilon \to 0$, one obtains

$$B_3 = b\left[2m\pi\left(\frac{T_0}{\eta}\sin y_0\right)\cos\eta y - 2m\pi\left(\frac{T_0}{\eta}\csc\eta b[\sin(b-y_0) - \sin y_0\cos\eta b]\right)\sin\eta y + \frac{T_0}{\eta}m\pi\sin(y-y_0)\right] \quad (75)$$

$$B_4 = 2\left(\frac{T_0}{\eta}sin(y_0)\right)(-1)^{m+1}(m\pi) + 2\left(\frac{T_0}{\eta}\csc\eta b[sin(b-y_0) - siny_0 cos\eta b]\right)sin\eta y(-1)^{m+1}(m\pi) + \frac{T_0}{\eta}sin(y - y_0)(-1)^{m+1} + \frac{2}{b}B_1 e^{\theta_1}\theta_1\sin\frac{n\pi y}{b} + \frac{2}{b}B_2 e^{\theta_2}\theta_2\sin\frac{n\pi y}{b} - \frac{2}{b}\eta_4\frac{\left[e^{-\left(\frac{s}{u}+\theta_1\right)}-1\right]}{(\theta_1-\theta_2)\left(\frac{s}{u}+\theta_1\right)}e^{\theta_1}\sin\frac{n\pi y}{b}\theta_1 - \frac{2}{b}\eta_4\frac{\left[e^{-\left(\frac{s}{u}+\theta_2\right)}-1\right]}{(\theta_2-\theta_1)\left(\frac{s}{u}+\theta_2\right)}e^{\theta_2}\theta_2\sin\frac{n\pi y}{b} \quad (76)$$

$$B_1 = \left[\left\{\tau_1 - \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]\varphi_1}{(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)}\right\}sin\varphi_1 y_0 cot\varphi_1 b_0\right]/\theta_2 \quad (77)$$

$$B_2 = \frac{\varphi_1 M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]\varphi_1 sin\varphi_1 y_0}{\theta_2(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)} - \frac{\tau_1 cos\varphi_1 y_0 cos\varphi_1 b}{\theta_2 sin\varphi_1 b sinm\pi x} - \frac{\tau_1 cos\varphi_1 b}{\theta_2 sin\varphi_1 b} + \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]\theta_1 cos^2\varphi_1 b sin\varphi_1 y_0}{\theta_2(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)sin\varphi_1 b} \quad (78)$$

In view of equations (75) – (78) equation (74) becomes

$$V_1^0(x,y,s) = \left\{2\left[\left\{\tau_1 - \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]\varphi_1}{\theta_2(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)}\right\}sin\varphi_1 y_0 cot\varphi_1 b\right]/\theta_2 * cos\varphi_1 y + 2\left\{\frac{\varphi_1 M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]\varphi_1 sin\varphi_1 y_0}{\theta_2(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)} - \frac{\tau_1 cos\varphi_1 y_0 cos\varphi_1 b}{\theta_2 sin\varphi_1 b sinm\pi x} - \frac{\tau_1 cos\varphi_1 b}{\theta_2 sin\varphi_1 b} + \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]\theta_1 cos^2\varphi_1 b sin\varphi_1 y_0}{\theta_2(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)sin\varphi_1 b}\right\}sin\varphi_1 y\right\}sinm\pi x + \left[\left\{\frac{2}{b}\frac{(-1)^m m\pi}{\theta_1(e^{\gamma_2}-e^{\gamma_1})sin\frac{n\pi y}{b}}\left[\frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]}{(\beta_1^2-\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)}\right]sin\varphi_1 y_0(cos\varphi_1 y - cot\varphi_1 b\ sin\varphi_1 y) + \frac{\tau_1}{\varphi_1}sin\varphi_1 y_0(cot\varphi_1 b\ sin\varphi_1 y + cos\varphi_1 y) - \frac{2\tau_1}{\varphi_1}cos\varphi_1 y_0 sin\varphi_1 y\right\}e^{\gamma_2}\right]sin\frac{n\pi y}{b} \quad (79)$$

Where $\tau_1 = \frac{M_0 gm\pi c^2[1-(-1)^m e^{-s/c}]}{(\beta_2^2+\alpha_{0t}s^2+G_0)(s^2+c^2m^2\pi^2)}$, $\varphi_1 = \sqrt{\frac{[\beta_1^2 m^2\pi^2 - s^2 + c^2m^2\pi^2\alpha_{0t}s^2 - k_0 + m^2\pi^2 G_0 - \Gamma_0\delta(y-y_0)(s^2+c^2m^2\pi^2)]}{\beta_2^2+\alpha_{0t}s^2+G_0}}$,

$\Lambda_1 = \frac{\beta_2^2+G_0}{\alpha_{0t}}$, $\Lambda_2 = c^2m^2\pi^2$, $\Lambda_3 = c^2m^2\pi^2 - 1 - \Gamma_0\delta(y-y_0)$, $\Lambda_4 = \frac{m^2\pi^2[\beta_1^2+G_0-c^2\Gamma_0\delta(y-y_0)]-k_0}{m^2\pi^2\alpha_{0t}-1-\Gamma_0\delta(y-y_0)}$ $\Lambda_5 = \frac{(\beta_1^2+G_0)}{\alpha_{0t}}$, $\Lambda_6 = \frac{bG_0+b\beta_1^2-\Gamma_0 c^2}{b\alpha_{0t}}$, $\Lambda_7 = \frac{n^2\pi^2\alpha_{0t}-b\Gamma_0-b^2}{b^2}$, $\Lambda_8 = \frac{n^2\pi^2\beta_2^2+n^2\pi^2G_0-b^2k_0}{n^2\pi^2\alpha_{0t}-b\Gamma_0-b^2}$, $\Lambda_9 = \frac{bG_0+b\beta_1^2-\Gamma_0c^2}{b\alpha_{0t}}$, $\Lambda_{10} = \frac{m^2\pi^2\left(\beta_1^2+\alpha_{0t}+G_0-c^2\Gamma_0\delta(y-y_0)\right)-k_0}{1-\Gamma_0\delta(y-y_0)}$,

$\Lambda_{15} = n^2\pi^2\alpha_{0t} - b\Gamma_0 - b^2$, $\varphi_2 = \frac{2\Gamma_0 sc}{b\left(\beta_1^2+\alpha_{0t}s^2+G_0-\frac{\Gamma_0c^2}{b}\right)}$, $\tau_2 = \frac{M_0 ge^{-sx/c}sin\frac{n\pi y_0}{b}}{\alpha_{0t}(s^2+\Lambda_6)}$,

Further simplification, rearrangement of equation (79) gives

$$\begin{aligned}V_1^0(x,y,s) = \Bigg\{&\frac{2}{\theta_2}\left[\frac{M_0gm\pi c^2[1-(-1)^m e^{-s/c}]}{\alpha_{0t}(s^2+\Lambda_1)} + \frac{M_0gm\pi c^2[1-(-1)^m e^{-s/c}]}{\alpha_{0t}(s^2-\Lambda_{10})(s^2+\Lambda_2)}\right.\\
&\left.\cdot\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]sin\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}y_0\ cot\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{0t}(s^2+\Lambda_1)}b\ cos\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}y\\
&+2\left[\frac{\Lambda_3(s^2+\Lambda_4)M_0gm\pi c^2[1-(-1)^m e^{-s/c}]}{\alpha_{0t}(s^2+\Lambda_1)\theta_2\alpha_{0t}(s^2-\Lambda_{10})(s^2+\Lambda_2)}sin\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}y_0\right.\\
&-\frac{M_0gm\pi c^2[1-(-1)^m e^{-s/c}]}{\theta_2\alpha_{0t}(s^2+\Lambda_1)}cos\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}y_0 cos\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}b_1\frac{1}{sinm\pi x}\\
&-\frac{M_0gm\pi c^2[1-(-1)^m e^{-s/c}]sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]b\ cos\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]b}{\theta_2\alpha_{0t}(s^2+\Lambda_1)sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]b}\\
&\left.+\frac{M_0gm\pi c^2[1-(-1)^m e^{-s/c}]cos^2\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]b\ sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y_0}{\theta_2\alpha_{0t}(s^2-\Lambda_{10})(s^2+\Lambda_2)sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]b}\right]sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y\Bigg\}sinm\pi x +\end{aligned}$$

$$\left\{\left[\frac{2}{b}\frac{(-1)^m m\pi e^{\gamma_1 x}}{\theta_1(e^{\gamma_1}-e^{\gamma_2})\sin\frac{n\pi y}{b}}\left[\frac{M_0gm\pi c^2[1-(-1)^m e^{-s/c}]\ sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y_0}{-\alpha_{0t}(s^2-\Lambda_{10})(s^2+\Lambda_2)}\ cos\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y - sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y\ cot\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{0t}(s^2+\Lambda_1)}\right]b + \frac{M_0gm\pi c^2[1-(-1)^m e^{-s/c}]\alpha_{0t}(s^2+\Lambda_1)\ sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y_0}{\alpha_{0t}(s^2+\Lambda_1)\Lambda_3(s^2+\Lambda_4)}\left(sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y\ cot\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{0t}(s^2+\Lambda_1)}\right]b + cos\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y\right) - 2\frac{M_0gm\pi c^2[1-(-1)^m e^{-s/c}]\alpha_{0t}(s^2+\Lambda_1)\ cos\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y_0\ sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y}{\alpha_{0t}(s^2+\Lambda_1)\Lambda_3(s^2+\Lambda_4)}\right] +\right.\right.$$

$$\left[\frac{2}{b}\frac{(-1)^m m\pi e^{\gamma_1 x}}{\theta_1(e^{\gamma_1}-e^{\gamma_2})\sin\frac{n\pi y}{b}}\right]\frac{M_0 g m\pi c^2[1-(-1)^m e^{-s/c}]\ \sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y_0}{-\alpha_{0t}(s^2-\Lambda_{10})\Lambda_3(s^2+\Lambda_2)}\left(\cos\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y-\sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y\cot\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{0t}(s^2+\Lambda_1)}\right]b\right)+$$

$$\frac{M_0 g m\pi c^2[1-(-1)^m e^{-s/c}](s^2+\Lambda_1)\sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y_0}{\Lambda_3(s^2+\Lambda_1)(s^2+\Lambda_4)}\left(\sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y\cot\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{0t}(s^2+\Lambda_1)}\right]b+\cos\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y\right)-$$

$$-2\frac{M_0 g m\pi c^2[1-(-1)^m e^{-s/c}]}{\Lambda_3(s^2+\Lambda_4)(s^2+\Lambda_2)}\cos\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y_0\sin\left[\frac{\Lambda_3[s^2+\Lambda_4]}{\alpha_{0t}(s^2+\Lambda_1)}\right]y\Bigg]\Bigg\}\sin\frac{n\pi y}{b} \tag{80}$$

The Laplace inversion of (80) is given by

$$V_1(x,y,t)=\Big\{\frac{m_0 g m\pi c^2}{\alpha_{ot}}[E_1(x,y,t)+E_2(x,y,t)+E_3(x,y,t)-E_4(x,y,t)-E_5(x,y,t)+E_6(x,y,t)]\Big\}\sin m\pi x+\left[\frac{2(-1)^m m\pi}{b\theta_1\sin\frac{n\pi y}{b}}\Big\{-\frac{M_0 g m\pi c^2}{\alpha_{ot}}E_7(x,y,t)-E_8(x,y,t)+M_0 g m\pi c^2[E_9(x,y,t)+E_{10}(x,y,t)]-2M_0 g m\pi c^2 E_{11}(x,y,t)\Big\}-\frac{2(-1)^m m\pi}{b\theta_1\sin\frac{n\pi y}{b}}\Big\{-\frac{M_0 g m\pi c^2}{\alpha_{ot}}E_{12}(x,y,t)-E_{13}(x,y,t)\Big\}+M_0 g m\pi c^2\{E_{14}(x,y,t)+E_{15}(x,y,t)\}-\frac{2M_0 g m\pi c^2}{\Lambda_3}E_{16}(x,y,t)\right]\sin\frac{n\pi y}{b} \tag{81}$$

Where

$$E_1(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}2\alpha_2 e^{st}\frac{\left[1-(-1)^m e^{\frac{-s}{c}}\right]}{\sqrt{\alpha_{ot}}(s^2+\Lambda_1)^{\frac{3}{2}}\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b}\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_{ds} \tag{82.1}$$

$$E_2(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{2\alpha_2 e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\Lambda_3(s^2+\Lambda_4)}{\sqrt{\alpha_{ot}}(s^2-\Lambda_{10})(s^2+\Lambda_2)(s^2+\Lambda_1)^{\frac{3}{2}}\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b}\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_{ds} \tag{82.2}$$

$$E_3(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{st}\frac{2\Lambda_3(s^2+\Lambda_4)\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_0\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]}{(s^2+\Lambda_1)(s^2-\Lambda_{10})(s^2+\Lambda_2)} \tag{82.3}$$

$$E_4(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}2e^{st}\frac{\left[1-(-1)^m e^{\frac{-s}{c}}\right]\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_0\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_{ds}}{(s^2+\Lambda_1)\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b\sin m\pi x} \tag{82.4}$$

$$E_5(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_{ds}}{(s^2+\Lambda_1)\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b} \tag{82.5}$$

$$E_6(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{st}\frac{\left[1-(-1)^m e^{\frac{-s}{c}}\right]\theta_1\cos^2\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_0\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_{ds}}{(s^2-\Lambda_{10})(s^2+\Lambda_2)\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b} \tag{82.6}$$

$$E_7(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}e^{\gamma_1 x}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_0\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_{ds}}{(e^{\gamma_1}-e^{\gamma_2})(s^2\pm)(s^2+\Lambda_2)} \tag{82.7}$$

$$E_8(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}e^{\gamma_1 x}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_0\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]ds}{(e^{\gamma_1}-e^{\gamma_2})(s^2-\Lambda_{10})(s^2+\Lambda_2)\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b} \tag{82.8}$$

$$E_9(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]ds}{\Lambda_3(s^2+\Lambda_4)} \quad (82.9)$$

