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Boundedness for the Transient Two-Dimensional Hydromagnetic Flow over an Impermeable Surface

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Abstract:
This paper examines the properties of the solution for the transient two-dimensional hydromagnetic flow over an impermeable surface with ohmic and viscous heat dissipation. Our results revealed that the fluid velocity $u(z,t)$, *species concentration* $\phi(z,t)$ *and medium temperature* $\theta(z,t)$ *are bounded.*

Keywords: *Boundedness, hydromagnetic flow, impermeable surface, transient, viscous fluid*

1. Introduction

Heat and mass transfer from a heated moving surface to a quiescent ambient medium occurred in many manufacturing processes such as hot rolling, wire drawing and crystal growing (Nag, 2006). Hydromagnetic flow is one of the fundamental problems in heat and mass transfer (Mohsen & Devood, 2016).

Investigations on hydrodynamic boundary layer flow and heat transfer over a stretching surface have gained appreciable attention due to their extensive applications in industry and their importance to several technological processes, which include the aerodynamic extrusion of plastic sheets, cooling of metallic sheets in a cooling bath, crystal growing. Crane (1970) investigated the steady boundary layer flow due to stretching surface with linear velocity. Many researchers, such as Gupta and Gupta (1977), Vleggaar (1997), and Chen and Char (1988), extended the work of Crane by considering the effects of heat and mass transfer analysis under different physical situations.

Simultaneous heat and mass transfer from different geometries may arise during industrial operations where the surface is sometimes stretched because of the process of drawing, for example, the process of cooling continuous strips or filament by drawing them through a quiescent fluid where simultaneous heat and mass transfer may occur during the cooling. Hence, it can be deduced that the combined heat and mass transfer can play a vital role in the problems of hydromagnetic flow over an impermeable surface. A new facet of approaching such problems can be given by considering the effect of thermal radiation. The thermal radiation effect might play a significant role in controlling the heat process in the polymer processing industry. The quality of the final product depends greatly on the heat-controlling factors and the knowledge of radiative heat transfer can perhaps lead to a desired product with a sought characteristic.

Many works have been reported on flow and heat transfer over a stretched surface in the presence of radiation. An analytical solution of MHD flow with radiation over a stretching sheet embedded in a porous medium was given by Anjali Devi and Kayalvizhi (2010). Makinde and Sibanda (2011) investigated the chemical reaction effects over the stretching surface in the presence of internal heat generation. Seini and Makinde (2013) studied the radiation and chemical reaction effects on MHD boundary layer flow over a stretching surface. Abdul Hakeem *et al*. (2014) investigated the thermal radiation effects on hydromagnetic flow over a stretching surface. The influence of thermal radiation on MHD flow over a stretching surface was studied by Jonnadula *et al*. (2015).

Durojaye and Ayeni (2011) consider a steady-state solution reaction kinetics model of polymerization in the presence of material diffusion. They obtained steady-state equations for the resulting partial differential equations. Criteria for the existence and uniqueness of solutions of the equations and numerical results were also provided. They concluded that the steady state equation is bounded and has a solution under reasonable physical conditions.

Durojaye and Agee (2019) investigated the one-dimensional, positive temperature coefficient (PTC) thermistor equation, using the hyperbolic-tangent function as an approximation to the electrical conductivity of the device. They observed that the steady state solution using the new approximation yielded a distribution of device temperature over the spatial dimension and all the phases of the temperature distribution of the device without having to look for a moving boundary. They analysed the steady state solution and the numerical solution of the unsteady state.

Taking consideration of dissipation effects in the study of heat and mass transfer boundary layer problems adds a new dimension to it. Gebhart (1962) was the first who studied the problem taking into account the viscous dissipation. Kayalvizhi *et al*. (2016) and Dessie and Kishan (2015) examined the effects of viscous dissipation and ohmic dissipation on MHD flow over a stretching surface with thermal radiation effects.