(82.10)

$$E_{10}(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\,ds}{\Lambda_3(s^2+\Lambda_4)}$$

$$E_{11}(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\,ds}{\Lambda_3(s^2+\Lambda_4)} \quad (82.11)$$

$$E_{12}(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}e^{\gamma_2 x}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\,ds}{(e^{\gamma_2}-e^{\gamma_1})(s^2-\Lambda_{10})(s^2+\Lambda_2)} \quad (82.12)$$

$$E_{13}(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}e^{\gamma_2 x}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]ds}{(e^{\gamma_2}-e^{\gamma_1})(s^2-\Lambda_{10})(s^2+\Lambda_2)\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b} \quad (82.13)$$

$$E_{14}(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]ds}{\Lambda_3(s^2+\Lambda_4)\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]b} \quad (82.14)$$

$$E_{15}(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]ds}{\Lambda_3(s^2+\Lambda_4)} \quad (82.15)$$

$$E_{16}(x,y,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y\,ds}{(s^2+\Lambda_4)(s^2+\Lambda_2)} \quad (82.16)$$

Now employing the Cauchy's residue theorem, one proceeds to the evaluation of integrals (82.1) – (82.16) to obtain the sum of the residues of $E_1(x,y,t)-E_{16}(x,y,t)$as

$$E_1(x,y,t)=2\alpha_2 e^{i\Lambda_1 t}\frac{\left[1-(-1)^m e^{\frac{-i\Lambda}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_1^2)}{\alpha_{ot}(\Lambda_1-\Lambda_1^2)}\right]y_o\cos\left[\frac{\Lambda_3(\Lambda_4-\Lambda_1^2)}{\alpha_{ot}(\Lambda_1-\Lambda_1^2)}\right]b\cos\left[\frac{\Lambda_3(\Lambda_4-\Lambda_1^2)}{\alpha_{ot}(\Lambda_1-\Lambda_1^2)}\right]y}{\sqrt{(2i\Lambda_1)^3}\,(i\Lambda_1-\Lambda_{22})(i\Lambda_1+\Lambda_{22})\sqrt{\alpha_{ot}}}+$$

$$\frac{e^{-i\Lambda_1 t}\left[1-(-1)^m e^{\frac{-i\Lambda}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_1^2+\Lambda_4)}{\alpha_{ot}(\Lambda_1^2-\Lambda_1)}\right]y_o\cos\left[\frac{\Lambda_3(\Lambda_1^2+\Lambda_4)}{\alpha_{ot}(\Lambda_1^2-\Lambda_1)}\right]b\cos\left[\frac{\Lambda_3(\Lambda_1^2+\Lambda_4)}{\alpha_{ot}(\Lambda_1^2-\Lambda_1)}\right]y}{\sqrt{(2i\Lambda_1)^3}\,(\Lambda_{22}+i\Lambda_1)(\Lambda_{22}-i\Lambda_1)}+$$

$$\frac{e^{\Lambda_{22} t}\left[1-(-1)^m e^{\frac{-\Lambda_{22}}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_{22}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{22}^2+\Lambda_1)}\right]y_o\cos\left[\frac{\Lambda_3(\Lambda_{22}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{22}^2+\Lambda_1)}\right]b\cos\left[\frac{\Lambda_3(\Lambda_{22}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{22}^2+\Lambda_1)}\right]y}{2\Lambda_{22}\sqrt{\left(\Lambda_{22}^2+\Lambda_1\right)^3}}+-$$

$$\frac{e^{-\Lambda_{22} t}\left[1-(-1)^m e^{\frac{\Lambda_{22}}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_{22}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{22}^2+\Lambda_1)}\right]y_o\cos\left[\frac{\Lambda_3(\Lambda_{22}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{22}^2+\Lambda_1)}\right]b\cos\left[\frac{\Lambda_3(\Lambda_{22}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{22}^2+\Lambda_1)}\right]y}{2\Lambda_{22}\sqrt{\left(\Lambda_{22}^2+\Lambda_1\right)^3}} \quad (83.1)$$

$$E_2(x,y,t) =$$

$$\frac{2\alpha_2 e^{\Lambda_{10} t}\left[1-(-1)^m e^{\frac{-\Lambda_{10}}{c}}\right]\Lambda_3(\Lambda_{10}+\Lambda_4)\sin\left[\frac{\Lambda_3(\Lambda_{10}+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}+\Lambda_1)}\right] y_0\, b\cos\left[\frac{\Lambda_3(\Lambda_4-\Lambda_1^2)}{\alpha_{ot}(\Lambda_1-\Lambda_1^2)}\right] y_o\cos\left[\frac{\Lambda_3(\Lambda_{10}+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}+\Lambda_1)}\right] b}{2\sqrt{\alpha_{ot}}\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_1\right)\left(\Lambda_{10}^2+\Lambda_1\right)^{\frac{3}{2}}\left(\Lambda_{10}^2+\Lambda_{22}\right)}\cos\left[\frac{\Lambda_3(\Lambda_{10}+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}+\Lambda_1)}\right] y +$$

$$\frac{2\alpha_2 e^{\Lambda_{10} t}\left[1-(-1)^m e^{\frac{\Lambda_{10}}{c}}\right]\Lambda_3(\Lambda_{10}+\Lambda_4)\sin\left[\frac{\Lambda_3(\Lambda_{10}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}^2+\Lambda_1)}\right] y_0}{2\sqrt{\alpha_{ot}}\Lambda_{10}(\Lambda_{10}+\Lambda_2)\left(\Lambda_{10}^2+\Lambda_1\right)^{\frac{3}{2}}\left(\Lambda_{10}^2-\Lambda_{22}\right)}\cos\left[\frac{\Lambda_3(\Lambda_{10}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}^2+\Lambda_1)}\right] b\cos\left[\frac{\Lambda_3(\Lambda_{10}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}^2+\Lambda_1)}\right] y +$$

$$\frac{2\alpha_2 e^{i\Lambda_2}\left[1-(-1)^m e^{\frac{-i\Lambda_2}{c}}\right]\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}\right] y_0}{2\sqrt{\alpha_{ot}}\left(\Lambda_{10}^2+\Lambda_1\right) i\Lambda_2\left(\Lambda_1-\Lambda_2^2\right)^{\frac{3}{2}}\left(\Lambda_2^2+\Lambda_{22}\right)}\cos\left[\frac{\Lambda_3\left(\Lambda_4+\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_2^2\right)}\right] b\cos\left[\frac{\Lambda_3\left(\Lambda_4+\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_2^2\right)}\right] y -$$

$$\frac{2\alpha_2 e^{-i\Lambda_2}\left[1-(-1)^m e^{\frac{i\Lambda_2}{c}}\right]\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}\right] y_0}{2i\Lambda_2\sqrt{\alpha_{ot}}\left(\Lambda_{10}^2+\Lambda_{10}\right)\left(\Lambda_1-\Lambda_2^2\right)^{\frac{3}{2}}\left(\Lambda_2^2+\Lambda_{22}\right)}\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] b\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y +$$

$$\frac{2\alpha_2 e^{i\Lambda_1 t}\left[1-(-1)^m e^{\frac{-i\Lambda_1}{c}}\right]\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_1^2)}{\alpha_{ot}(\Lambda_1-\Lambda_1^2)}\right] y_0}{2i\Lambda_1\sqrt{\alpha_{ot}}\left(\Lambda_1^2+\Lambda_{10}\right) i\Lambda_2\left(\Lambda_2-\Lambda_1^2\right)^{\frac{3}{2}}\left(\Lambda_1^2+\Lambda_{22}\right)}\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right] b\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right] y -$$

$$\frac{2\alpha_2 e^{-i\Lambda_1 t}\left[1-(-1)^m e^{\frac{i\Lambda_1}{c}}\right]\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_1^2)}{\alpha_{ot}(\Lambda_1-\Lambda_1^2)}\right] y_0}{2i\Lambda_1\sqrt{\alpha_{ot}}\left(\Lambda_1^2+\Lambda_{10}\right)\left(\Lambda_2-\Lambda_1^2\right)^{\frac{3}{2}}\left(\Lambda_1^2+\Lambda_{22}\right)}\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right] b\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right] y +$$

$$\frac{2\alpha_2 e^{\Lambda_{22} t}\left[1-(-1)^m e^{\frac{-\Lambda_{22}}{c}}\right]\Lambda_3\left(\Lambda_2^2+\Lambda_4\right)\sin\left[\frac{\Lambda_3(\Lambda_{22}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{22}^2+\Lambda_1)}\right] y_0}{2\Lambda_{22}\sqrt{\alpha_{ot}}\left(\Lambda_{22}^2-\Lambda_{10}\right)\left(\Lambda_{22}^2+\Lambda_2\right)\left(\Lambda_{22}^2+\Lambda_1\right)^{\frac{3}{2}}}\cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] b\cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y -$$

$$\frac{2\alpha_2 e^{-\Lambda_{22} t}\left[1-(-1)^m e^{\frac{\Lambda_{22}}{c}}\right]\Lambda_3\left(\Lambda_2^2+\Lambda_4\right)\sin\left[\frac{\Lambda_3(\Lambda_{22}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{22}^2+\Lambda_1)}\right] y_0}{2\Lambda_{22}\sqrt{\alpha_{ot}}\left(\Lambda_{22}^2-\Lambda_{10}\right)\left(\Lambda_{22}^2+\Lambda_2\right)\left(\Lambda_{22}^2+\Lambda_1\right)^{\frac{3}{2}}}\cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] b\cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y \qquad (83.2)$$

$$E_{3f}(x,y,t) = -\frac{2e^{i\Lambda_1 t}\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)\left[1-(-1)^m e^{\frac{-i\Lambda_1}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_4+\Lambda_1^2)}{\alpha_{ot}(\Lambda_1+\Lambda_1^2)}\right] y_0\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_1^2)}{\alpha_{ot}(\Lambda_1-\Lambda_1^2)}\right] y}{2i\Lambda_1\left(\Lambda_1^2+\Lambda_{10}\right)\left(\Lambda_2+\Lambda_1^2\right)} +$$

$$\frac{2e^{i\Lambda_1 t}\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)\left[1-(-1)^m e^{\frac{i\Lambda_1}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_4+\Lambda_1^2)}{\alpha_{ot}(\Lambda_1+\Lambda_1^2)}\right] y_0\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_1^2)}{\alpha_{ot}(\Lambda_1-\Lambda_1^2)}\right] y}{2i\Lambda_1\left(\Lambda_1^2+\Lambda_{10}\right)\left(\Lambda_2-\Lambda_1^2\right)} +$$

$$\frac{2e^{\Lambda_{10} t}\Lambda_3\left(\Lambda_{10}^2-\Lambda_4\right)\left[1-(-1)^m e^{\frac{-\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_{10}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}^2+\Lambda_1)}\right] y_0\sin\left[\frac{\Lambda_3(\Lambda_{10}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}^2+\Lambda_1)}\right] y}{2\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_1\right)\left(\Lambda_{10}^2+\Lambda_2\right)} -$$

$$\frac{2e^{\Lambda_{10} t}\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)\left[1-(-1)^m e^{\frac{\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_{10}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}^2+\Lambda_1)}\right] y_0\sin\left[\frac{\Lambda_3(\Lambda_{10}^2+\Lambda_4)}{\alpha_{ot}(\Lambda_{10}^2+\Lambda_1)}\right] y}{2\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_1\right)\left(\Lambda_{10}^2+\Lambda_2\right)} -$$

$$\frac{2e^{i\Lambda_2 t}\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)\left[1-(-1)^m e^{\frac{-i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}\right] y_0\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}\right] y}{2i\Lambda_2\left(\Lambda_1-\Lambda_2^2\right)\left(\Lambda_2^2+\Lambda_{10}\right)} +$$

$$\frac{2e^{-i\Lambda_2 t}\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)\left[1-(-1)^m e^{\frac{i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}\right] y_0\sin\left[\frac{\Lambda_3(\Lambda_4-\Lambda_2^2)}{\alpha_{ot}(\Lambda_1-\Lambda_2^2)}\right] y}{2i\Lambda_2\left(\Lambda_1-\Lambda_2^2\right)\left(\Lambda_2^2+\Lambda_{10}\right)} \qquad (83.3)$$

$$E_4(x,y,t) =- \ 2e^{i\Lambda_1 t}\frac{\left[1-(-1)^m e^{\frac{-i\Lambda_1}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]\mathrm{b}\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]y_o\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]y}{2i\Lambda_1\left(\Lambda_1^2+\Lambda_{22}\right)}+-$$
$$2e^{i\Lambda_1 t}\frac{\left[1-(-1)^m e^{\frac{i\Lambda_1}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]\mathrm{b}\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]y_o\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]y}{2i\Lambda_1\left(\Lambda_1^2+\Lambda_{22}\right)}+$$
$$\lim_{s\to\Lambda_{22}}\frac{2(s-\Lambda_{22})e^{st}\left[1-(-1)^m e^{\frac{-s}{c}}\right]\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]\mathrm{b}\cos\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y_o\sin\left[\frac{\Lambda_3(s^2+\Lambda_4)}{\alpha_{ot}(s^2+\Lambda_1)}\right]y}{(s^2+\Lambda_1)(s-\Lambda_{22})(s-\Lambda_{22})}+-$$
$$\frac{2e^{-\Lambda_{22}}\left[1-(-1)^m e^{\frac{\Lambda_{22}}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]\mathrm{b}\cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]y_o\sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]y}{2\Lambda_{22}\left(\Lambda_{22}^2+\Lambda_1\right)}$$