All the above-mentioned studies are confined to the steady state flow problems. However, in certain practical problems, the motion of the stretched surface may start impulsively from rest. In such cases, the transient or unsteady aspects become more interesting. Effects of radiation and heat transfer over an unsteady stretching surface in the presence of a heat source or sink were studied by Elbashbeshy and Emam (2011). Makinde (2012) analyzed the chemically reacting hydromagnetic unsteady flow of a radiating fluid. Yusof *et al*. (2012) analyzed the radiation effect on unsteady MHD flow over a stretching surface. Mass transfer and MHD effect on an unsteady stretching surface were investigated by Ramana Reddy and Bhaskar Reddy (2013). Seini and Makinde (2013) analyzed the radiation and recently, unsteady MHD flow and heat transfer over a stretching permeable surface were investigated by Choudhary *et al*. (2015). Reddy *et al*. (2015) considered the thermal radiation and viscous dissipation effects on unsteady MHD flow over a stretching surface.

Durojaje *et al.* (2013) also presented a mathematical model for free racial polymerization in the presence of material diffusion. They proved the existence and uniqueness of the solution of the model. They used the parameter expanding method and sought direct eigen functions expansion to obtain an analytical solution to the model. The results were presented graphically and discussed. It was discovered that the mixture temperature and monometer concentration were significantly influenced by Kamenetskii number and thermal diffusivity of the mixture.

However, to the best of the author's knowledge, no attempt has been made to investigate the effects of thermal radiation, viscous dissipation, ohmic dissipation and heat and mass transfer effects on transient hydromagnetic flows over an impermeable surface.

Being motivated by the extensive applications, this paper seeks to investigate the heat and mass transfer effects on transient two-dimensional hydromagnetic flow over an impermeable surface.

The objectives of this paper are to establish the criteria for the existence of a unique solution of the transient two-dimensional hydromagnetic flow over an impermeable surface with ohmic and viscous heat dissipation and to examine the properties of the solution under certain conditions.

2. Model Formulation

The two-dimensional, transient hydromagnetic flow of a viscous, incompressible, electrically conducting, and radiating fluid, along with heat and mass transfer over an impermeable surface with ohmic and viscous dissipation, is considered. A constant magnetic field B is applied in the direction perpendicular to that of the fluid flow.

The formulation of our model is guided by the following assumptions:

- The fluid is considered to be grey.
- The radiative heat flux in the x-direction is negligible in comparison with that in the y-direction.

Under these assumptions, the governing equations that are based on the laws balancing mass, linear momentum, energy, and concentration for the present investigation are given as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\mathrm{V}\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2 u}{\rho} \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_0\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho c_P}\frac{\partial q_r}{\partial y}+\frac{\mathrm{V}}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B^2 u^2}{\rho c_P} \tag{3}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2} \tag{4}$$

Rosseland approximation is used to simplify the radiative heat flux term in the energy equation, which has the form:

$$q_r=-\frac{4\sigma^*}{3k_1}\frac{\partial T^4}{\partial y} \tag{5}$$

The temperature difference within the flow is assumed to be sufficiently small such that T^4 may be expressed as a linear function of temperature. On expanding T^4 in a Taylor series about T_∞ and thereby neglecting the higher-order terms, it is obtained as follows:

$$T^4\approx 4T_\infty^3 T-3T_\infty^4 \tag{6}$$

The above-governing equations are associated with the following initial and boundary conditions:

$$\left.\begin{array}{l} u(x,y,t)=0,\ T(x,y,t)=T_0,\quad C(x,y,t)=C_0\quad x\geq 0, y\geq 0,\ \ t=0. \\ \qquad\left.\begin{array}{lll} \frac{\partial u}{\partial x}=0, & \frac{\partial u}{\partial y}=0, & \frac{\partial T}{\partial x}=0, \\ \frac{\partial T}{\partial y}=0, & \frac{\partial C}{\partial x}=0, & \frac{\partial C}{\partial y}=0 \end{array}\right\}\ x=0,\ y=0,\ t>0 \\ u(x,y,t)=U_\infty,\ T(x,y,t)=T_\infty,\quad C(x,y,t)=C_\infty,\ \ x=L, y=H, t\geq 0. \end{array}\right\} \tag{7}$$

Where: u is the velocity component along the x-axis, v is the velocity component along the y-axis, V is the kinematic coefficient of viscosity, σ is the electrical conductivity of the fluid, B is the strength of the applied variable magnetic field, ρ is the fluid density, T is the temperature of the fluid, $\alpha_0 = K/_{\rho}\, c_\rho$ is the thermal diffusivity with K as the thermal conductivity of the fluid, c_ρ is the specific heat capacity at constant pressure, q_r is the radiative heat flux, C is the concentration of the fluid, D is the coefficient of mass diffusivity, q_r is the radiative heat flux, σ^* is the Stefan -Boltzman constant, k_1 is the mean absorption coefficient.

3. Method of Solution

3.1. Transformation

Introducing the following new space variable (Olayiwola *et al*., 2013):

$$z = x + y \tag{8}$$

The equations (1–6), together with initial and boundary conditions (7), become:

$$\frac{\partial U}{\partial z} = 0 \tag{9}$$

$$\frac{\partial u}{\partial t} + U\,\frac{\partial u}{\partial z} = \mathrm{V}\,\frac{\partial^2 u}{\partial z^2} - \frac{\sigma B^2 u}{\rho} \tag{10}$$

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial z} = \alpha_0\frac{\partial^2 T}{\partial z^2} = \frac{16\sigma^* T_\infty}{3k_1\rho c_p}\frac{\partial^2 T}{\partial z^2} + \frac{\mathrm{V}}{c_p}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{\sigma B^2 u^2}{\rho c_p} \tag{11}$$

$$\frac{\partial C}{\partial t} + U\,\frac{\partial C}{\partial z} = D\,\frac{\partial^2 C}{\partial z^2} \tag{12}$$

$$\left.\begin{array}{l} U(z,t) = 0,\ T(z,t) = T_0,\quad C(z,t) = C_0\quad z \ge 0, t = 0. \\ \dfrac{\partial U}{\partial z} = 0,\quad \dfrac{\partial T}{\partial z} = 0,\ \dfrac{\partial C}{\partial z} = 0,\quad z = 0,\ t > 0 \\ U(z,t) = U_\infty,\ T(z,t) = T_\infty,\quad C(z,t) = C_\infty\quad z = h,\quad t \ge 0. \end{array}\right\} \tag{13}$$

Where: $U = u + v$

3.2. Dimensional Analysis

Introducing the following non-dimensional variables:

$$t' = \frac{Ut}{h},\quad z' = \frac{z}{h},\quad u' = \frac{u}{U},\quad \theta = \frac{T-T_0}{T_\infty - T_0},\quad \varphi = \frac{C-C_0}{C_\infty - C_0} \tag{14}$$

Then, equations (9) – (13) become:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = \frac{1}{R_e}\frac{\partial^2 u}{\partial z^2} - Mu \tag{15}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial z} = \frac{1}{p_e}\frac{\partial^2 \theta}{\partial z^2} - R\,\frac{\partial^2 \theta}{\partial z^2} + \frac{E_c}{R_e}\left(\frac{\partial u}{\partial z}\right)^2 + E_c M\, u^2 \tag{16}$$

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial z} = \frac{1}{p_{em}}\frac{\partial^2 \varphi}{\partial z^2} \tag{17}$$

$$\left.\begin{array}{l} u(z,0) = 0,\quad \left.\dfrac{\partial u}{\partial z}\right|_{z=0} = 0,\ u(1,t) = \alpha \\ \theta(z,0) = 0,\quad \left.\dfrac{\partial \theta}{\partial z}\right|_{z=0} = 0,\ \theta(1,t) = 1 \\ \varphi(z,0) = 0,\quad \left.\dfrac{\partial \varphi}{\partial z}\right|_{z=0} = 0,\ \varphi(1,t) = 1 \end{array}\right\} \tag{18}$$

Where:

$$R_e = \frac{hv}{\mathrm{V}},\ M = \frac{\sigma B^2 h}{\rho v},\ p_e = \frac{hv}{\alpha_0},\quad E_c = \frac{v^2}{c_p(T_\infty - T_0)},\ p_{em} = \frac{hv}{D},\ R = \frac{16\sigma^* T_\infty}{3k_1\rho c_p hv}$$

3.3. Properties of Solution

First, we extend the domain from $0 \le z \le 1$ to $0 \le z \le \infty$ and consider equations (15) – (18) when u is constant and $\left(\frac{1}{P_e} - R\right) = \frac{1}{P_{em}}$.