(83.4)

$$E_5(x,y,t) =- \ e^{i\Lambda_1 t}\frac{\left[1-(-1)^m e^{\frac{-i\Lambda_1}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]y}{\left(\Lambda_1-\Lambda_1^2\right)\left(\Lambda_1^2+\Lambda_{22}\right)}-$$
$$e^{-i\Lambda_1 t}\frac{\left[1-(-1)^m e^{\frac{i\Lambda_1}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_1^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_1^2\right)}\right]y}{\left(\Lambda_1-\Lambda_1^2\right)\left(\Lambda_1^2+\Lambda_{22}\right)}+\frac{e^{\Lambda_{22}t}\left[1-(-1)^m e^{\frac{-\Lambda_{22}}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]y}{2\Lambda_{22}\left(\Lambda_{22}^2+\Lambda_1\right)}-$$
$$\frac{2e^{-\Lambda_{22}t}\left[1-(-1)^m e^{\frac{\Lambda_{22}}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]y}{2\Lambda_{22}\left(\Lambda_{22}^2+\Lambda_1\right)}$$

(83.5)

$$E_6(x,y,t) = \frac{\sqrt{\alpha_{ot}}e^{\Lambda_{10}t}\left[1-(-1)^m e^{\frac{-\Lambda_{10}}{c}}\right]\left(\Lambda_{10}^2+\Lambda_{23}\right)^{\frac{1}{2}}\cos^2\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y}{2\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_2\right)\left(\Lambda_{10}^2+\Lambda_{22}\right)}-$$
$$\sqrt{\alpha_{ot}}e^{-\Lambda_{10}t}\left[1-(-1)^m e^{\frac{-\Lambda_{10}}{c}}\right]\left(\Lambda_{10}^2+\Lambda_{23}\right)^{\frac{1}{2}}$$
$$\frac{\cos^2\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y_0}{2\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_2\right)\left(\Lambda_{10}^2+\Lambda_{22}\right)}\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y+$$
$$\frac{\sqrt{\alpha_{ot}}e^{i\Lambda_2 t}\left[1-(-1)^m e^{\frac{-i\Lambda_2}{c}}\right]\left(\Lambda_{23}-\Lambda_2^2\right)^{\frac{1}{2}}\cos^2\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y}{2i\Lambda_2\left(\Lambda_2^2+\Lambda_{10}\right)\left(\Lambda_2^2+\Lambda_{22}\right)}+$$
$$\frac{\sqrt{\alpha_{ot}}e^{-i\Lambda_2 t}\left[1-(-1)^m e^{\frac{i\Lambda_2}{c}}\right]\left(\Lambda_{23}-\Lambda_2^2\right)^{\frac{1}{2}}\cos^2\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y}{2i\Lambda_2\left(\Lambda_2^2+\Lambda_{10}\right)\left(\Lambda_2^2+\Lambda_{22}\right)}+$$
$$\frac{\sqrt{\alpha_{ot}}e^{\Lambda_{22}t}\left[1-(-1)^m e^{\frac{i\Lambda_{22}}{c}}\right]\left(\Lambda_{22}^2+\Lambda_{23}\right)^{\frac{1}{2}}\cos^2\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]b\sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]y}{2\Lambda_{22}^2\left(\Lambda_2^2-\Lambda_{10}\right)\left(\Lambda_2^2+\Lambda_2\right)}-$$
$$\frac{\sqrt{\alpha_{ot}}e^{\Lambda_{22}t}\left[1-(-1)^m e^{\frac{\Lambda_{22}}{c}}\right]\left(\Lambda_{22}^2+\Lambda_{23}\right)^{\frac{1}{2}}\cos^2\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] b\sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right]y}{2\Lambda_{22}\left(\Lambda_2^2-\Lambda_{10}\right)\left(\Lambda_2^2+\Lambda_2\right)}$$

(83.6)

$$
\begin{aligned}
E_7(x,y,t) = {} & \frac{e^{\Lambda_{10}t}e^{\gamma_{1a}x}\left[1-(-1)^m e^{\frac{-\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y}{2\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_2\right)\left(\Lambda_{10}^2-\Lambda_{29}\right)\left(\Lambda_{10}^2+\Lambda_{30}\right)}+ \\
& \frac{e^{-\Lambda_{10}t}e^{\gamma_{1b}x}\left[1-(-1)^m e^{\frac{\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y}{-2\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_2\right)\left(\Lambda_{10}^2-\Lambda_{29}\right)\left(\Lambda_{10}^2+\Lambda_{30}\right)}- \\
& \frac{e^{i\Lambda_2 t}e^{\gamma_{1c}x}\left[1-(-1)^m e^{\frac{-i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y}{2i\Lambda_2\left(\Lambda_2^2+\Lambda_{10}\right)\left(\Lambda_2^2+\Lambda_{29}\right)\left(\Lambda_2^2+\Lambda_{30}\right)}+ \\
& \frac{e^{-i\Lambda_2 t}e^{\gamma_{1d}x}\left[1-(-1)^m e^{\frac{i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y}{2i\Lambda_2\left(\Lambda_2^2+\Lambda_{10}\right)\left(\Lambda_2^2+\Lambda_{29}\right)\left(\Lambda_2^2+\Lambda_{30}\right)}+ \\
& \frac{e^{\Lambda_{29}t}e^{\gamma_{1e}x}\left[1-(-1)^m e^{\frac{-\Lambda_{29}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]y}{2\Lambda_{29}\left(\Lambda_{29}^2-\Lambda_{10}\right)\left(\Lambda_{29}^2+\Lambda_2\right)\left(\Lambda_{29}^2-\Lambda_{30}\right)}+ \\
& \frac{e^{-\Lambda_{29}t}e^{\gamma_{1f}}\left[1-(-1)^m e^{\frac{\Lambda_{29}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]y}{2\Lambda_{29}\left(\Lambda_{29}^2-\Lambda_{10}\right)\left(\Lambda_{29}^2+\Lambda_2\right)\left(\Lambda_{29}^2-\Lambda_{30}\right)}+ \\
& \frac{e^{\Lambda_{30}t}e^{\gamma_{1g}}\left[1-(-1)^m e^{\frac{-\Lambda_{30}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]y}{2\Lambda_{30}\left(\Lambda_{30}^2-\Lambda_{10}\right)\left(\Lambda_{30}^2+\Lambda_2\right)\left(\Lambda_{30}^2-\Lambda_{29}\right)}- \\
& \frac{e^{-\Lambda_{30}t}e^{\gamma_{1h}}\left[1-(-1)^m e^{\frac{\Lambda_{30}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]y}{2\Lambda_{30}\left(\Lambda_{30}^2-\Lambda_{10}\right)\left(\Lambda_{30}^2+\Lambda_2\right)\left(\Lambda_{30}^2-\Lambda_{29}\right)}
\end{aligned}
\qquad (83.7)
$$

$$E_8(x,y,t)=\frac{e^{\Lambda_{10}t}e^{\gamma_{1a}x}\left[1-(-1)^m e^{\frac{-\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]b}{2\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_2\right)\left(\Lambda_{10}^2-\Lambda_{29}\right)\left(\Lambda_{10}^2-\Lambda_{30}\right)}-$$

$$\frac{e^{-\Lambda_{10}t}e^{\gamma_{1b}x}\left[1-(-1)^m e^{\frac{\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]b}{2\Lambda_{10}\left(\Lambda_{10}^2+\Lambda_2\right)\left(\Lambda_{10}^2-\Lambda_{29}\right)\left(\Lambda_{10}^2-\Lambda_{30}\right)}-$$

$$\frac{e^{i\Lambda_2t}e^{\gamma_{1c}x}\left[1-(-1)^m e^{\frac{-i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]b}{2i\Lambda_2\left(\Lambda_2^2+\Lambda_{10}\right)\left(\Lambda_2^2+\Lambda_{29}\right)\left(\Lambda_2^2+\Lambda_{30}\right)}-$$

$$\frac{e^{-i\Lambda_2t}e^{\gamma_{1d}x}\left[1-(-1)^m e^{\frac{i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right]b}{2i\Lambda_2\left(\Lambda_2^2+\Lambda_{10}\right)\left(\Lambda_2^2+\Lambda_{29}\right)\left(\Lambda_2^2+\Lambda_{30}\right)}+$$

$$\frac{e^{\Lambda_{29}t}e^{\gamma_{1e}x}\left[1-(-1)^m e^{\frac{-\Lambda_{29}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]b}{2\Lambda_{29}\left(\Lambda_{29}^2-\Lambda_{10}\right)\left(\Lambda_{29}^2+\Lambda_2\right)\left(\Lambda_{29}^2-\Lambda_{30}\right)}-$$

$$\frac{e^{-\Lambda_{29}t}e^{\gamma_{1e}x}\left[1-(-1)^m e^{\frac{\Lambda_{29}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right]b}{2\Lambda_{29}\left(\Lambda_{29}^2-\Lambda_{10}\right)\left(\Lambda_{29}^2+\Lambda_2\right)\left(\Lambda_{29}^2-\Lambda_{30}\right)}+$$

$$\frac{e^{\Lambda_{30}t}e^{\gamma_{1g}}\left[1-(-1)^m e^{\frac{-\Lambda_{30}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]b}{2\Lambda_{30}\left(\Lambda_{30}^2-\Lambda_{10}\right)\left(\Lambda_{30}^2+\Lambda_2\right)\left(\Lambda_{30}^2-\Lambda_{29}\right)}+-$$

$$\frac{e^{-\Lambda_{30}t}e^{\gamma_{1h}}\left[1-(-1)^m e^{\frac{\Lambda_{30}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right]b}{2\Lambda_{30}\left(\Lambda_{30}^2-\Lambda_{10}\right)\left(\Lambda_{30}^2+\Lambda_2\right)\left(\Lambda_{30}^2-\Lambda_{29}\right)}\tag{83.8}$$

$$E_9(x,y,t)=\frac{e^{i\Lambda_4}\left[1-(-1)^m e^{\frac{-i\Lambda_4}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]b}{2i\Lambda_4 i\Lambda_3}-$$

$$\frac{e^{-i\Lambda_4}\left[1-(-1)^m e^{\frac{i\Lambda_4}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]y_0\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]y\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]b}{2i\Lambda_4 i\Lambda_3}\tag{83.9}$$

$$E_{10}(x,y,t)=\frac{e^{i\Lambda_4t}\left[1-(-1)^m e^{\frac{-i\Lambda_4}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]y}{2i\Lambda_4 i\Lambda_3}-\frac{e^{-i\Lambda_4t}\left[1-(-1)^m e^{\frac{i\Lambda_4}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]y_0\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right]y}{2i\Lambda_4 i\Lambda_3}\tag{83.10}$$

$$E_{11}(x,y,t) = \frac{e^{i\Lambda_4 t}\left[1-(-1)^m e^{\frac{-i\Lambda_4}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right] y}{2i\Lambda_4 i\Lambda_3} - \frac{e^{-i\Lambda_4 t}\left[1-(-1)^m e^{\frac{i\Lambda_4}{c}}\right]\cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right] y}{2i\Lambda_4 i\Lambda_3} \tag{83.11}$$

$$\begin{aligned}
E_{12}(x,y,t) = {} & \frac{e^{\Lambda_{10} t}.e^{\gamma_{2a} x}\left[1-(-1)^m e^{\frac{-\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right] y}{2\Lambda_{10}\left(\Lambda_{10}^2-\Lambda_{29}^2\right)\left(\Lambda_{10}^2-\Lambda_{30}^2\right)\left(\Lambda_{10}^2+\Lambda_2^2\right)} - \\
& \frac{e^{-\Lambda_{10} t}.e^{\gamma_{2b} x}\left[1-(-1)^m e^{\frac{\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right] y}{2\Lambda_{10}\left(\Lambda_{10}^2-\Lambda_{29}^2\right)\left(\Lambda_{10}^2-\Lambda_{30}^2\right)\left(\Lambda_{10}^2+\Lambda_2^2\right)} - \\
& \frac{e^{i\Lambda_2 t}.e^{\gamma_{2c} x}\left[1-(-1)^m e^{\frac{-i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y}{2i\Lambda_2\left(\Lambda_{10}^2-\Lambda_{29}^2\right)\left(\Lambda_{10}^2-\Lambda_{30}^2\right)\left(\Lambda_{10}^2+\Lambda_{10}^2\right)} + \\
& \frac{e^{-i\Lambda_2 t}.e^{\gamma_{2d} x}\left[1-(-1)^m e^{\frac{i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y}{2i\Lambda_2\left(\Lambda_{10}^2-\Lambda_{29}^2\right)\left(\Lambda_{10}^2-\Lambda_{30}^2\right)\left(\Lambda_{10}^2+\Lambda_{10}^2\right)} + \\
& \frac{e^{\Lambda_{29} t}.e^{\gamma_{2e} x}\left[1-(-1)^m e^{\frac{-\Lambda_{29}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] y}{2\Lambda_{29}\left(\Lambda_{29}^2-\Lambda_{30}^2\right)\left(\Lambda_{29}^2-\Lambda_2^2\right)\left(\Lambda_{29}^2+\Lambda_{10}^2\right)} - \\
& \frac{e^{-\Lambda_{29} t}.e^{\gamma_{2f} x}\left[1-(-1)^m e^{\frac{\Lambda_{29}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] y}{2\Lambda_{29}\left(\Lambda_{29}^2-\Lambda_{30}^2\right)\left(\Lambda_{29}^2+\Lambda_2^2\right)\left(\Lambda_{29}^2-\Lambda_{10}^2\right)} + \\
& \frac{e^{\Lambda_{30} t}.e^{\gamma_{2g} x}\left[1-(-1)^m e^{\frac{-\Lambda_{30}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] y}{2\Lambda_{30}\left(\Lambda_{30}^2-\Lambda_{29}^2\right)\left(\Lambda_{30}^2+\Lambda_2^2\right)\left(\Lambda_{30}^2-\Lambda_{10}^2\right)} - \\
& \frac{e^{-\Lambda_{30} t}.e^{\gamma_{2h} x}\left[1-(-1)^m e^{\frac{\Lambda_{30}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] y}{2\Lambda_{30}\left(\Lambda_{30}^2-\Lambda_{29}^2\right)\left(\Lambda_{30}^2+\Lambda_2^2\right)\left(\Lambda_{30}^2-\Lambda_{10}^2\right)}
\end{aligned} \tag{83.12}$$

$$\begin{aligned}
E_{13}(x,y,t) = {} & \frac{e^{\Lambda_{10} t}.e^{\gamma_{2a} x}\left[1-(-1)^m e^{\frac{-\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]}{2\Lambda_{10}\left(\Lambda_{10}^2-\Lambda_2^2\right)\left(\Lambda_{10}^2-\Lambda_{29}^2\right)\left(\Lambda_{10}^2+\Lambda_{30}^2\right)\left(\Lambda_{10}^2+\Lambda_{22}^2\right)} \\
& - \frac{e^{-\Lambda_{10} t}.e^{\gamma_{2b} x}\left[1-(-1)^m e^{\frac{\Lambda_{10}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{10}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{10}^2+\Lambda_1\right)}\right]}{2\Lambda_{10}\left(\Lambda_{10}^2-\Lambda_2^2\right)\left(\Lambda_{10}^2-\Lambda_{29}^2\right)\left(\Lambda_{10}^2+\Lambda_{30}^2\right)\left(\Lambda_{10}^2+\Lambda_{22}^2\right)} + \\
& \frac{e^{i\Lambda_2 t}.e^{\gamma_{2c} x}\left[1-(-1)^m e^{\frac{-i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] b}{2i\Lambda_2\left(\Lambda_2^2-\Lambda_{29}^2\right)\left(\Lambda_2^2-\Lambda_{30}^2\right)\left(\Lambda_2^2+\Lambda_{22}^2\right)\left(\Lambda_2^2+\Lambda_{10}^2\right)} + \\
& \frac{e^{-i\Lambda_2 t}.e^{\gamma_{2d} x}\left[1-(-1)^m e^{\frac{i\Lambda_2}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_2^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] b}{2i\Lambda_2\left(\Lambda_2^2+\Lambda_{29}^2\right)\left(\Lambda_2^2+\Lambda_{30}^2\right)\left(\Lambda_2^2+\Lambda_{22}^2\right)\left(\Lambda_2^2+\Lambda_{10}^2\right)} + \\
& \frac{e^{\Lambda_{29} t}.e^{\gamma_{2e} x}\left[1-(-1)^m e^{\frac{-\Lambda_{29}}{c}}\right]\sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] b}{2\Lambda_{29}\left(\Lambda_{29}^2-\Lambda_{10}^2\right)\left(\Lambda_{29}^2-\Lambda_2^2\right)\left(\Lambda_{29}^2-\Lambda_{30}^2\right)\left(\Lambda_{29}^2+\Lambda_{22}^2\right)} -
\end{aligned}$$

$$\frac{e^{-\Lambda_{29} t}.e^{\gamma_{2f} x}\left[1-(-1)^m e^{\frac{\Lambda_{29}}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{29}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{29}^2+\Lambda_1\right)}\right] b}{2\Lambda_{29}\left(\Lambda_{29}^2-\Lambda_{10}^2\right)\left(\Lambda_{29}^2+\Lambda_2^2\right)\left(\Lambda_{29}^2-\Lambda_{30}^2\right)\left(\Lambda_{29}^2-\Lambda_{22}^2\right)}+$$

$$\frac{e^{\Lambda_{30} t}.e^{\gamma_{2g} x}\left[1-(-1)^m e^{\frac{-\Lambda_{30}}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] b}{2\Lambda_{30}\left(\Lambda_{30}^2-\Lambda_{10}^2\right)\left(\Lambda_{30}^2+\Lambda_2^2\right)\left(\Lambda_{30}^2-\Lambda_{29}^2\right)\left(\Lambda_{30}^2-\Lambda_{22}^2\right)}-$$

$$\frac{e^{-\Lambda_{30} t}.e^{\gamma_{2h} x}\left[1-(-1)^m e^{\frac{\Lambda_{30}}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{30}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{30}^2+\Lambda_1\right)}\right] b}{2\Lambda_{30}\left(\Lambda_{30}^2-\Lambda_{10}^2\right)\left(\Lambda_{30}^2+\Lambda_2^2\right)\left(\Lambda_{30}^2-\Lambda_{29}^2\right)\left(\Lambda_{30}^2-\Lambda_{22}^2\right)}+$$

$$\frac{e^{\Lambda_{22} t}.e^{\gamma_{2i} x}\left[1-(-1)^m e^{\frac{-\Lambda_{22}}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] b}{2\Lambda_{22}\left(\Lambda_{22}^2-\Lambda_{10}^2\right)\left(\Lambda_{22}^2+\Lambda_2^2\right)\left(\Lambda_{22}^2-\Lambda_{29}^2\right)\left(\Lambda_{22}^2-\Lambda_{30}^2\right)}-$$

$$\frac{e^{-\Lambda_{22} t}.e^{\gamma_{2j} x}\left[1-(-1)^m e^{\frac{\Lambda_{22}}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] b}{2\Lambda_{22}\left(\Lambda_{22}^2-\Lambda_{10}^2\right)\left(\Lambda_{22}^2+\Lambda_2^2\right)\left(\Lambda_{22}^2-\Lambda_{29}^2\right)\left(\Lambda_{22}^2-\Lambda_{30}^2\right)} \qquad (83.13)$$

$$E_{14}(x,y,t) = -\frac{e^{i\Lambda_4 t}\left[1-(-1)^m e^{\frac{-i\Lambda_4}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] b}{2i\Lambda_3\Lambda_4\left(\Lambda_4^2+\Lambda_{22}^2\right)}+$$

$$\frac{e^{-i\Lambda_4 t}\left[1-(-1)^m e^{\frac{i\Lambda_4}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] b}{2i\Lambda_3\Lambda_4\left(\Lambda_4^2+\Lambda_{22}^2\right)}+$$

$$\frac{e^{\Lambda_{22} t}\left[1-(-1)^m e^{\frac{-\Lambda_{22}}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] b}{2\Lambda_{22}\Lambda_3\left(\Lambda_{22}^2+\Lambda_4^2\right)}-$$

$$\frac{e^{-\Lambda_{22} t}\left[1-(-1)^m e^{\frac{\Lambda_{22}}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] y \cos\left[\frac{\Lambda_3\left(\Lambda_{22}^2+\Lambda_4\right)}{\alpha_{ot}\left(\Lambda_{22}^2+\Lambda_1\right)}\right] b}{2\Lambda_{22}\Lambda_3\left(\Lambda_{22}^2+\Lambda_4^2\right)} \qquad (83.14)$$

$$E_{15}(x,y,t) = \frac{e^{i\Lambda_4 t}\left[1-(-1)^m e^{\frac{-i\Lambda_4}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right] y}{2i\Lambda_4 i\Lambda_3} - \frac{e^{-i\Lambda_4 t}\left[1-(-1)^m e^{\frac{i\Lambda_4}{c}}\right] \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right] y_0 \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1+\Lambda_4^2\right)}\right] y}{2i\Lambda_4 i\Lambda_3} \qquad (83.15)$$

$$E_{16}(x,y,t) = \frac{e^{i\Lambda_4 t}\left[1-(-1)^m e^{\frac{-i\Lambda_4}{c}}\right] \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] y}{2i\Lambda_4\left(\Lambda_2^2-\Lambda_4^2\right)} - \frac{e^{-i\Lambda_4 t}\left[1-(-1)^m e^{\frac{i\Lambda_4}{c}}\right] \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_4^2\right)}\right] y}{2i\Lambda_4\left(\Lambda_2^2-\Lambda_4^2\right)}+$$

$$\frac{e^{i\Lambda_2 t}\left[1-(-1)^m e^{\frac{-i\Lambda_2}{c}}\right] \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y}{2i\Lambda_2\left(\Lambda_4^2-\Lambda_2^2\right)} - \frac{e^{-i\Lambda_2 t}\left[1-(-1)^m e^{\frac{i\Lambda_2}{c}}\right] \cos\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y_0 \sin\left[\frac{\Lambda_3\left(\Lambda_4-\Lambda_4^2\right)}{\alpha_{ot}\left(\Lambda_1-\Lambda_2^2\right)}\right] y}{2i\Lambda_2\left(\Lambda_4^2-\Lambda_2^2\right)} \qquad (83.16)$$

Substitution of evaluated integrals $E_1(x,y,t) - E_{16}(x,y,t)$ into equation (81) produces the complete inversion of $W_1(x,y,t)$

From equation (14), the perturbation scheme of a uniformly valid solution in the entire domain of definition of the plate problem is given as

$$W(x,y,t) = W_o(x,y,t) + \varepsilon W_1(x,y,t) \qquad (84)$$

where $W_o(x,y,t)$ and $W_1(x,y,t)$ are respectively the leading order solution and the first order correction. These are given as (55) and (81) in that order. In view of equations (55) and (81), equation (84) becomes the required uniformly valid approximate analytical solution of the plate dynamical problem.

6.1. Remarks on Theory

Equations (55) and (81) are the leading order and the first order (transformed) solutions of the problem. The leading order and the first order solutions are combined in equation (84) to form the composite solution which is uniformly valid in the entire domain of the highly prestressed orthotropic rectangular plate.

It is observed from the leading order and first order solutions that fully clamped highly prestressed orthotropic rectangular plate traversed by moving concentrated masses and resting on Pasternak foundation reached the resonant state whenever

$$\Lambda_2 = \Lambda_1^2 \qquad (85)$$

Other conditions when the system operates at a frequency which equals the natural frequency to display an enhanced oscillation are

$$\Lambda_2^2 = \Lambda_{10}, \quad \Lambda_{29} = \Lambda_{10}^2, \ \Lambda_{10} = \Lambda_{30}^2, \ \Lambda_{29} = \Lambda_{30}^2 \text{ and } \Lambda_{10} = \Lambda_2 . \qquad (86)$$

From (85) and (86), it is observed that the resonance conditions of the plate are dependent on anisotropic prestress, mass ratio, rotatory inertia correction factor, shear and foundation moduli.

At this juncture, the critical speeds for the system of a highly prestressed orthotropic rectangular plate under the action of travelling masses are sought. Few of the critical speeds that exist in the dynamical system are given as

$$c_1(m,\pi) = \sqrt{\frac{(m\pi)^2(\beta_1^2+G_0)-K_0}{(m\pi)^2\Gamma_{0\delta(y-y_0)}}} \qquad (87)$$

$$c_2(m,\pi) = \frac{1}{m\pi}\sqrt[4]{\frac{\beta_1^2+G_0}{\alpha_{ot}}} \qquad (88)$$

$$c_3(m,\pi) = \frac{J_1}{J_2} \pm \frac{1}{2}\sqrt{J_3 - 4J_4} \qquad (89)$$

where

$$J_1 = -(m\pi)^2 \propto_{ot} - 1 - \Gamma_{0\delta(y-y_0)}$$

$$J_2 = 2m\pi\Gamma_{0\delta(y-y_0)}$$

$$J_3 = \left(\frac{(m\pi)^2\propto_{ot} - 1 - \Gamma_{0\delta(y-y_0)}}{m\pi\Gamma_{0\delta(y-y_0)}}\right)^2$$

$$J_4 = \frac{(m\pi)^2\beta_1^2 + (m\pi)^2 G_0 - K_0}{(m\pi)^2\Gamma_{0\delta(y-y_0)}}$$

6.2. Numerical Simulation

In order to illustrate the analytical results, for instance, the orthotropic rectangular plate is taken to be of length 1.0 m and width 0.9 m. Other values used for the analysis are b = 0.65 m, g = 9.81, $\pi = \frac{22}{7}$, $y_0 = 0.2$, $c = 8.128\frac{m}{s}$. The values of the prestress ratio in the x-direction range between 0 and 2 000 N. The critical speeds are plotted against prestress for various values of other parameters. The process is repeated for mass ratio, shear and foundation moduli in turn.