Then equation (15) - (17) reduces to:

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial z} = \frac{1}{p_{em}}\frac{\partial^2 \theta}{\partial z^2} + E_c M\, u^2 \tag{19}$$

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial z} = \frac{1}{p_{em}}\frac{\partial^2 \varphi}{\partial z^2} \tag{20}$$

Adding (19) and (20) leads to:

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial z} = \frac{1}{p_{em}}\frac{\partial^2 \psi}{\partial z^2} + E_c M\, u^2 \tag{21}$$

Where:

$$\psi = (\theta + \varphi)$$

We make a change of variable by introducing (Olayiwola, 2011).

$$\eta = \left(\frac{\rho^2}{P_{em}}\right)^{-\frac{1}{2}}\int_0^z \rho\, ds \tag{22}$$

Using (22), the coordinate transformation becomes:

$$\frac{\partial}{\partial z} \rightarrow \frac{\partial}{\partial \eta}, \ \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \frac{\partial}{\partial \eta} \tag{23}$$

Using (23), equation (21) in terms of η *and* t are:

$$\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial \eta^2} = E_c M\, u^2 \tag{24}$$

With initial and boundary conditions:

$$\psi(\eta, 0) = 0,\ \psi(0,t) = C_1 + C_2,\ \psi(\infty, t) \rightarrow 2 \tag{25}$$

Theorem 1: Let$M \rightarrow 0$. Then, there exists at most one bounded solution of equations (19) and (20) satisfying (18).

Proof: We add (19) and (20) and obtain (24) satisfying (25) as earlier done.

Using the Fourier sine transform (see Myint-U and Debnath, (1987)), we obtain the solution to the problem (24) in compact form as:

$$\psi(\eta, t) = \frac{2}{\pi}(C_1 + C_2) \int_0^\infty s \sin s\eta \int_0^t e^{-s^2(t-\tau)}\, d\tau ds \tag{26}$$

That is:

$$\psi(z, t) = \frac{2}{\pi}(C_1 + C_2) \int_0^\infty s \sin \sqrt{p_{em}}\, zs \int_0^t e^{-s^2(t-\tau)}\, d\tau ds \tag{27}$$

Then, we obtain:

$$\varphi(z, t) = \frac{2}{\pi}(C_1 + C_2) \int_0^\infty s \sin \sqrt{p_{em}}\, zs \int_0^t e^{-s^2(t-\tau)}\, d\tau ds - \theta(z,t) \tag{28}$$

$$\theta(z, t) = \frac{2}{\pi}(C_1 + C_2) \int_0^\infty s \sin \sqrt{p_{em}}\, zs \int_0^t e^{-s^2(t-\tau)}\, d\tau ds - \varphi(z,t) \tag{29}$$

Hence, there exists a unique solution to problems (19) and (20). This completes the proof.

Next, we shall examine the properties of the solution of equations (15) - (18).

Here, we show that $u(z,t)$, $\theta(z,t)$, $\psi(z,t)$ are bounded. We consider equation (15) – (18) when $u(0,t) = a$, $\theta(0,t) = b$, and $\varphi(0,t) = c$. Then equations (15) - (18) become:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = \frac{1}{R_e}\frac{\partial^2 u}{\partial z^2} - Mu \\ &u(z,0) = 0,\quad u(0,t) = a,\ u(1,t) = \alpha \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} &\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial z} = \frac{1}{p_e}\frac{\partial^2 \theta}{\partial z^2} - R\frac{\partial^2 \theta}{\partial z^2} + \frac{E_c}{R_e}\left(\frac{\partial u}{\partial z}\right)^2 + E_c M\, u^2 \\ &\theta\ (z,0) = 0,\ \theta(0,t) = b,\ \theta(1,t) = 1 \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} &\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial z} = \frac{1}{p_{em}}\frac{\partial^2 \varphi}{\partial z^2} \\ &\varphi(z,0) = 0\ \varphi(0,t) = c,\ \varphi(1,t) = 1, \end{aligned}\right\} \tag{32}$$

Theorem 2: Let $M > 0,\ a > 0,\ b > 0,\ c > 0,\ \ \alpha > 0$, then the equations (30) – (32) have a solution for all. $t \geq 0$.
In the proof, we shall need the following Definitions:

Definition 1 (Olayiwola, 2011): A smooth function $\underline{u}$ is said to be a lower solution to the problem $Lu = f(x,t,u)$
Where:

$$L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} + c(x,t)$$

If $\underline{u}$ satisfies,

$$L\underline{u} \leq f(x,t,\underline{u})$$

$$\underline{u}(x,0) \leq f(x),\quad \underline{u}(0,t) \leq h_1(t),\quad \underline{u}(L,t) \leq h_2(t)$$

Definition 2 (Olayiwola, 2011): A smooth function $\overline{u}$ is said to be an upper solution to the problem.
$Lu = f(x,t,u)$
Where:

$$L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} + c(x,t)$$

If $\overline{u}$ satisfies,

$$L\overline{u} \geq f(x,t,\overline{u})$$

$$\overline{u}(z,0) \geq f(x),\quad \overline{u}(0,t) \geq h_1(t),\quad \overline{u}(L,t) \geq h_2(t)$$

Proof of Theorem 2: Equations (30) – (32) can be written respectively as:

$$Lu = f(z,t,u),\ L\theta = f(z,t,\theta),\ L\varphi = F(z,t,\varphi) \tag{33}$$

Where:

$$Lu = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = \frac{1}{R_e}\frac{\partial^2 u}{\partial z^2} - Mu, f(z,t,u) = 0$$

$$L\theta = \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial z} - \left(\frac{1}{p_e} - R\right)\frac{\partial^2 \theta}{\partial z^2}, f(z,t,\theta) = \frac{E_c}{R_e}\left(\frac{\partial u}{\partial z}\right)^2 + E_c M\, u^2$$

$$L\varphi = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial z} = \frac{1}{p_{em}}\frac{\partial^2 \varphi}{\partial z^2}, f(z,t,\varphi) = 0$$

Consider:

$$\underline{u}(z,t) = 0,\ \underline{\theta}(z,t) = 0,\ \underline{\varphi}(z,t) = 0 \tag{34}$$

We shall show that (34) are the lower solutions.

Clearly,

$$\left.\begin{array}{l} \underline{u}(z,0)=0,\ \underline{u}(0,t)=0,\ \underline{u}(1,t)=0 \\ \underline{\theta}(z,0)=0,\ \underline{\theta}(0,t)=0,\ \underline{\theta}(1,t)=0 \\ \underline{\varphi}(z,0)=0,\ \underline{\varphi}(0,t)=0,\ \underline{\varphi}(1,t)=0 \end{array}\right\} \quad (35)$$

Now,

$$\frac{\partial \underline{u}}{\partial t}=\frac{\partial \underline{u}}{\partial z}=\frac{\partial^2 \underline{u}}{\partial z^2}=\underline{u}=0 \quad (36)$$

$$\frac{\partial \underline{\theta}}{\partial t}=\frac{\partial \underline{\theta}}{\partial z}=\frac{\partial^2 \underline{\theta}}{\partial z^2}=0 \quad (37)$$

$$\frac{\partial \underline{\varphi}}{\partial t}=\frac{\partial \underline{\varphi}}{\partial z}=\frac{\partial^2 \underline{\varphi}}{\partial z^2}=0 \quad (38)$$