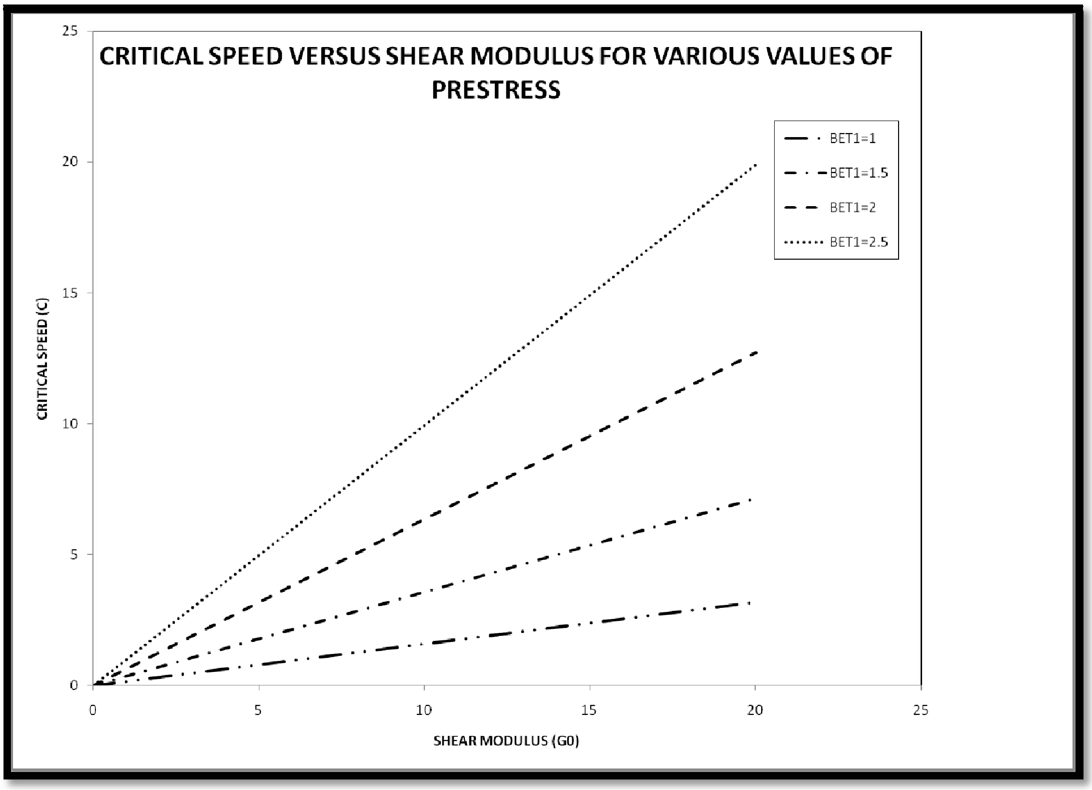

Figure 1: The Graph of Critical Speed (1) against Shear Modulus for Various Values of Prestress

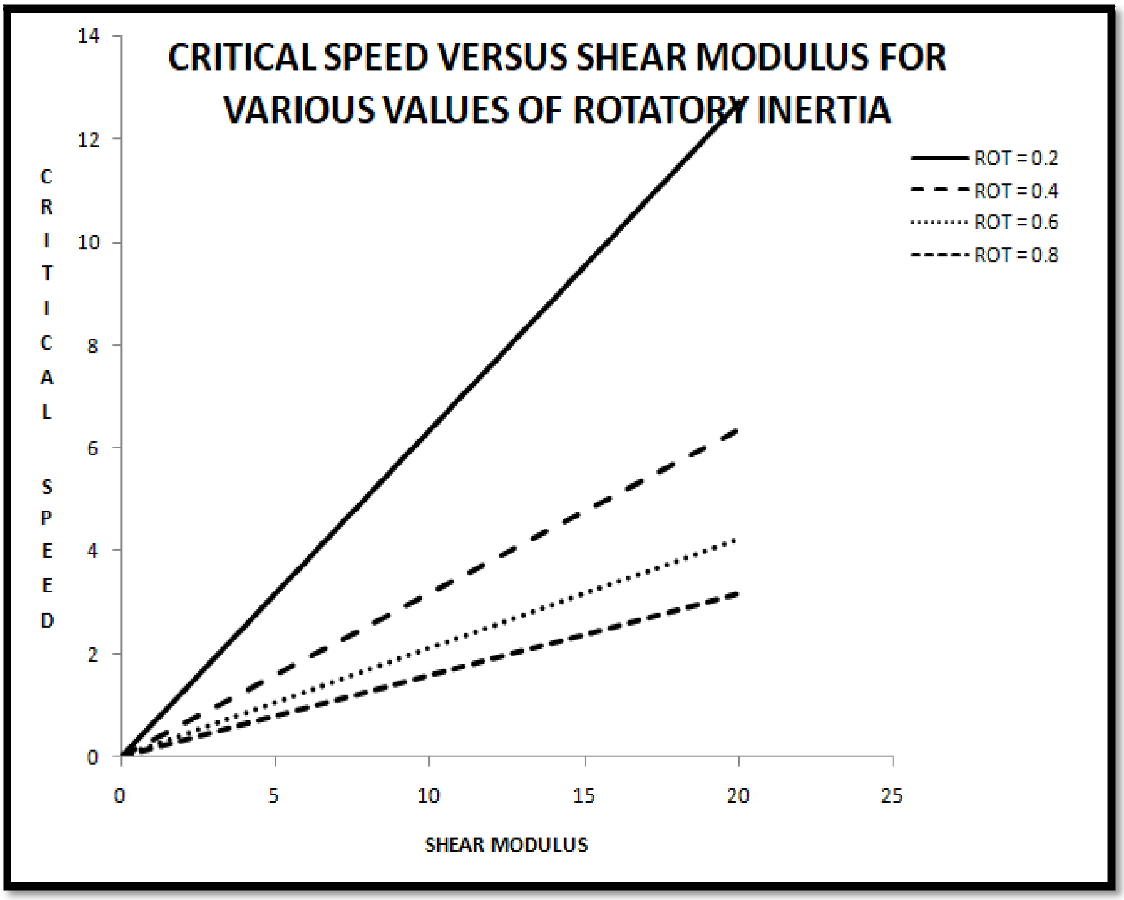

Figure 2: The Graph of Critical Speed (1) against Shear Modulus for Various Values of Rotatory Inertia Correction Factor (Rot= $\propto_{ot}$)

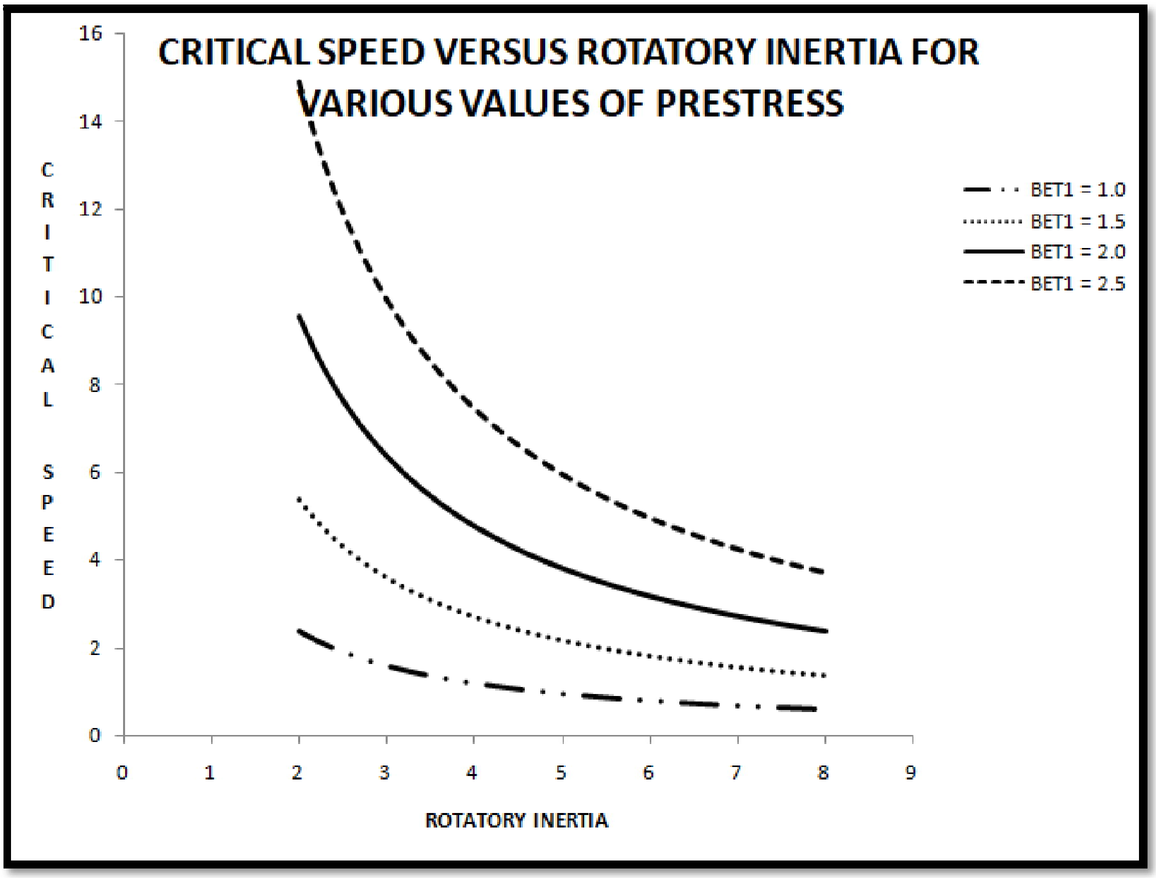

Figure 3: Graph Showing Critical Speed against Rotatory Inertia Correction Factor for Various Values of Prestress

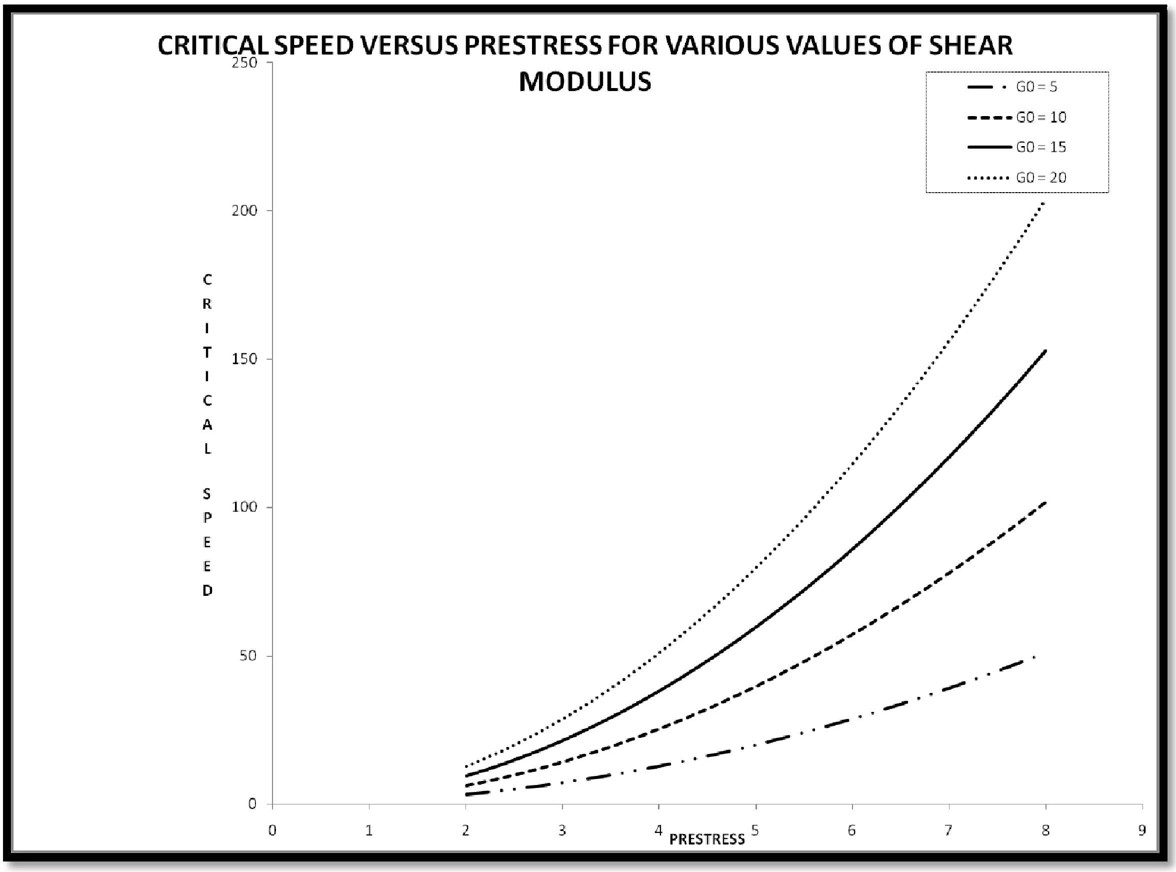

Figure 4: Graph Showing Critical Velocity against Prestress for Various Values of Mass Ratio