These imply:

$$L\underline{u}=0,\quad L\underline{\theta}=0,\quad L\underline{\varphi}=0 \quad (39)$$

and

$$f(z,t,\underline{u})=0,\ f(z,t,\underline{\theta})=0,\ f\left(z,t,\underline{\varphi}\right)=0 \quad (40)$$

Hence,

$$L\underline{u}\le f(z,t,\underline{u}),\ L\underline{\theta}\le f(z,t,\underline{\theta}),\ L\underline{\varphi}\le F\left(z,t,\underline{\varphi}\right)$$

By definition 1, $\underline{u}(z,t)=0,\ \underline{\theta}(z,t)=0,\ \underline{\varphi}(z,t)=0$ are the lower solutions.

Also, consider,

$$\left.\begin{array}{l} \overline{u}(z,t)=(1+a)+t \\ \overline{\theta}(z,t)=(1+b)+t\left(E_cM\left((1+a)+t\right)^2\right) \\ \overline{\varphi}(z,t)=(1+c)+t \end{array}\right\} \quad (41)$$

We shall show that (41) is the upper solution.

Clearly,

$$\left.\begin{array}{l} \overline{u}(z,0)=(1+a),\overline{u}(0,t)=(1+a)+t,\ \overline{u}(1,t)=(1+a)+t \\ \overline{\theta}(z,t)=(1+b),\overline{\theta}(0,t)=(1+b)+t\ \left(E_cM\left((1+a)+t\right)^2\right), \\ \qquad \overline{\theta}(1,t)=(1+b)+t\ \left(E_cM\left((1+a)+t\right)^2\right) \\ \overline{\varphi}(z,0)=(1+c),\overline{\varphi}(0,t)=(1+c)+t,\ \overline{\varphi}(1,t)=(1+c)+t \end{array}\right\} \quad (42)$$

Now,

$$\frac{\partial \overline{u}}{\partial t}=1,\ \frac{\partial \overline{u}}{\partial z}=\frac{\partial^2 \overline{u}}{\partial z^2}=0\ ,\ \overline{u}=(1+a)+t \quad (43)$$

$$\frac{\partial \overline{\theta}}{\partial t}=E_cM\left((1+a)+t\right)^2+2E_cM\left((1+a)+t\right)t\ ,\ \ \frac{\partial \overline{\theta}}{\partial z}=\frac{\partial^2 \overline{\theta}}{\partial z^2}=0 \quad (44)$$

$$\frac{\partial \overline{\varphi}}{\partial t}=1,\ \frac{\partial \overline{\varphi}}{\partial z}=\frac{\partial^2 \overline{\varphi}}{\partial z^2}=0 \quad (45)$$

These imply:

$$\left.\begin{array}{l} L\overline{u}=M\left((1+a)+t\right)^2 \\ L\overline{\theta}=E_cM\left((1+a)+t\right)^2+2E_cM\left((1+a)+t\right)t \\ L\overline{\varphi}=1 \end{array}\right\} \quad (46)$$

and

$$f(z,t,\overline{u})=0,\ f\left(z,t,\overline{\theta}\right)=E_cM\left((1+a)+t\right)^2,\ f(z,t,\overline{\varphi})=0 \quad (47)$$

Hence,

$$L\overline{u}\ge f(z,t,\overline{u}),\quad L\overline{u}\ge f\left(z,t,\overline{\theta}\right),\quad L\overline{u}\ge F(z,t,\overline{\varphi})$$

By definition 2, (41) are the lower solutions.

Thus, there exists a solution to problems $(30)-(32)$. This completes the proof.

4. Conclusion

To examine the properties of the solution of the transient two-dimensional hydromagnetic flow over an impermeable surface with ohmic and viscous heat dissipation, we used the actual solution approach and method of upper and lower solution as presented in the work of Olayiwola (2011). The results obtained revealed that the fluid velocity $u(z,t)$, species concentration $\phi(z,t)$ and temperature $\theta(z,t)$ are bounded.

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