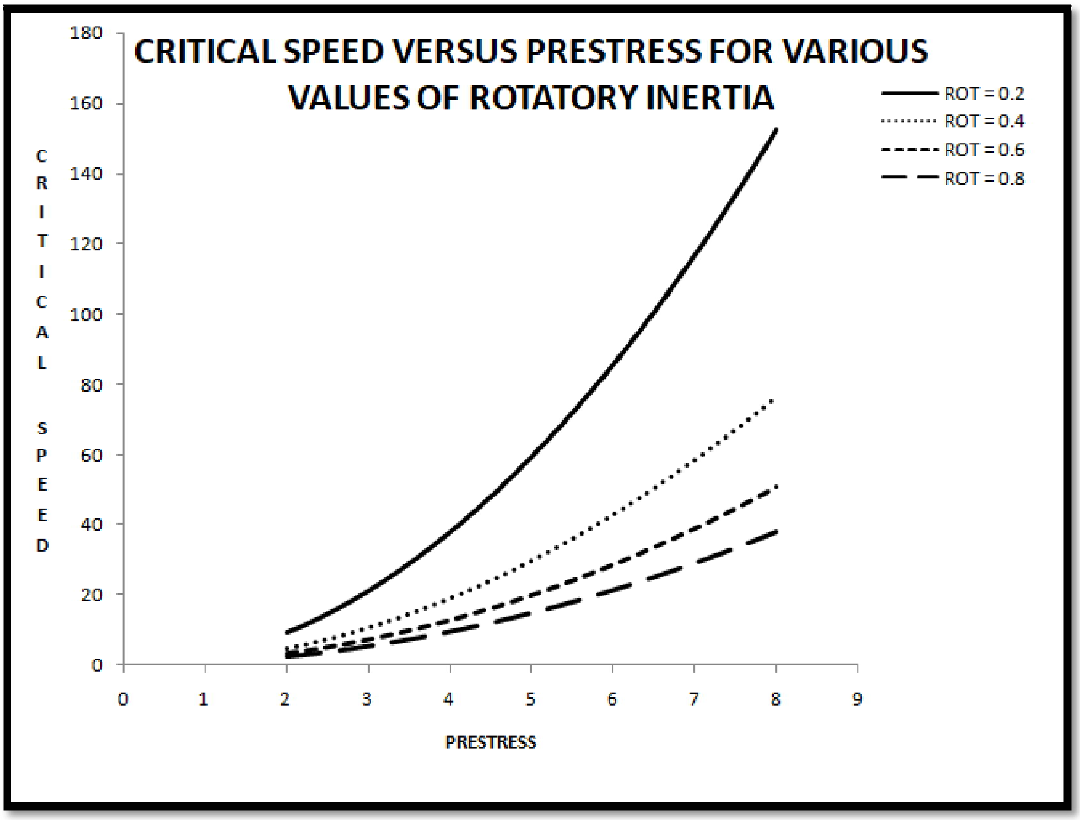

Figure 5: Graph Showing Critical Speed versus Prestress for Various Values of Rotatory Inertia Correction Factor

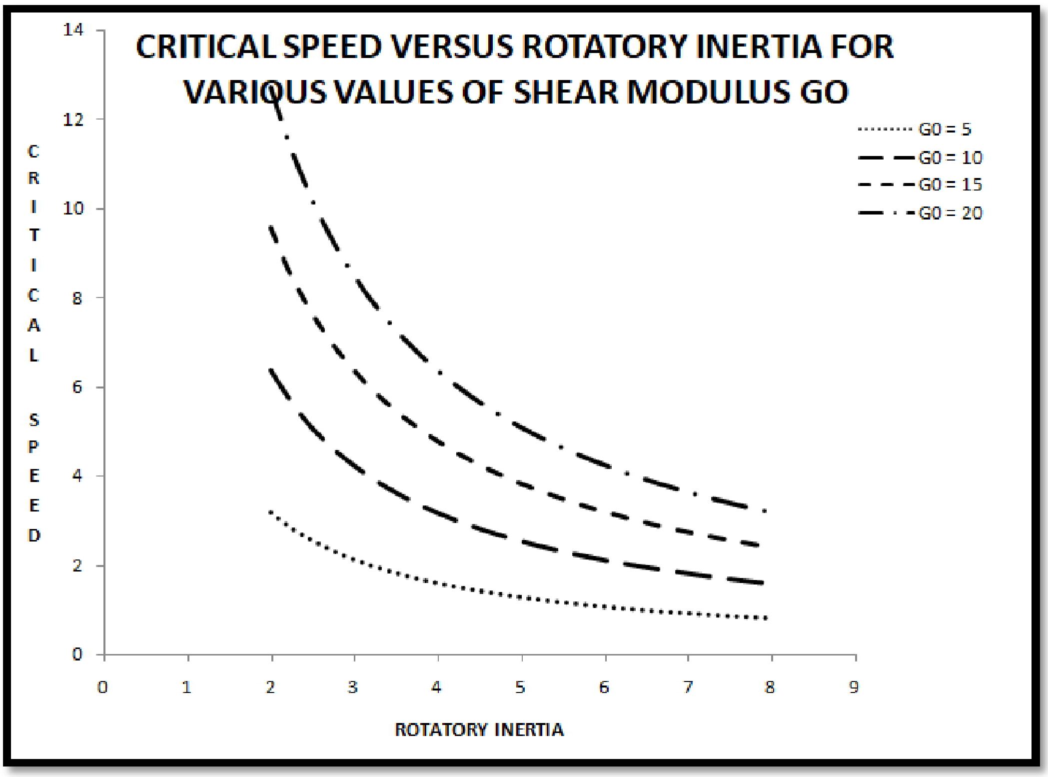

Figure 6: Graph Showing Critical Speed (1) against Rotatory Inertia Correction Factor for Various Values of Shear Modulus

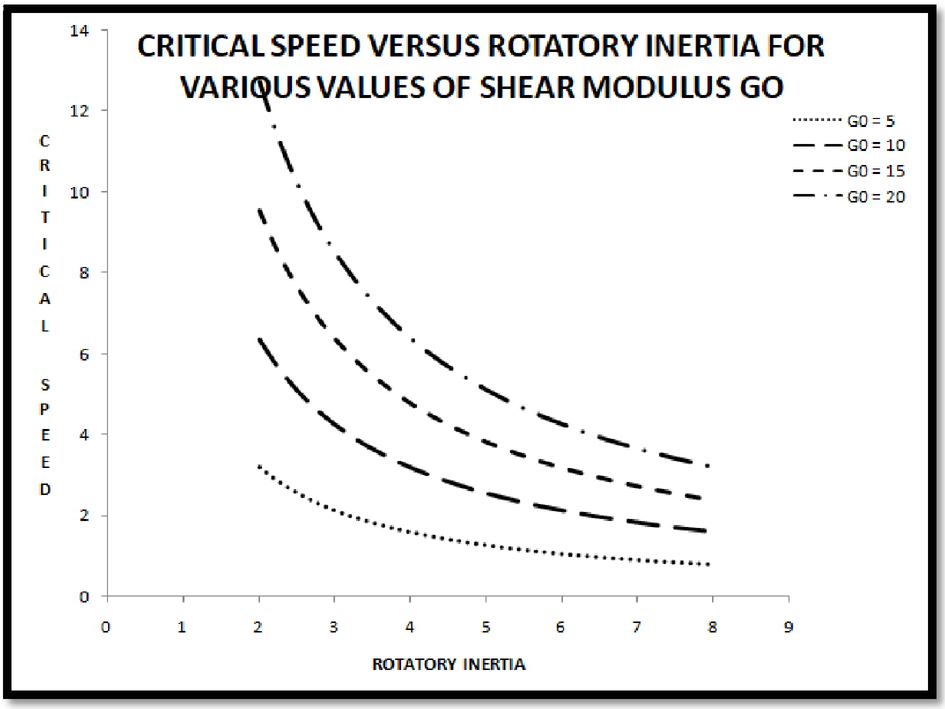

Figure 7: Graph Displaying Critical Speed against Rotatory Inertia for Various Values of Shear Modulus

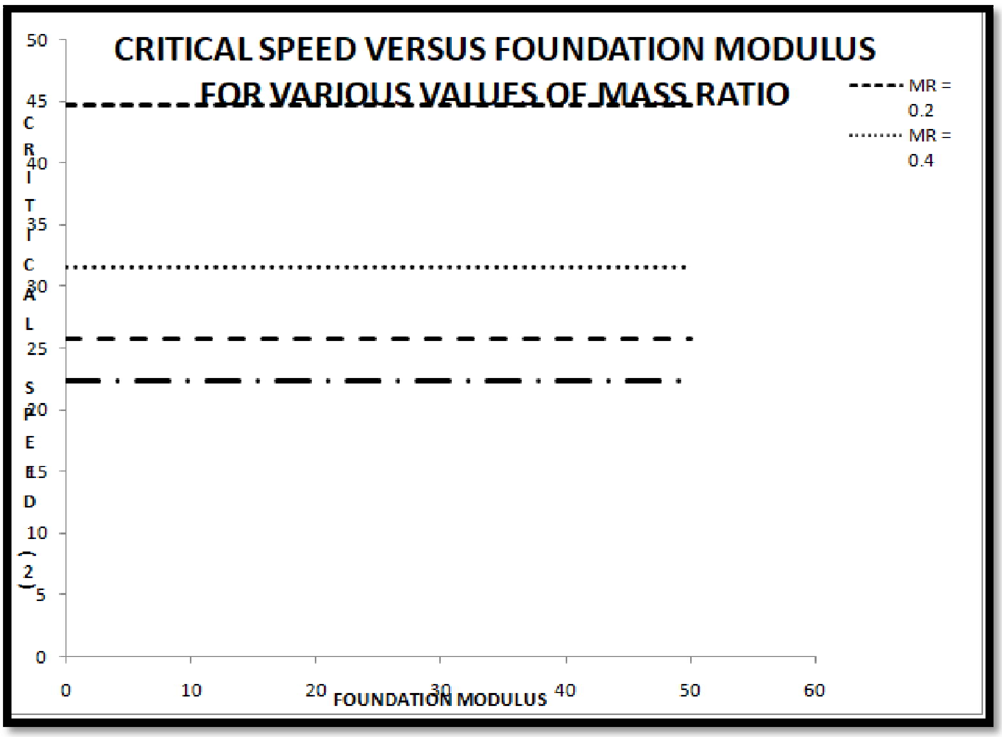

Figure 8: Graph Showing Critical Speed (2) against Foundation Modulus for Various Values of Mass Ratio

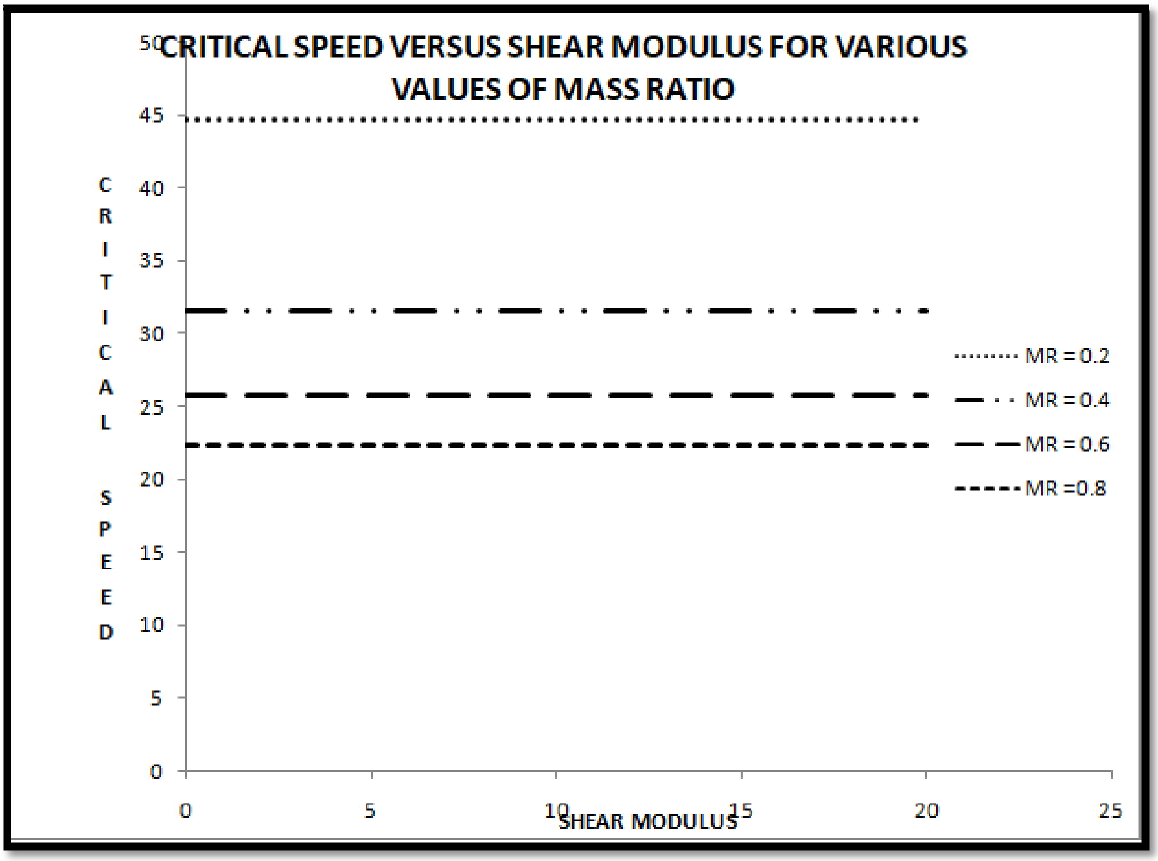

Figure 9: Showing the Graph of Critical Speed (2) against Shear Modulus for Various Values of Mass Ratio

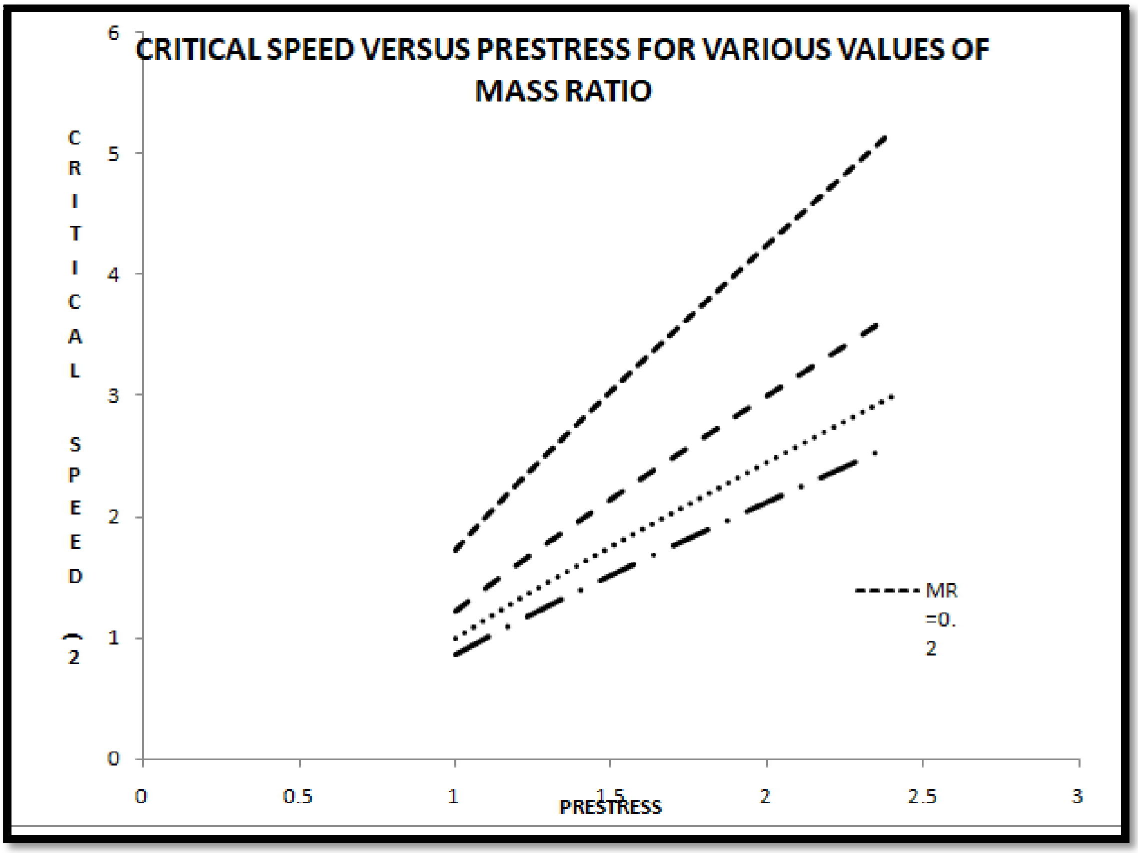

Figure 10: Showing the Graph of Critical Speed against Prestress for Various Values of Mass Ratio

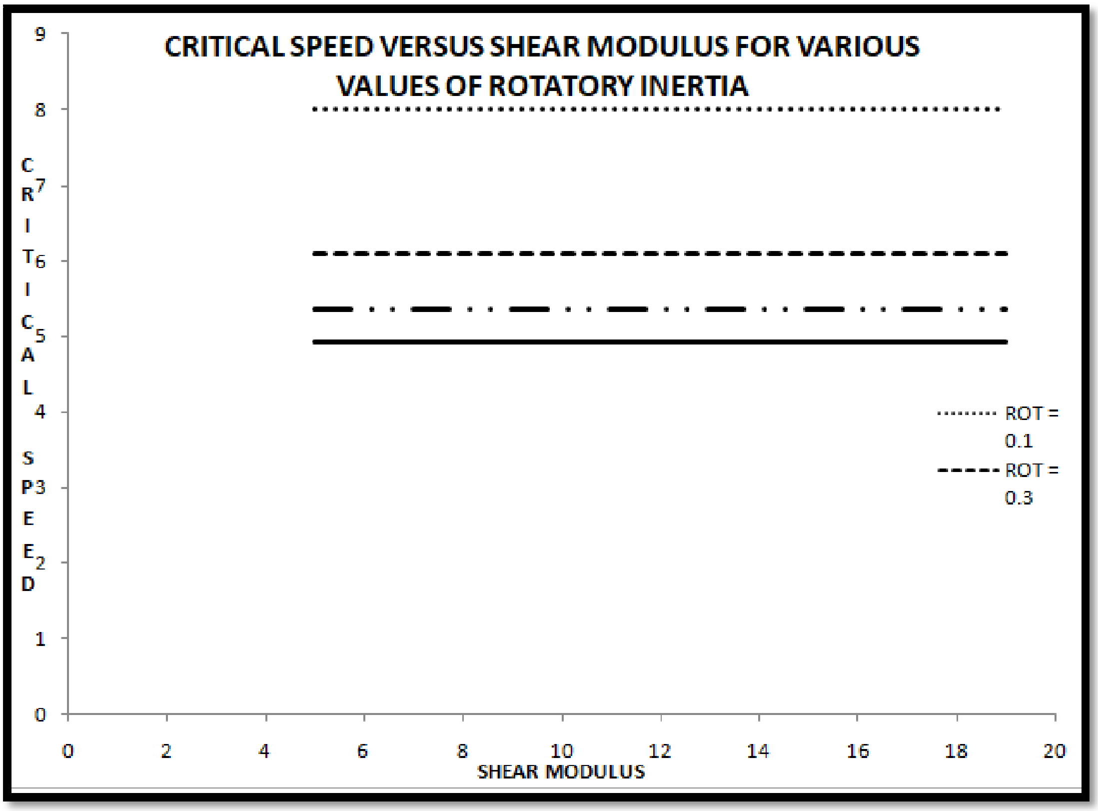

Figure 11: Showing the Graph of Critical Speed against Shear Modulus for Various Values of Rotatory Inertia Correction Factor

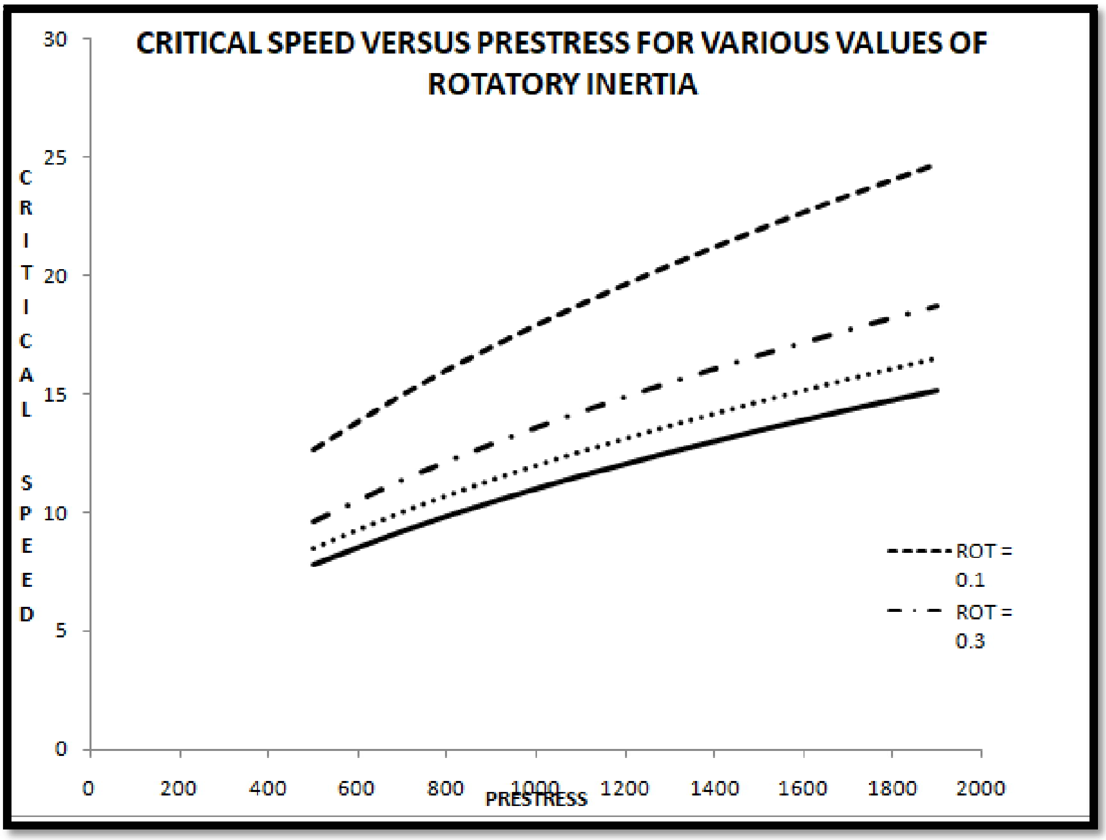

Figure 12: Showing the Graph of Critical Speed against Prestress for Various Values of Rotatory Inertia Correction Factor

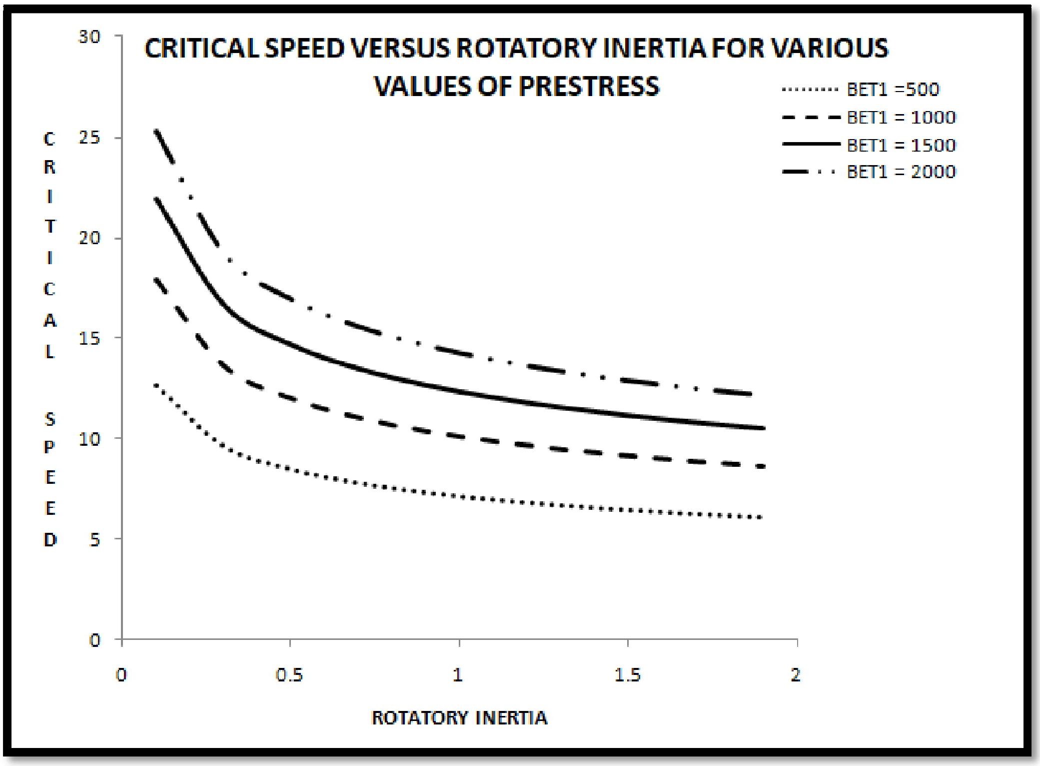

Figure 13: Showing the Graph of Critical Speed against Rotatory Inertia Correction Factor

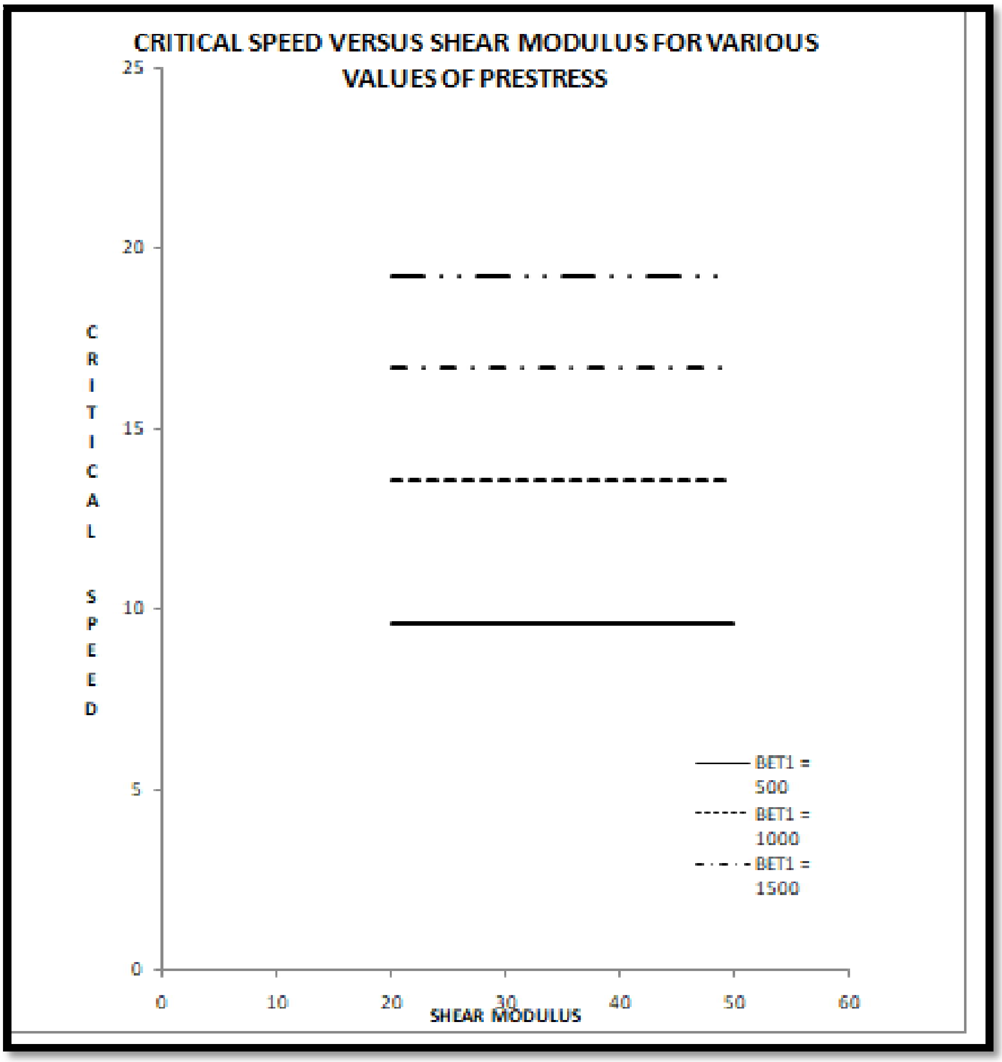

Figure 14: Showing the Graph of Critical Speed against Shear Modulus for Various Values of Prestress

From the figures displayed, it is discovered that at whatever critical speed, increase in prestress results in increase in critical speed (i.e. figures 1, 3, 6, 13). In the same vein, increase in the value of shear modulus yielded increase in critical speed (see figures 4, 7, 14). On the other hand, increase in either rotatory inertia correction factor or mass ratio, produces a reduction in critical speed (see figures 2, 5, 8, 9, 10, 11, 12). Thus, for high value of prestress, the structural design under consideration is more stable and reliable.

Evidently, the critical speed increases with prestress for all values of rotatory inertia correction factor used. Thus, resonance is reached earlier for lower values of prestress than for high values of prestress. Thus, the design is more stable and the risk of resonance is remote for high values of prestress.

It is clearly seen that increase in the values of rotatory inertia produce decrease in critical speed which connotes lower risk of resonance.

For smaller values of mass ratio, the critical speed is higher indicating that the durability and stability of the structure is guaranteed.

The rotatory inertia correction factor does not affect the system significantly as the prestress values increase.

It is also observed that the smaller the mass ratio, the greater the critical speed, indicating that the lightness or heaviness of bodies on structures is significant in the consideration of critical speed.

7. Conclusion

This study concerns the problem of the dynamic response of a highly prestressed orthotropic rectangular plate resting on Pasternak elastic foundation and traversed by concentrated moving mass. The problem is governed by a fourth order non-homogeneous partial differential equation. For the purpose of solution, the equation of motion of the plate problem is presented in a non-dimensional form. It is observed that a small parameter multiplied the highest derivatives in the governing differential equation. In accordance with the principle that the behaviour of solutions is governed by the highest order term, the choice of a suitable method of solution is made. Thus, this type of dynamical problem is usually amenable to singular perturbation technique. In particular the Method of Matched Asymptotic Expansions (MMAE) is used. This technique constructs outer (core) and inner (boundary layer) solutions that are valid in partly disjoint domains. These solutions are then matched in order to obtain a composite solution that is uniformly valid in the entire domain of definition of the rectangular plate. This solution is analysed for some resonant states in the dynamical system. A numerical simulation is carried out and the study reveals the following results:

- The leading order solution and the first order correction are affected by the mass ratio, anisotropic prestress, shear and foundation modulus. However, the effects of rotatory inertia correction factor are not appreciably noticed.
- As the prestress or shear modulus increases, the critical speed of the orthotropic rectangular plate traversed by moving concentrated mass also increases. Also, as the rotatory inertia or mass ratio increases, the critical speed decreases.
- There may be more than one resonance condition in a dynamical system such as this which involves plate flexure under moving concentrated masses.
- The smaller the mass ratio, the better the improvement in critical speed.

Finally, this work has showcased the use of a valuable method for the solution of this class of dynamical problems, and the valuable results will be useful for the design and construction of engineers.

8. References

i. Willis *et al*: Preliminary essay to appendix B.: Experiments for determining the effects produced by causing weights to travel over bars with different velocities. In: Barlow P., (1851): Treatise on the strength of timber, cast iron and malleable iron. London.
ii. Stokes, G. G. (1849): Discussion of a differential equation relating to the breaking of railway bridges. *Trans. Cambridge Phil. Soc.* 815, 707 – 735.
iii. Timoshenko S. (1964): Theory of plates and shells, Mcgraw Hill
iv. Lowan, A. N., (1935): On transverse oscillations of beams under the action of moving variable loads. *Philosophical magazine*, ser 7(19). No 127, 708 – 715.
v. Bondar, N. G., (1954): Dynamic calculations of beams subjected to a moving load (in Russian). *Issledovaniyapoteoriisooruzheni*, vol. 6. 11 – 23, Strolizdat, Moscow.
vi. Reissmann, H. (1963): Response of elastic plates strip to line load, *AIAA Journal* 1, 354.
vii. Reissmann Herbert (1988): *Elastic plates – Theory and application*. John Wiley and Sons, Inc. New York
viii. Fryba, Ladislav (1972): *Vibration of solids and structures under moving loads*, printing house of the Academy of science, Prague.
ix. Leissa A. W. (1969): *Vibrations of plates, Scientific and technical information division*, NASASP - 160, Washington D. C.
x. Holl, D. K. (1950): Dynamic loads on thin plates on elastic foundations. *Proc. Of symposium in applied mathematic*s, Vol13, McGraw-Hill. New York.
xi. Livesley, R. K., (1953): Some notes on the mathematical theory of a loaded elastic plate resting on an elastic foundation. *Quart. J. Mech. Appl. Math.* 6, 32 – 44.
xii. Stanisic, M. M., Hardin, J. C. and Lou, Y. C. (1968): On the response of the plate to a multi-masses moving system. *Actamechanica* 5, 37 – 53.

xiii. Stanisic, M. M., Euler, J. A. and Mongomery, S. T. (1974): On a theory concerning the behaviour of structures carrying moving masses. *Ingenieurachiv* 43, 295 – 305.
xiv. Stanisic, M. M. and Hardin, J. C. (1969): On the response of beams to an arbitrary number of concentrated moving mass. *J. Franklin Inst.* 287, 115 – 123.
xv. Milormir, M., Stanisic, M. M. and Hardin, J. C. (1969): On the response of beams to an arbitrary number of concentrated masses. *Journal of the Franklin institute* 287(2), 115 – 123.
xvi. Sadiku, S. and Leipholz, H. H. E. (1987): On the dynamics of elastic systems with moving concentrated masses. *Ingenieur Archiv* 57, 223 – 242.
xvii. Oni, S. T. (1991): On the dynamic response of elastic structures to moving multi-mass system. *Ph.D. thesis*, University of Ilorin.
xviii. Gbadeyan, J. A. and Oni, S. T. (1995): Dynamic behaviour of beams and rectangular plates under moving loads. *Journal of sound and vibration* 182(5), 677 – 695.
xix. Stanisic, M. M., Hardin, J. C. and Lou, Y. C. (1968): On the response of the plate to a multi-masses moving system. *Actamechanica* 5, 37 – 53.
xx. Stanisic, M. M., Euler, J. A. and Mongomery, S. T. (1974): On a theory concerning the behaviour of structures carrying moving masses. *Ingenieurachiv* 43, 295 – 305.
xxi. Aiyesimi, Y. M. (1990): Dynamic response of elastic structures continuously supported by visco –elastic foundation and traversed by moving loads. *M.Sc. Thesis,* University of Ilorin.
xxii. Ramkumar R. L., Kamat, M. P. and Nayfeh, A. H. (1977): Vibration of highly prestressed anisotropic plates via numerical-perturbation technique. *Int. J. Solids & Structures*, Vol. 13.
xxiii. Gbadeyan, J. A. and Oni, S. T. (1992): Dynamic response to moving concentrated masses of elastic plates on a non-Winkler elastic plates. *Journal of sound and vibration* 154, 343 – 358.
xxiv. Shadnam, M. R., Mofid, M. and Akin, J. E. (2001): On the dynamic response of rectangular plate with moving mass. *Thin-walled structures* 39, 797 – 806.
xxv. Shadnam, M. R., Mofid, M. and Akin, J. E. (2002): Periodicity in the response of non-linear plate under moving mass. *Thin-walled structures* 40, 283 – 295.
xxvi. Oni, S. T. and Tolorunshagba J. M. (2005): Rotatory inertial influence on the highly prestressed orthotropic rectangular plates under travelling loads. *Journal of the Nigerian Association of Mathematical Physics*. Vol. 9, 103 – 126.
xxvii. Awodola, T. O and Omolofe, B. (2014): Response to concentrated moving masses of elastically supported rectangular plates resting on Winkler elastic foundation, *Journal of theoretical and applied mechanics,* vol. 44(3), pp 65 – 90. Sofia.
xxviii. 28 Awodola, T. O (2014): Flexural motions under moving concentrated masses of elastically supported rectangular plates resting on variable Winkler elastic foundation, *Latin American Journal of Solids and Structures,* 11(9), 1515 – 1540, Brazil.
xxix. Awodola, T. O and , Oni, S. T. (2013): Dynamic response to moving masses of rectangular plates with general boundary conditions and resting on variable Winkler foundation, *Latin American Journal of Solids and Structures,* 10(2), 301 – 322, Brazil.
xxx. Erich Zauderer (1989): *Partial differential equation of applied mathematics*, A Wiley inter-science publications, John Wiley and Sons, 601 – 615, New York.
xxxi. Cole, J. D. (1968): *Perturbation methods in applied mathematics*, Blaisdell: Waltham.
xxxii. Cole, J. D. (1975): Singular perturbation, *Advances in mathematics* 16, 380 – 392, California.
xxxiii. Hutter, K. and Olunloyo, V. O. S. (1974): Vibration of an anisotropically prestressed thick rectangular membrane with small bending rigidity. *Actamechanica* 20, 1 – 22.
xxxiv. Hutter, K. and Olunloyo, V. O. S. (1974): The transient and steady state response of a thick membrane to static and dynamic loading. *Zamm* 54, 795 – 806.
xxxv. Schneider, W. (1972): Einflubeinerkleinen Biegesteifikeitaufdie Querschwingungeneinereingespanntenrechtedkigenmembrain. *Acta Mechanica*13, 293 – 302.
xxxvi. Olunloyo, V. O. S. and Hutter, K. (1977): The response of an anisotropically prestressed thick rectangular membrane to dynamic loading. *Acta Mechanica* 28, 293 – 311.
xxxvii. Oyediran, A. A. and Gbadeyan J. A. (1986): Vibration of a prestressed orthotropic rectangular thin plate via singular perturbation technique. *Actamechanica* 64, 165 – 178.
xxxviii. Gbadeyan, J. A. and Oyediran A. A. (1988): A comparison of two singular perturbation techniques for initially stressed thin rectangular plate. *Journal of sound and vibration* 124(3), 517 – 528.
xxxix. Olunloyo, V. O. S. and Hutter, K. (1979): Forced vibration of prestressed rectangular membrane: near resonance response. *Actamechanica* 32, 63 – 77.
xl. Oni, S. T. (1994): Dynamic response to a moving load of a fully clamped prestressed orthotropic rectangular plate. *Journal of modelling, measurement and control* 53(3), 12 – 38.
xli. Oni, S. T. and Tolorunshagba J. M. (2005): Rotatory inertial influence on the highly prestressed orthotropic rectangular plates under travelling loads. *Journal of the Nigerian Association of Mathematical Physics*. Vol. 9, 103 – 126.

xlii. Oni, S. T. and Ogunbamike O. K. (2008): Transverse vibration under a moving load of a highly prestressed isotropic rectangular plate on a bi-parametric subgrade. *Journal of the Nigerian association of mathematical physics*. Vol. 13, 141 – 160.

xliii. Oni, S. T. and Adedowole, A. (2008): Influence of prestress on the response to moving loads of rectangular plates incorporating rotatory inertia correction factor. *Journal of the Nigerian association of mathematical physics*. Vol. 8, 127 – 140.

xliv. Are, E. B., Idowu, A. S. and Gbadeyan, J. A. (2013): Vibration of damped simply supported orthotropic rectangular plate resting on elastic foundation subjected to moving loads. *Advances in applied science research* 4(5), 387 – 393. U.S.A.